Chapter 4 Filtering in the Frequency Domain

4.1 Background

A Brief History of the Fourier Series and Transform

Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist best known for initiating the investigation of Fourier series and their application to problems of heat transfer.

One of the most important Fourier's contributions states that any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. We now call this sum a Fourier series.

It does not matter how complicated the function is, if it is periodic and satisfies some mild mathematical conditions, it can be represented by Fourier series.

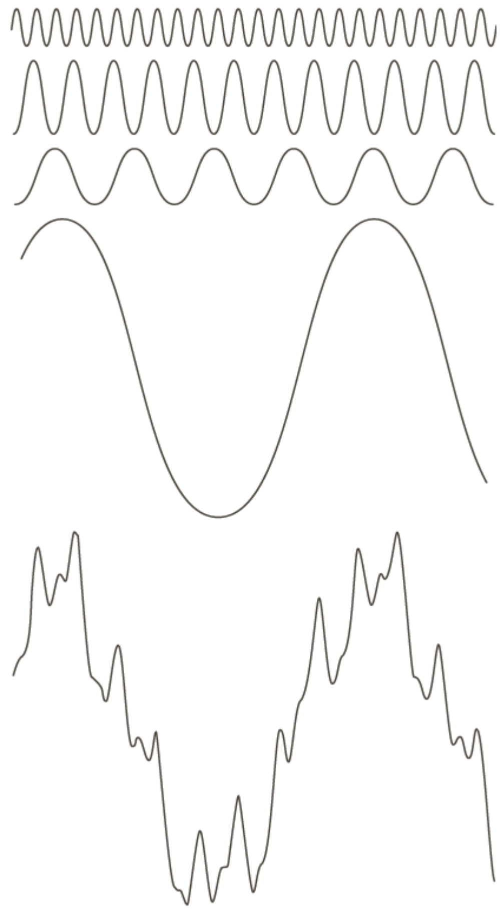


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function. The formulation in this case is the Fourier transform, and its utility is even greater than the Fourier series in many theoretical and applied disciplines.

One of the most important characteristics of these representations is that a function, expressed in either a Fourier series or transform, can be reconstructed completely via an inverse process with no loss of information. This characteristic allows us to work in the Fourier domain and then return to the original domain of the function without losing any information.

The initial application of Fourier's ideas was in the field of heat diffusion. The advent of digital computers and the "discovery" of a fast Fourier transform (FFT) algorithm in the early 1960s revolutionized the field of signal processing.

We will show that Fourier techniques will provide a meaningful and practical way to study and implement a host of image processing approaches.

4.2 Preliminary Concepts

Complex Numbers

A complex number, C, is defined as

$$C = R + jI \quad (4.2\text{-}1)$$

where R and I are real numbers, and j is an imaginary number equal to $\sqrt{-1}$.

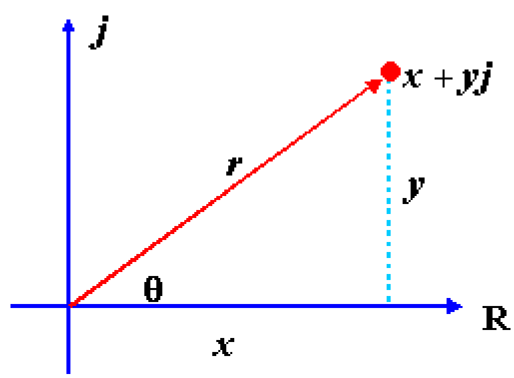
The conjugate of a complex number C, denoted C^*, is defined as

$$C^* = R - jI \quad (4.2\text{-}2)$$

Sometimes, it is useful to represent complex numbers in polar coordinates,

$$C = |C|(\cos\theta + j\sin\theta) \quad (4.2\text{-}3)$$

where $|C| = \sqrt{R^2 + I^2}$ is the length of the vector extending from the origin of the complex plane to the point (R, I), and θ is the angle between the vector and the real axis.



Using Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta, \quad (4.2\text{-}4)$$

we have the following familiar representation of complex numbers in polar coordinates

$$C = |C|e^{j\theta}\,. \qquad (4.2\text{-}5)$$

The above equations are applicable also to complex functions. For example, a complex function, $F(u)$, of a variable u, can be expressed as the sum $F(u) = R(u) + jI(u)$, where $R(u)$ and $I(u)$ are the real and imaginary component functions.

The complex conjugate of $F(u)$ is $F^*(u) = R(u) - jI(u)$, the magnitude is $|F(u)| = \sqrt{R(u)^2 + I(u)^2}$, and the angle is $\theta(u) = \arctan[I(u)/R(u)]$.

Fourier Series

A function $f(t)$ of a continuous variable t that is periodic with period, T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients. The sum, known as a Fourier series, has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \qquad (4.2\text{-}6)$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \qquad (4.2\text{-}7)$$

are the coefficients.

Impulses and Their Sifting Property

Central to the study of linear systems and the Fourier transform is the concept of an impulse and its sifting property.

A unit impulse of a continuous variable t located at $t = 0$, denoted $\delta(t)$, is defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \cdot \qquad (4.2\text{-}8a)$$

It is also to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 . \qquad (4.2\text{-}8b)$$

An impulse has the sifting property with respect to integration

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \qquad (4.2\text{-}9)$$

Provided that $f(t)$ is continuous at $t = 0$.

A more general statement of the sifting property involves an impulse located at an arbitrary point t_0, denoted by $\delta(t - t_0)$. Then, the sifting property becomes

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0) , \qquad (4.2\text{-}10)$$

which yields the value of the function at the impulse location, t_0.

Example: If $f(t) = \cos(t)$, using the impulse $\delta(t - \pi)$, then (4.2-10) yields the result $f(\pi) = \cos(\pi) = -1$.

The unit discrete impulse, $\delta(x)$, is defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}. \qquad (4.2\text{-}11a)$$

Clearly, this definition also satisfies the discrete equivalent of (4.2-8b):

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1 . \qquad (4.2\text{-}11b)$$

The sifting property for discrete variables has the form

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0) , \qquad (4.2\text{-}12)$$

and more generally

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0) . \qquad (4.2\text{-}13)$$

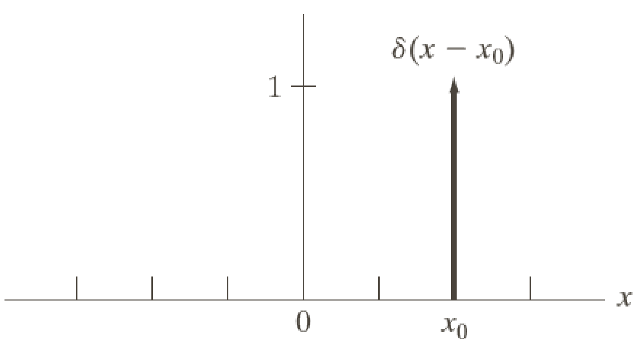


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

An impulse train, $s_{\Delta T}(t)$, defined as the sum of infinitely many periodic impulses ΔT units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) . \qquad (4.2\text{-}14)$$

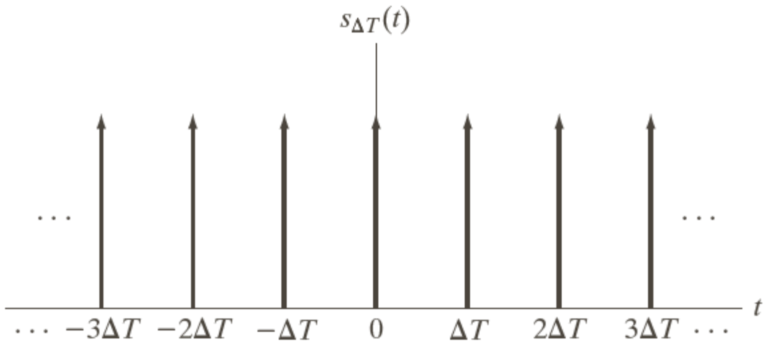


FIGURE 4.3 An impulse train.

The Fourier Transform of Functions of One Continuous Variable

The Fourier transform of a continuous function $f(t)$ of a continuous variable, t, is defined by

$$\mathscr{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt \qquad (4.2\text{-}15)$$

where μ is also a continuous variable. By writing

$$\mathscr{F}\{f(t)\} = F(\mu)$$

the Fourier transform may be written as

$$F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt\,. \qquad (4.2\text{-}16)$$

Conversely, given $F(\mu)$, we can obtain $f(t)$ back using the inverse Fourier transform, written as

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu\,. \qquad (4.2\text{-}17)$$

Equations (4.2-16) and (4.2-17) comprise the so-called Fourier transform pair. They indicate the important fact that a function can be recovered from its transform.

Using Euler's formula, (4.2-16) can be expressed as

$$F(\mu) = \int_{-\infty}^{\infty} f(t)[\cos(2\pi\mu t) - j\sin(2\pi\mu t)]\,dt\,. \quad (4.2\text{-}18)$$

Because the only variable left after integration is frequency, we say that the domain of the Fourier transform is the frequency domain.

Example 4.1: Obtaining the Fourier transform of a simple function

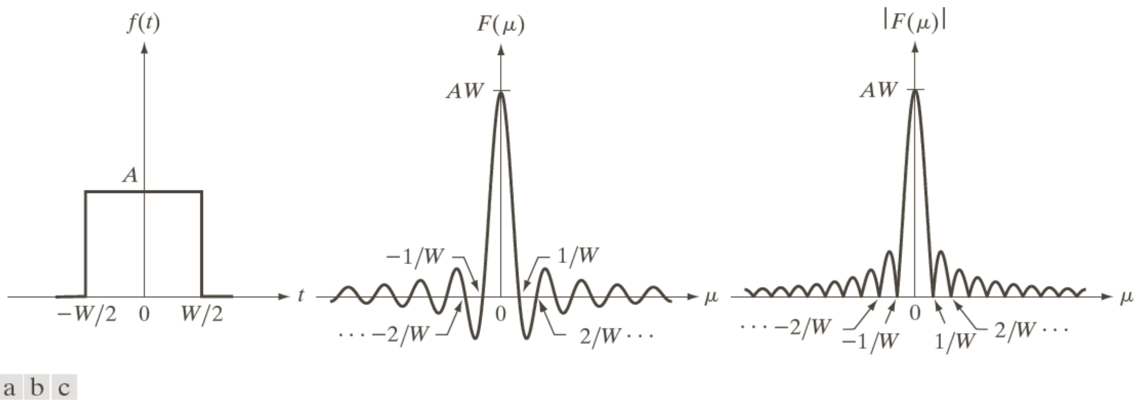


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

The Fourier transform of the function in Figure 4.4 (a) is

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt = \int_{-W/2}^{W/2} Ae^{-j2\pi\mu t}dt \\ &= \frac{-A}{j2\pi\mu}\left[e^{-j2\pi\mu t}\right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\left[e^{-j\pi\mu W} - e^{j\pi\mu W}\right] \\ &= \frac{A}{j2\pi\mu}\left[e^{j\pi\mu W} - e^{-j\pi\mu W}\right] = AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}\,, \end{aligned}$$

where we applied the trigonometric identity

$$\sin\theta = \left(e^{j\theta} - e^{-j\theta}\right)/2j\,.$$

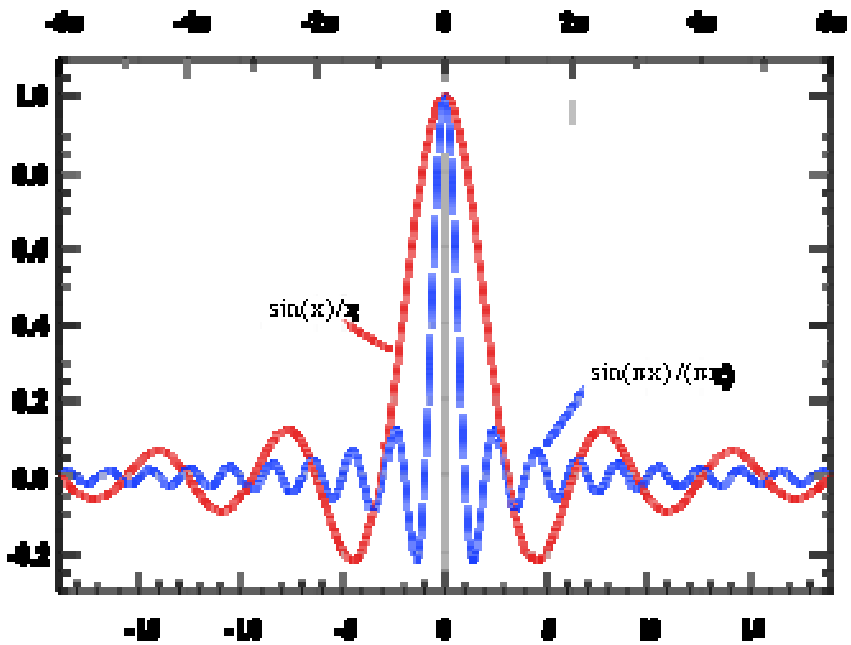
Note that $\frac{\sin(\pi m)}{(\pi m)}$ is known as the normalized $\operatorname{sinc}(m)$ function, with

$$\operatorname{sinc}(0) = 1$$

and

$$\operatorname{sinc}(m) = 0$$

for all other integer values of m.



In general, the Fourier transform contains complex terms, and we work with the magnitude of the transform, which is called the Fourier spectrum or the frequency spectrum:

$$|F(\mu)| = A\,W\left|\frac{\sin(\pi\mu W)}{(\pi\mu W)}\right|,$$

which is shown in Figure 4.4 (c).

Example 4.2: Fourier transform of an impulse

The Fourier transform of a unit impulse located at the origin follows from (4.2-16)

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt$$
$$= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t) dt$$

By using the sifting property in

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)\,, \qquad (4.2\text{-}9)$$

we have

$$F(\mu) = e^{-j2\pi\mu 0} = e^{0} = 1$$

Thus, we see that the Fourier transform of an impulse located at the origin of the spatial domain is a constant in the frequency domain.

The Fourier transform of a unit impulse located at $t = t_0$

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt$$
$$= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t - t_0) dt$$

Again, by using the sifting property stated in (4.2-9), we get

$$F(\mu) = e^{-j2\pi\mu t_0}$$
$$= \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)$$

which are equivalent representations of a unit circle centered on the origin of the complex plane.

Convolution

We are interested in the convolution of two continuous functions, $f(t)$ and $h(t)$, of one continuous variable t.

The convolution of these two functions is defined as

$$f(t)\star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \,. \qquad (4.2\text{-}20)$$

By a few steps starting from (4.2-15), we can find the so-called Fourier transform pair:

$$f(t)\star h(t) \Leftrightarrow H(\mu)F(\mu) \qquad (4.2\text{-}21)$$

and

$$f(t)h(t) \Leftrightarrow H(\mu)\star F(\mu) \,. \qquad (4.2\text{-}22)$$

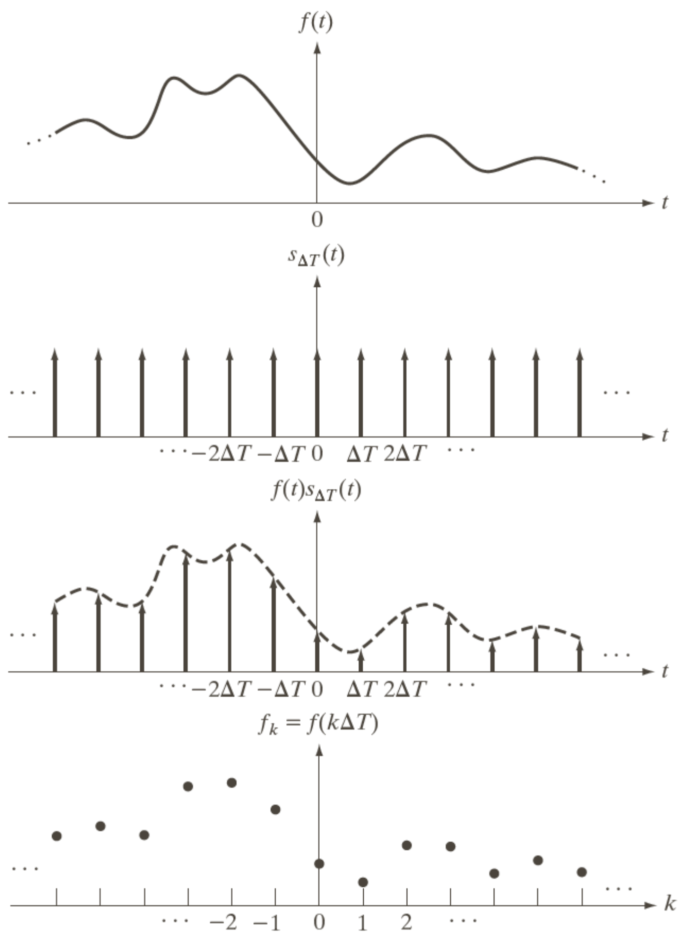
The double arrow ($\Leftrightarrow$) is used to indicate that the expression on the right is obtained by taking Fourier transform of the expression of the left.

The convolution theorem is the foundation for filtering in the frequency domain.

4.3 Sampling and the Fourier Transform of Sampled Functions

Sampling

Continuous functions have to be converted into a sequence of discrete values before they can be processed in a computer.



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

One way to model sampling is to multiply $f(t)$ by a sampling function equal to a train of impulses ΔT unit apart

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) \qquad (4.3\text{-}1)$$

where $\tilde{f}(t)$ denotes the sampled function. $\tilde{f}(t)$ is shown in Figure 4.5 (c).

The value of each sample is then given by the "strength" of the weighted impulses, and the value, f_k, of an arbitrary sample in the sequence is given by

$$f_k(t) = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt = f(k\Delta T). \qquad (4.3\text{-}2)$$

Equation (4.3-2) holds for any integer value $k = \ldots, -2, -1, 0, 1, 2, \ldots$.

The Fourier Transform of Sampled Functions

Let $F(\mu)$ denote the Fourier transform of a continuous function $f(t)$. Then, the Fourier transform, $\tilde{F}(\mu)$, of the sampled function $\tilde{f}(t)$ is

$$\tilde{F}(\mu) = F(\mu) \star S(\mu), \qquad (4.3\text{-}3)$$

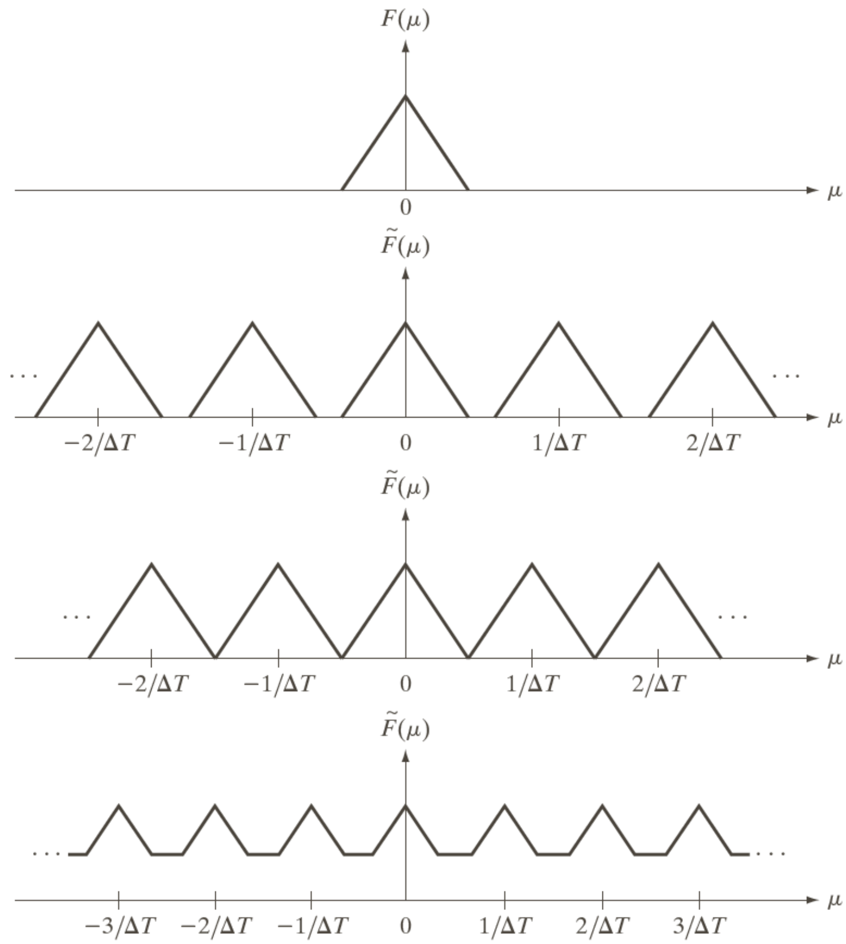
where

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right) \qquad (4.3\text{-}4)$$

is the Fourier transform of the impulse train $s_{\Delta T}(t)$. Then we have

$$\begin{aligned}
\tilde{F}(\mu) = F(\mu) \star S(\mu) &= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau) d\tau \\
&= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \qquad (4.3\text{-}5)\\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}$$

The equation (4.3-5) shows that the Fourier transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$ is an infinite, periodic sequence of copies of $F(\mu)$. The separation between copies is determined by the value of $1/\Delta T$.



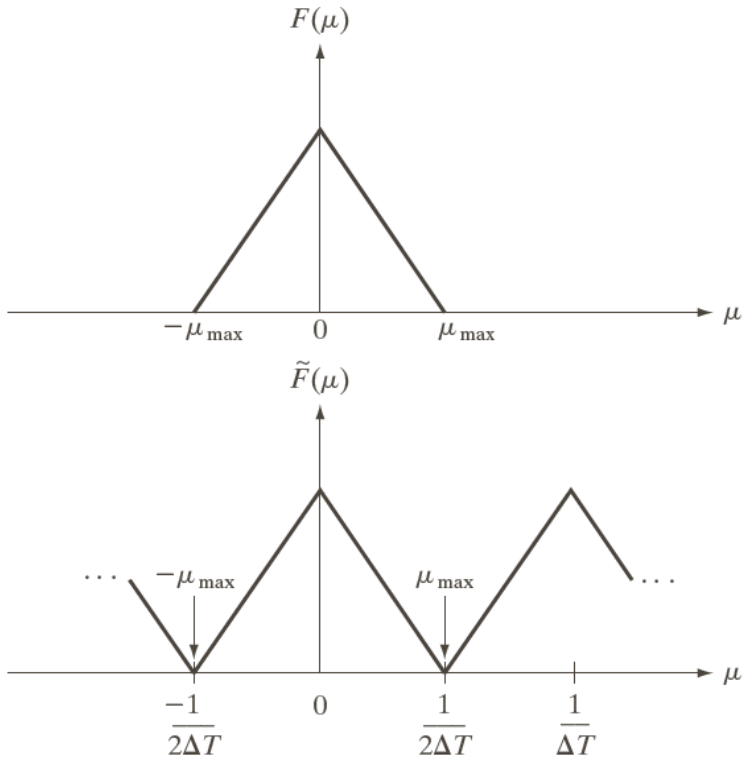
a
b
c
d

FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

The Sampling Theorem

We consider the sampling process and establish the conditions under which a continuous function can be recovered uniquely from a set of its samples.

A function $f(t)$ whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{\max}, \mu_{\max}]$ is called a band-limited function. Figure 4.7 (a) is such a function.



a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

We can recover $f(t)$ from its sampled version if we can isolate a copy of $F(\mu)$ from the periodic sequence of copies of this function contained in $\tilde{F}(\mu)$.

Extracting from $\tilde{F}(\mu)$ a single period that is equal to $F(\mu)$ is possible if the separation between copies is sufficient. In terms of Figure 4.7 (b), sufficient separation is guaranteed if $1/2\Delta T > \mu_{\max}$, or

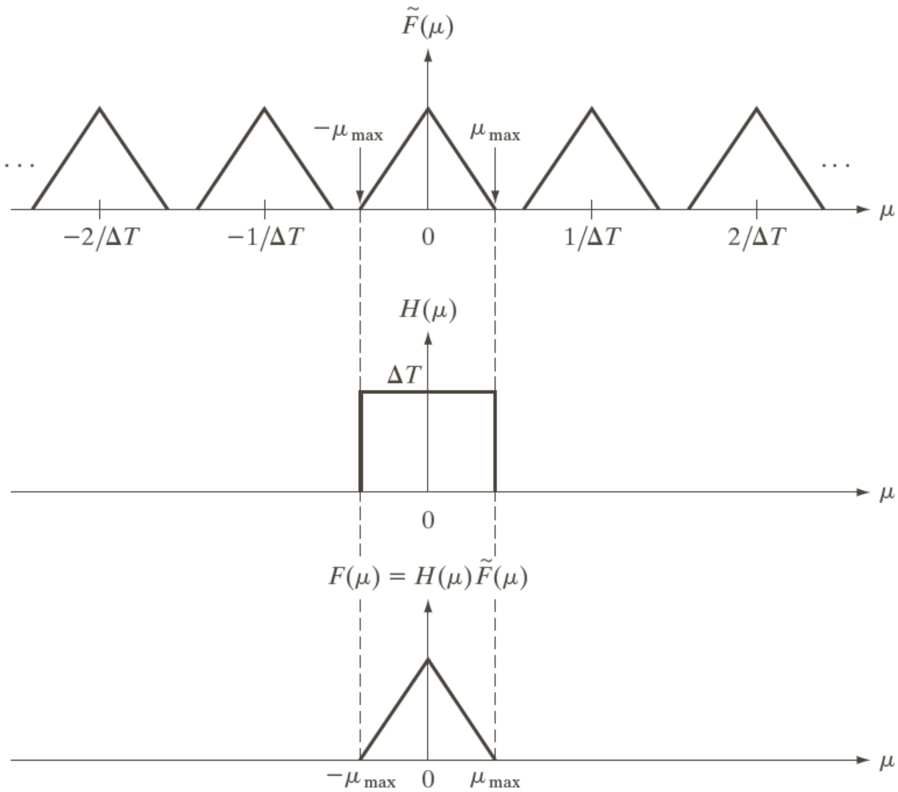
$$\frac{1}{\Delta T} > 2\mu_{\max} . \qquad (4.3\text{-}6)$$

Equation (4.3-6) indicates that a continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.

This result is known as the sampling theorem.

A sampling rate equal to exactly twice the highest frequency is called the Nyquist rate.

Figure 4.8 shows the Fourier transform of a function sampled at a rate slightly higher than the Nyquist rate.



a
b
c
FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

The function in Figure 4.8 (b) is defined by

$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \leq \mu \leq \mu_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{4.3-7}$$

When multiplied by the periodic sequence in Figure 4.8 (a), this function isolates the period centered on the origin. Then, we obtain $F(\mu)$ by multiplying $\tilde{F}(\mu)$ by $H(\mu)$:

$$F(\mu) = H(\mu)\tilde{F}(\mu)\,. \tag{4.3-8}$$

Once we have $F(\mu)$, we can recover $f(t)$ by using the inverse Fourier transform:

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu . \qquad (4.3\text{-}9)$$

Equations (4.3-7) through (4.3-9) have proved theoretically that it is possible to recover a band-limited function from samples of the function obtained at a rate exceeding twice the highest frequency content of the function. Function $H(\mu)$ is called a lowpass filter.

Aliasing

What happens if a band-limited function is sampled at a rate that is less than twice its highest frequency? Figure 4.9 shows the under-sampled case.

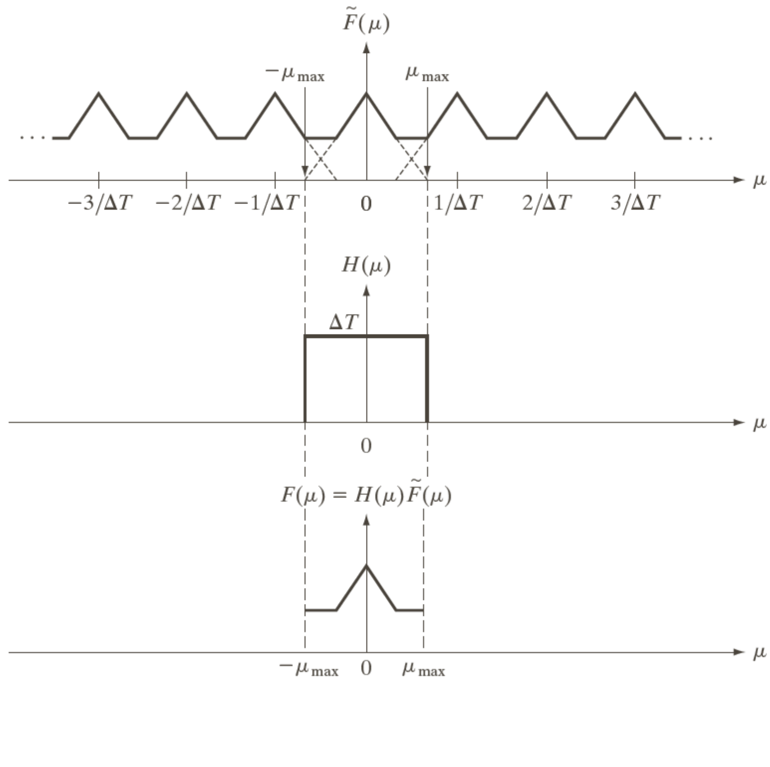


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.