Gaussian Lowpass Filters

The form of Gaussian lowpass filters (GLPFs) in two dimensions is given by

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}, \qquad (4.8\text{-}6)$$

where $D(u,v)$ is the distance from the center of the frequency rectangle. As mentioned previously, σ is a measure of spread about the center. Let $\sigma = D_0$, we can express the filter using the notation of other filters

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}, \qquad (4.8\text{-}7)$$

where D_0 is the cutoff frequency. When $D(u,v) = D_0$, the GLPF is down to 0.607 of its maximum value.

As shown in Table 4.3, the inverse Fourier transform of the GLPF is Gaussian as well. So a spatial Gaussian filter, obtained by the IDFT of (4.8-6) and (4.8-7), will have no ringing.

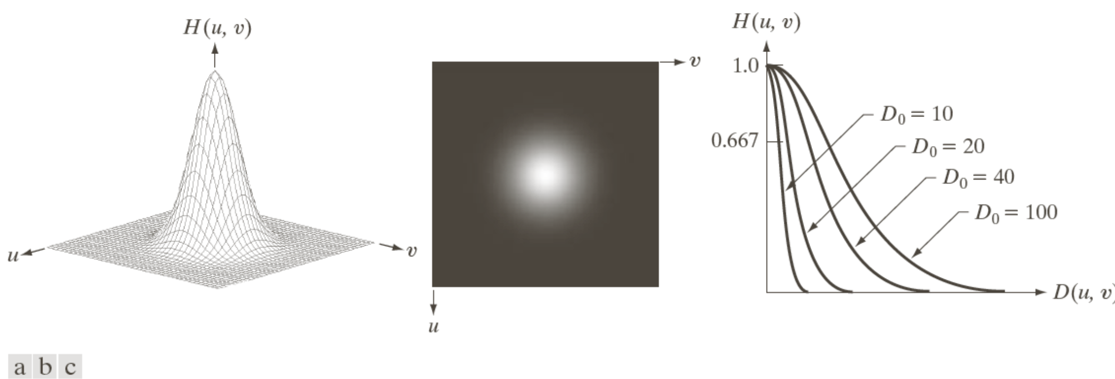


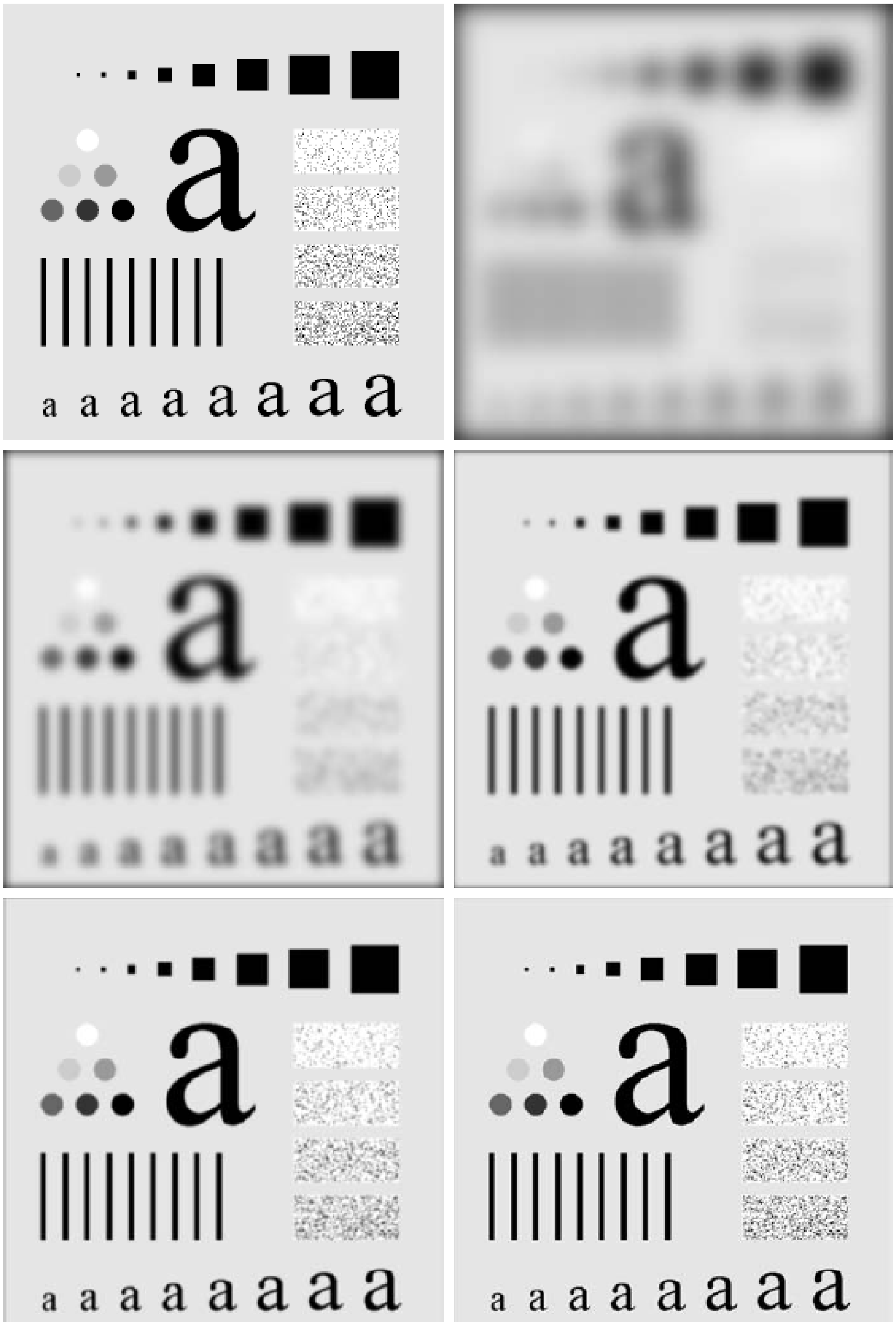
FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Table 4.4 summarizes the lowpass filters we have discussed.

TABLE 4.4
Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

Example 4.18: Image smoothing with a Gaussian lowpass filter

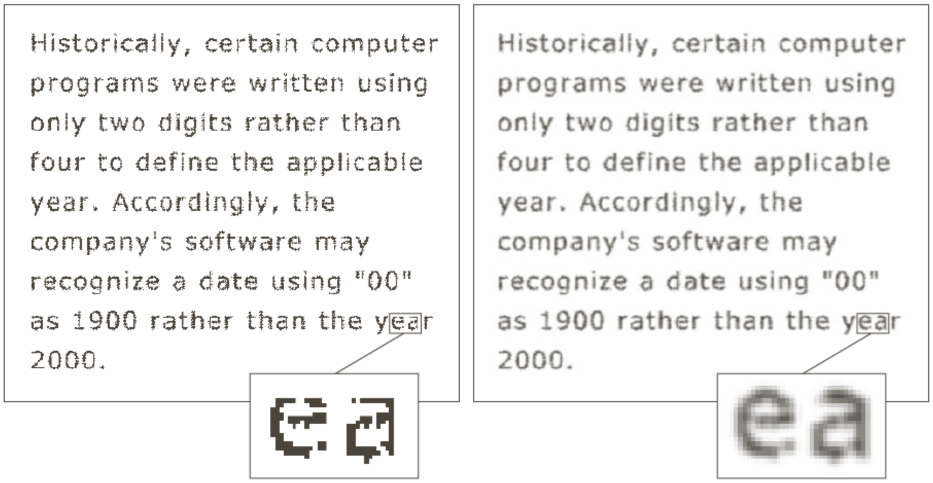


a b
c d
e f

FIGURE 4.48 (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

Additional Examples of Lowpass Filtering

Figure 4.49 (a) shows a sample of text of poor resolution.



a b

FIGURE 4.49 (a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

To deal with the characters that have distorted shapes due to lack of resolution, one approach is to bridge small gaps in the input image by blurring them.

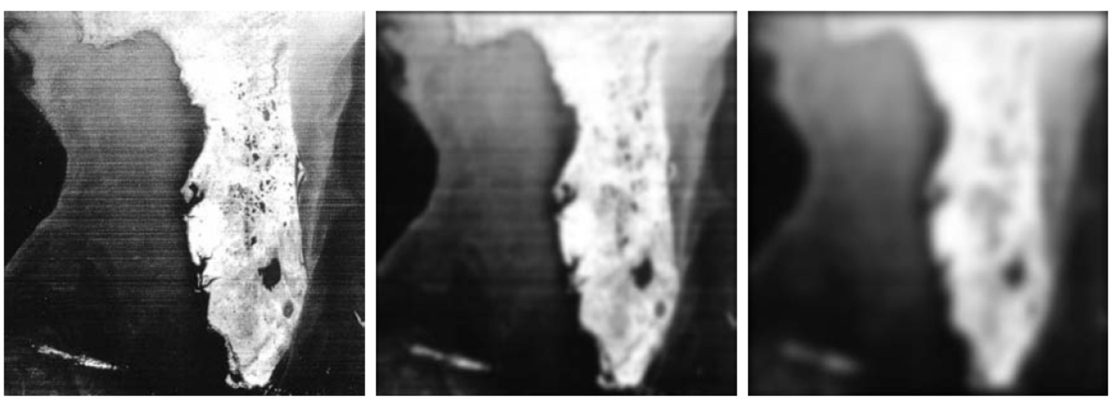
Figure 4.49 (b) shows a result of using a Gaussian lowpass filter with $D_0 = 80$. The images are of size 444×508.

Figure 4.50 shows an application of lowpass filtering for producing a smoother, softer-looking result from a sharp original.

a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Figure 4.51 shows two applications of lowpass filtering on the same image with $D_0 = 50$ and $D_0 = 20$.



a b c

FIGURE 4.51 (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

4.9 Image Sharpening Using Frequency Domain Filters

A highpass filter is obtained from a given lowpass filter using the equation

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)\,, \qquad (4.9\text{-}1)$$

where $H_{LP}(u,v)$ is the transfer function of the lowpass filter.

As in the previous section, we will also discuss ideal, Butterworth, and Gaussian highpass filters, which are shown in Figure 4.52.

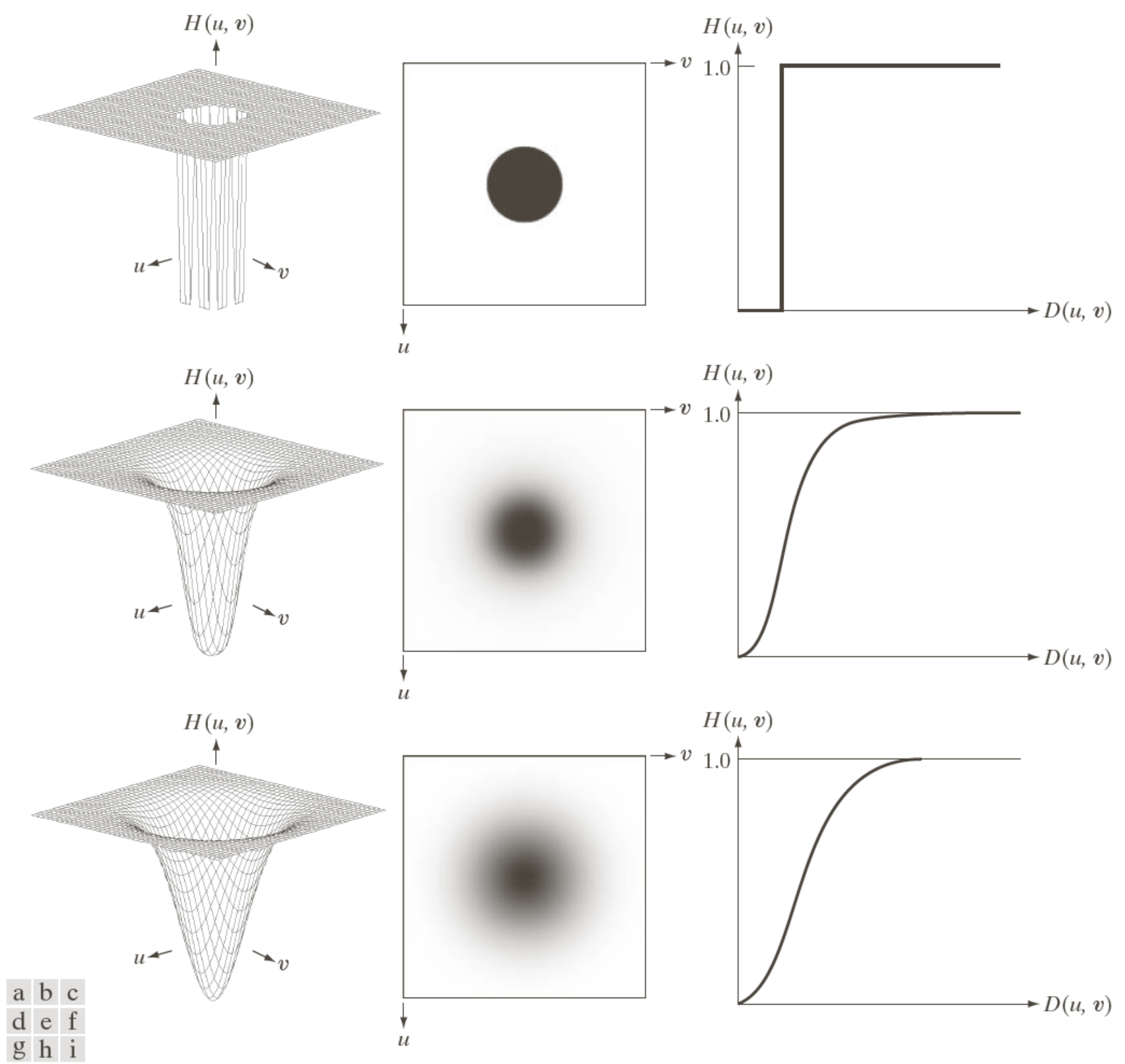
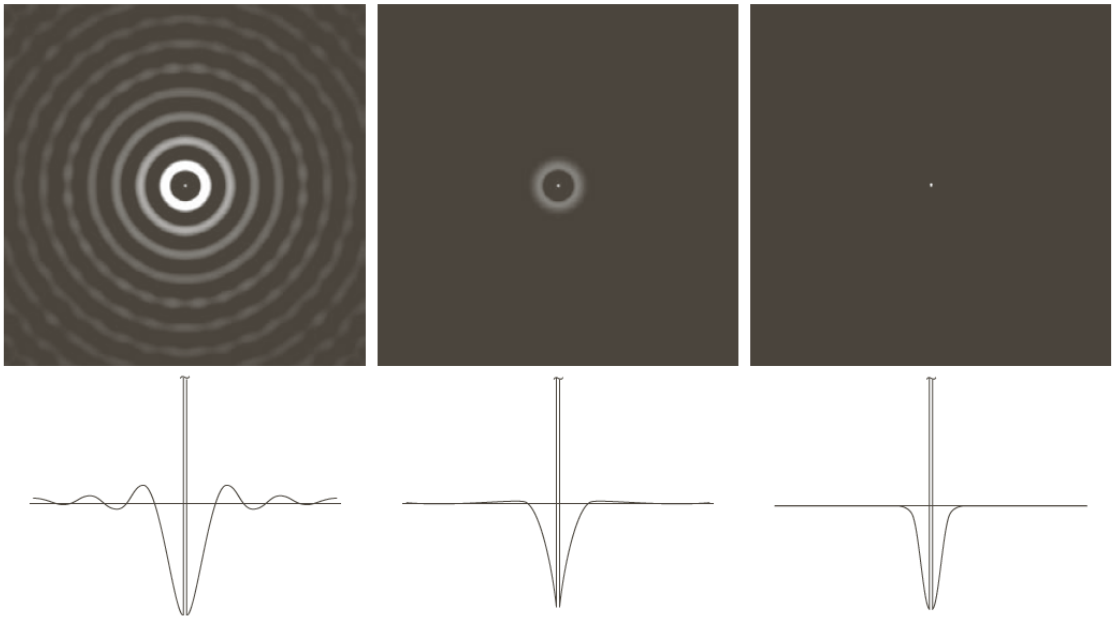


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Figure 4.53 shows what these filters look like in the spatial domain.



a b c

FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Ideal Highpass Filters

A 2-D ideal highpass filter (IHPF) is defined as

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases} \qquad (4.9\text{-}2)$$

Because of the way they are related, we can expect IHPFs to have the same ringing properties as ILPFs.

Figure 4.54 shows the various IHPF results of using the original image in Figure 4.41 (a) with D_0 equal to 30, 60, and 160 pixels.

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30$, 60, and 160.

Butterworth Highpass Filters

A 2-D Butterworth highpass filter (BHPF) of order n and cutoff frequency D_0 is defined as

$$H(u,v) = \frac{1}{1 + [D_0 \,/\, D(u,v)]^{2n}}, \qquad (4.9\text{-}3)$$

where $D(u,v)$ is given by

$$D(u,v) = \sqrt{(u - P/2)^2 + (v - Q/2)^2}\,. \qquad (4.8\text{-}2)$$

Figure 4.55 shows the performance of a BHPF.

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

Gaussian Highpass Filters

The transfer function of the Gaussian highpass filter (GHPF) with cutoff frequency locus at a distance D_0 from the center of the frequency rectangle is defined as

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}. \quad (4.9\text{-}4)$$

Figure 4.56 shows some comparable results from using GHPF s.

a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30$, 60, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

As expected, the results are more gradual than with the IHPFs and BHPFs.

Table 4.5 contains a summary of the highpass filters we have discussed.

TABLE 4.5
Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

where the expression for Ideal should read as

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Example 4.19: Using highpass filtering and thresholding for image enhancement

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

The Laplacian in the Frequency Domain

The Laplacian was used for image enhancement in the spatial domain. Now, we show that the Laplacian can yield equivalent results using frequency domain techniques.

The Laplacian can be implemented in the frequency domain using the filter

$$H(u,v) = -4\pi^2(u^2 + v^2), \tag{4.9-5}$$

or, with respect to the center of the frequency rectangle, using the filter

$$\begin{aligned} H(u,v) &= -4\pi^2\left[(u - P/2)^2 + (v - Q/2)^2\right] \\ &= -4\pi^2 D^2(u,v) \end{aligned} \tag{4.9-6}$$

Then, the Laplacian image is obtained as

$$\nabla^2 f(x,y) = -\mathscr{F}^{-1}\{H(u,v)F(u,v)\}, \tag{4.9-7}$$

where $F(u,v)$ is the DFT of $f(x,y)$.

As explained in Chapter 3, the enhancement is achieved using the question

$$g(x,y) = f(x,y) + c\nabla^2 f(x,y), \tag{4.9-8}$$

where $c = -1$ because $H(u,v)$ is negative.

In the frequency domain, (4.9-8) is written as

$$\begin{aligned} g(x,y) &= \mathscr{F}^{-1}\{F(u,v) - H(u,v)F(u,v)\} \\ &= \mathscr{F}^{-1}\{[1 - H(u,v)]F(u,v)\} \\ &= \mathscr{F}^{-1}\left\{\left[1 + 4\pi^2 D^2(u,v)\right]F(u,v)\right\} \end{aligned} \tag{4.9-9}$$

Although (4.9-9) is an elegant result, there are the scaling issues that would make the computation more difficult. Therefore, equation (4.9-8) is the preferred implementation in the frequency domain.

Example 4.20: Image sharpening in the frequency domain using the Laplacian

Figure 4.58 (a) is the same image as Figure 3.38 (a), and Figure 4.58 (b) shows the result of using

$$g(x,y) = f(x,y) + c\nabla^2 f(x,y)\,, \qquad (4.9\text{-}8)$$

in which the Laplacian was computed in the frequency domain using

$$\nabla^2 f(x,y) = -\mathscr{F}^{-1}\left\{H(u,v)F(u,v)\right\}. \qquad (4.9\text{-}7)$$

By comparing Figure 4.58 (a) and Figure 3.38 (a), we see that the frequency domain and spatial results are identical visually.

a b

FIGURE 4.58 (a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

Unsharp Masking, Highboost Filtering, and High-Frequency-Emphasis Filtering

Using frequency domain methods, the mask defined in

$$g_{\text{mask}}(x,y) = f(x,y) - \overline{f}(x,y) \qquad (3.6\text{-}8)$$

is given by

$$g_{\text{mask}}(x,y) = f(x,y) - f_{LP}(x,y) \qquad (4.9\text{-}10)$$

where

$$f_{LP}(x,y) = \mathscr{F}^{-1}[H_{LP}(u,v)F(u,v)] \qquad (4.9\text{-}11)$$

is a smoothed image analogous to $\overline{f}(x,y)$ in (3.6-8), $H_{LP}(u,v)$ is a lowpass filter, and $F(u,v)$ is the Fourier transform of $f(x,y)$.

Then, as in (3.6-9)

$$g(x,y) = f(x,y) + k * g_{\text{mask}}(x,y), \qquad (4.9\text{-}12)$$

which defines unsharp masking when $k = 1$ and highboost filtering when $k > 1$.

Equation (4.9-12) can be expressed in terms of frequency domain computations involving a lowpass filter

$$g(x,y) = \mathscr{F}^{-1}\{[1 + k * [1 - H_{LP}(u,v)]]F(u,v)\} \qquad (4.9\text{-}13)$$

We also can express this result in terms of a highpass filter

$$g(x,y) = \mathscr{F}^{-1}\{[1 + k * H_{HP}(u,v)]F(u,v)\} \qquad (4.9\text{-}14)$$

In (4.9-14), the term $[1 + k * H_{HP}(u,v)]$ is called a high-frequency-emphasis filter.

A slightly more general formulation of high-frequency-emphasis filtering is given in the expression:

$$g(x,y) = \mathscr{F}^{-1}\{[k_1 + k_2 * H_{HP}(u,v)]F(u,v)\} \qquad (4.9\text{-}15)$$

where $k_1 \geq 0$ gives controls of the offset from the origin and $k_2 \geq 0$ controls the contribution of high frequencies.

Example 4.21: Image enhancement using high-frequency-emphasis filtering.

Figure 4.59 (a) shows a 416×596 chest X-ray with a narrow range of intensity levels. Figure 4.59 (b) shows the result of highpass filtering using a Gaussian filter with $D_0 = 40$.

Figure 4.59 (c) shows the advantage of high-emphasis filtering with $k_1 = 0.5$ and $k_2 = 0.75$. Figure 4.59 (d) shows the result of applying the histogram equalization on Figure 4.59 (c).

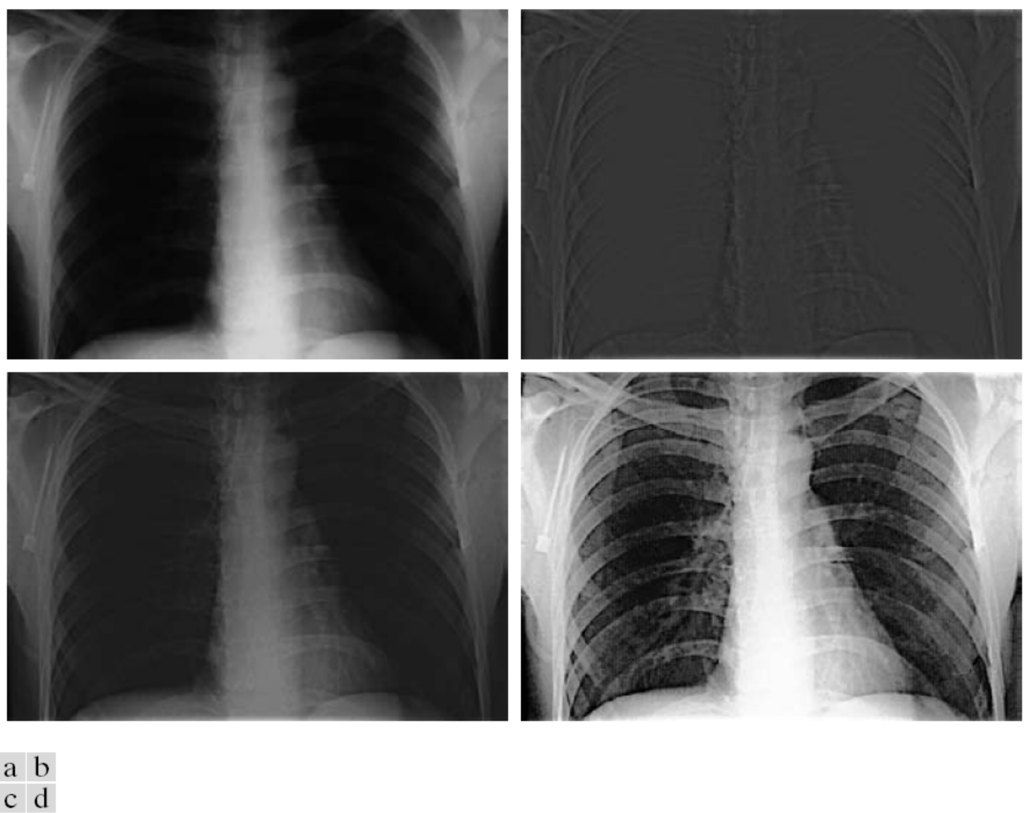


FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

4.10 Selective Filtering

All filters we have discussed in the previous two sections operate over the entire frequency rectangle.

There are filters to process specific bands of frequencies or small regions of the rectangle, which are called bandreject or bandpass filters.

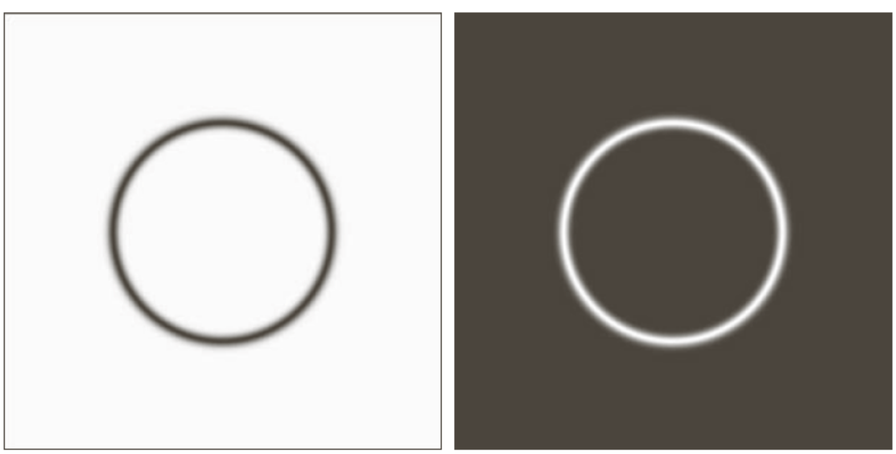
Bandreject and Bandpass Filters

Table 4.6 shows expressions for ideal, Butterworth, and Gaussian bandreject filters.

TABLE 4.6
Bandreject filters. W is the width of the band, D is the distance $D(u, v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u, v)$ to simplify the notation in the table.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

Figure 4.63 (a) shows a Gaussian bandreject filter in image form.



a b
FIGURE 4.63
(a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

A bandpass filter is obtained from a bandreject filter

$$H_{BP}(u,v) = 1 - H_{BR}(u,v)\,. \qquad (4.10\text{-}1)$$

Figure 4.63 (b) shows a Gaussian bandpass filter in image form.

Notch Filters

A notch filter will reject (or pass) frequencies in a predefined neighbourhood.

Since zero-phase-shift filters must be symmetric about the origin, a notch with center at (u_0, v_0) must have a corresponding notch at location $(-u_0, -v_0)$.

Notch reject filters are constructed as products of highpass filters whose centers have been translated to the centers of the notches:

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v), \qquad (4.10\text{-}2)$$

where $H_k(u,v)$ and $H_{-k}(u,v)$ are highpass filters whose centers are at (u_k, v_k) and $(-u_k, -v_k)$. These "centers" are specified to the center of the frequency rectangle, $(M/2, N/2)$.

The distance computations for each filter are

$$D_k(u,v) = \sqrt{(u - M/2 - u_k)^2 + (v - N/2 - v_k)^2} \qquad (4.10\text{-}3)$$

and

$$D_{-k}(u,v) = \sqrt{(u - M/2 + u_k)^2 + (v - N/2 + v_k)^2} \qquad (4.10\text{-}3)$$

For example, the following is a Butterworth notch reject filter of order *n*, containing three notch pairs

$$H_{NR}(u,v) = \prod_{k=1}^{3} \left[\frac{1}{1 + [D_{0k} / D_k(u,v)]^{2n}} \right] \left[\frac{1}{1 + [D_{0k} / D_{-k}(u,v)]^{2n}} \right] \quad (4.10\text{-}5)$$

The constant D_{0k} is the same for each pair of notches, but can be different for different pairs.

Other notch reject filters are constructed in the same manner, depending on the highpass filter chosen.

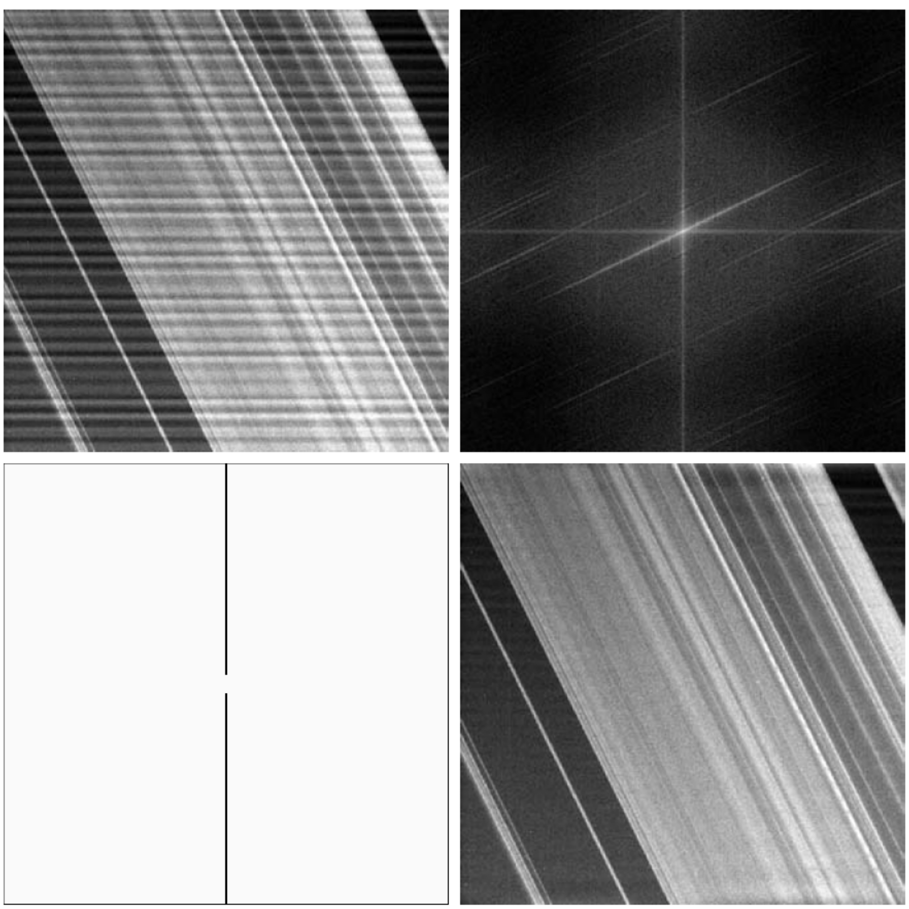
A notch pass filter is obtained from a notch reject filter by

$$H_{NP}(u,v) = 1 - H_{NR}(u,v)\,. \qquad (4.10\text{-}6)$$

Example 4.24: Enhancement of corrupted Cassini Saturn image by notch filtering

Figure 4.65 (a) shows an image of part of the rings surrounding the planet Saturn. Figure 4.65 (b) shows the DFT spectrum.

Figure 4.65 (c) shows a narrow notch rectangle filter (white represents 1 and black 0). Figure 4.65 (d) shows the result of filtering the corrupted image with this filter with a significant improvement over the original image.

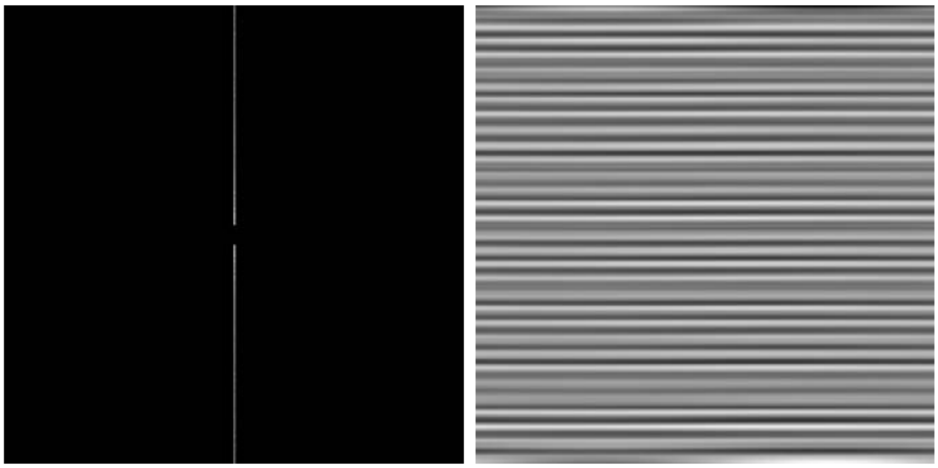


a b
c d

FIGURE 4.65 (a) 674 × 674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

Figure 4.66 (a) shows the result of using a notch pass version of the same filter to the DFT of Figure 4.65 (a).

Figure 4.66 (b) shows the spatial pattern obtained by computing the IDFT of Figure 4.66 (a).



a b

FIGURE 4.66 (a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).