Algebraic Cycles and the Mixed Hodge Structure on the Fundamental Group of a Punctured Curve

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ABSTRACT. Let X be a smooth projective curve of genus ≥ 1 over $\mathbb{C}$, and $e, \infty \in X(\mathbb{C})$ be distinct points. Let L_n be the mixed Hodge structure of functions on $\pi_1(X-\{\infty\}, e)$ given by iterated integrals of length $\leq n$ (as defined by Hain). Building on a work of Darmon, Rotger, and Sols **[7]**, we express the mixed Hodge extension $\mathbb{E}^{\infty}_{n,e}$ given by the weight filtration on $\frac{L_n}{L_{n-2}}$ as the Abel-Jacobi image of a null-homologous algebraic cycle on X^{2n-1}. This algebraic cycle is constructed using the different embeddings of X^{n-1} into X^n. As a corollary, we show that the extension $\mathbb{E}^{\infty}_{n,e}$ determines the point $\infty \in X - \{e\}$. When $n = 2$, our main result is a strengthening of a theorem of Darmon et al. **[7]**. In the final section we assume that X, e, ∞ are defined over a subfield K of $\mathbb{C}$. Generalizing a construction in **[7]**, we use the extension $\mathbb{E}^{\infty}_{n,e}$ to define a family of K-rational points on the Jacobian of X parametrized by $(n-1)$-dimensional algebraic cycles on X^{2n-2} defined over K.

CONTENTS

1. Introduction

Let X be a smooth (connected) projective curve over $\mathbb{C}$ of genus ≥ 1. Let $e, \infty \in X(\mathbb{C})$ be distinct. Let $L_n(X - \{\infty\}, e)$ be Hain's mixed Hodge structure with integral lattice

$$\left(\frac{\mathbb{Z}[\pi_1(X-\{\infty\}, e)]}{I^{n+1}}\right)^{\vee},$$

where $I \subset \mathbb{Z}[\pi_1(X - \{\infty\}, e)]$ is the augmentation ideal (see Paragraph 3.4). Darmon, Rotger, and Sols in **[7]** consider the extension

$$0 \to \frac{L_1}{L_0}(X - \{\infty\}, e) \to \frac{L_2}{L_0}(X - \{\infty\}, e) \to \frac{L_2}{L_1}(X - \{\infty\}, e) \to 0.$$

They relate this extension to the Abel-Jacobi image of the modified diagonal cycle of Gross, Kudla, and Schoen in X^3. If the curve X and the points e, ∞ are defined over a subfield $K \subset \mathbb{C}$, consequently they are able to define a family of rational points on the Jacobian of X parametrized by algebraic cycles in X^2 defined over K. The goal of this paper is to generalize this picture to higher weights. We will discuss this in more detail shortly.

Let us fix some notation. We use $CH_i(-)$ for Chow groups. (As usual, the subscript is the dimension.) By $CH_i^{hom}(-)$ we mean the subgroup of $CH_i(-)$ consisting of homologically trivial cycles. We denote by $\underline{Hom}$ the internal Hom in the category of mixed Hodge structures, and for a pure Hodge structure A of odd weight $2k-1$, by JA we refer to the intermediate Jacobian

$$JA := \frac{A_{\mathbb{C}}}{F^k A_{\mathbb{C}} + A_{\mathbb{Z}}},$$

where $F^{\cdot}$ denotes the Hodge filtration. We write H^1 for $H^1(X)$, the Hodge structure associated to the degree one cohomology of X.

We start with a brief account of the main result of **[7]**. Denote the extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{L_1}{L_0}(X - \{\infty\}, e) & \longrightarrow & \frac{L_2}{L_0}(X - \{\infty\}, e) & \longrightarrow & \frac{L_2}{L_1}(X - \{\infty\}, e) & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & H^1 & & & & (H^1)^{\otimes 2} & & \end{array}$$

by $\mathbb{E}_{2,e}^{\infty}$. Let $\Delta_{2,e}$ be the modified diagonal cycle of Gross, Kudla and Schoen in X^3:

$$\begin{aligned} \Delta_{2,e} \;:=\; & \{(x,x,x) : x \in X\} - \{(e,x,x) : x \in X\} - \{(x,e,x) : x \in X\} - \{(x,x,e) : x \in X\} \\ + \;& \{(e,e,x) : x \in X\} + \{(e,x,e) : x \in X\} + \{(x,e,e) : x \in X\} \in CH_1^{hom}(X^3). \end{aligned}$$

(The reason for this non-standard choice of notation will be clear shortly.) Let

$$Z_{2,e}^{\infty} := \{(x,x,\infty) : x \in X\} - \{(x,x,e) : x \in X\} \in CH_1^{hom}(X^3).$$

Let h_2 be the composition

$$CH_1^{hom}(X^3) \longrightarrow J\underline{Hom}(H^3(X^3), \mathbb{Z}(0)) \longrightarrow J\underline{Hom}((H^1)^{\otimes 3}, \mathbb{Z}(0)), \tag{1}$$

where the first arrow is the Abel-Jacobi map (see Paragraph 8.1) and the second arrow is the restriction map (induced by the Kunneth inclusion $(H^1)^{\otimes 3} \subset H^3(X^3)$). Identify

$$Ext((H^1)^{\otimes 2}, H^1) \cong J\underline{Hom}((H^1)^{\otimes 2}, H^1) \cong J\underline{Hom}((H^1)^{\otimes 2} \otimes H^1, \mathbb{Z}(0)),$$

where the first isomorphism is that of Carlson **[1]** (see Paragraph 2.3), and the second is given by Poincare duality. For a Hodge class $\xi \in H^1 \otimes H^1$, let

$$\xi^{-1} : J\underline{Hom}((H^1)^{\otimes 2} \otimes H^1, \mathbb{Z}(0)) \to J\underline{Hom}(H^1, \mathbb{Z}(0)) \cong Jac(\mathbb{C})$$

be the map that sends the class of $f : (H^1_{\mathbb{C}})^{\otimes 3} \to \mathbb{C}$ to the class of $f(\xi \otimes -)$. (Here Jac is the Jacobian of X and the isomorphism $J\underline{Hom}(H^1, \mathbb{Z}(0)) \cong Jac(\mathbb{C})$ is given by the classical Abel-Jacobi map.) Theorem 2.5 of **[7]** asserts that for every Hodge class ξ,

$$\xi^{-1}(\mathbb{E}_{2,e}^{\infty}) = \xi^{-1}(h_2(-\Delta_{2,e} + Z_{2,e}^{\infty})). \tag{2}$$

Consequently Darmon, Rotger and Sols show that if X, e, ∞ and ξ are all defined over a subfield K of $\mathbb{C}$, then the point $\xi^{-1}(\mathbb{E}^{\infty}_{2,e})$ is a K-rational point of the Jacobian. Thus by varying ξ one gets a family of rational points on the Jacobian, parametrized by Hodge classes in $H^1 \otimes H^1$ that are defined over K (or by divisors on $X \times X$ defined over K). These points have a nice analytic description in terms of Chen type iterated integrals (as the extension $\mathbb{E}^{\infty}_{2,e}$ is described using such iterated integrals). Our main goal in this article is to generalize this picture to higher weight parts of the fundamental group. For each $n \geq 2$, we consider the extension $\mathbb{E}^{\infty}_{n,e}$

$$0 \longrightarrow \underset{\substack{\wr\| \\ (H^1)^{\otimes n-1}}}{\frac{L_{n-1}}{L_{n-2}}(X-\{\infty\}, e)} \longrightarrow \frac{L_n}{L_{n-2}}(X-\{\infty\}, e) \longrightarrow \underset{\substack{\wr\| \\ (H^1)^{\otimes n}}}{\frac{L_n}{L_{n-1}}(X-\{\infty\}, e)} \longrightarrow 0$$

of mixed Hodge structures as an element of $\operatorname{Ext}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right)$. This extension comes from the weight filtration on

$$\frac{L_n}{L_{n-2}}(X-\{\infty\}, e),$$

which is given by

$$W_{n-2} = 0, \quad W_{n-1} = \frac{L_{n-1}}{L_{n-2}}(X-\{\infty\}, e), \quad \text{and} \quad W_n = \frac{L_n}{L_{n-2}}(X-\{\infty\}, e).$$

Let h_n be the composition

$$CH^{\text{hom}}_{n-1}(X^{2n-1}) \overset{\text{Abel-Jacobi}}{\longrightarrow} J\underline{\operatorname{Hom}}(H^{2n-1}(X^{2n-1}), \mathbb{Z}(0)) \overset{\text{Kunneth}}{\longrightarrow} J\underline{\operatorname{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0)),$$

and identify

$$\operatorname{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1}) \overset{\text{Carlson}}{\cong} J\underline{\operatorname{Hom}}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right) \overset{\text{Poincare duality}}{\cong} J\underline{\operatorname{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0)).$$

For each n, we define algebraic cycles

$$\Delta_{n,e}, Z^{\infty}_{n,e} \in CH^{\text{hom}}_{n-1}(X^{2n-1})$$

such that (2) generalizes to the following result.

THEOREM 1.

$$\mathbb{E}^{\infty}_{n,e} = (-1)^{\frac{n(n-1)}{2}} h_n\left(\Delta_{n,e} - Z^{\infty}_{n,e}\right)$$

Note that when $n = 2$, this is a strengthening of Darmon-Rotger-Sols' (2).

The cycle $\Delta_{n,e}$ is constructed by first taking an alternating sum

$$\sum_i (-1)^{i-1}\, {}^t\Gamma_{\delta_i}$$

of the transposes of the graphs of the diagonal embeddings $\delta_i : X^{n-1} \longrightarrow X^n$ defined by

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_i, x_i, \ldots, x_{n-1}), \tag{3}$$

and then using the method of Gross and Schoen **[19]** to produce a null-homologous cycle (see Paragraphs 7.2 and 7.3). The cycle $Z^{\infty}_{n,e}$ is defined as

$$\sum_{i=1}^{n-1} (-1)^{i-1} \left((\pi_{n+i,\infty})_* - (\pi_{n+i,e})_*\right) ({}^t\Gamma_{\delta_i}),$$

where for $x \in X$, $\pi_{i,x}$ is the map $X^{2n-1} \to X^{2n-1}$ that replaces the i^{th} coordinate by x, and leaves the other coordinated unchanged.

Note that the fact that the diagonal embeddings $\delta_i : X^{n-1} \to X^n$ appear in the constructions is not surprising. Wojtkowiak used these maps in **[31]** to form a cosimplicial variety that gives rise to the de Rham fundamental group[†], and Deligne and Goncharov used these maps in **[13]** to construct their motivic fundamental group.

Theorem 1 has the following corollaries:

(1) The function
$$X(\mathbb{C}) - \{e\} \to \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1}) \qquad \infty \mapsto \mathbb{E}^{\infty}_{n,e}$$
is injective.

(2) Suppose X has genus 1, or that X is hyperelliptic and e is a ramification point of X. Then $\mathbb{E}^{\infty}_{n,e}$ is torsion if and only if $\infty - e \in CH_0^{hom}(X)$ is torsion.

We should mention that one motivation for considering extensions of the form
$$0 \longrightarrow \frac{L_{n-1}}{L_{n-2}} \longrightarrow \frac{L_n}{L_{n-2}} \longrightarrow \frac{L_n}{L_{n-1}} \longrightarrow 0,$$
rather than
$$0 \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow \frac{L_n}{L_{n-1}} \longrightarrow 0,$$
is that the quotients $\{\frac{L_n}{L_{n-1}}\}$ are independent of the base point, so that we can think of extensions coming from different base points as elements of the same Ext group. The reason for looking at extensions coming from π_1 of the punctured curve, rather than the curve X itself, is that the successive quotients $\frac{L_n}{L_{n-1}}(X, e)$ for $n > 2$ are much more complicated than their counterparts for $X - \{\infty\}$. (See **[29]**.)

In the last section, following the ideas of **[7]** we give a number theoretic application of Theorem 1. Suppose $K \subset \mathbb{C}$ is a subfield, $X = X_0 \otimes_K \mathbb{C}$, where X_0 is a smooth projective curve over K, and $e, \infty \in X_0(K)$. Let g be the genus. Denote the Jacobian of X_0 by Jac. Generalizing a construction in **[7]**, we associate to the extension $\mathbb{E}^{\infty}_{n,e}$ a family of points in $\mathrm{Jac}(K)$ parametrized by algebraic cycles of dimension $n-1$ in X_0^{2n-2}. Our approach is in line with Darmon's general philosophy of constructing rational points on Jacobians of curves using algebraic cycles on higher dimensional varieties.

For a Hodge class
$$\xi \in (H^1)^{\otimes 2n-2},$$
let ξ^{-1} be the map
$$J\underline{\mathrm{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0)) \to J\underline{\mathrm{Hom}}(H^1, \mathbb{Z}(0)) \cong \mathrm{Jac}(\mathbb{C})$$
defined by
$$\left(\text{class of } f : (H^1_{\mathbb{C}})^{\otimes 2n-1} \to \mathbb{C} \right) \mapsto \left(\text{class of } f(\xi \otimes -) \right).$$
For $Z \in CH_{n-1}(X_0^{2n-2})$, let ξ_Z be the $(H^1)^{\otimes 2n-2}$ Kunneth component of the class of Z. In Section 12 we prove the following result.

THEOREM 2. Let $Z \in CH_{n-1}(X_0^{2n-2})$. Then $\xi_Z^{-1}(\mathbb{E}^{\infty}_{n,e}) \in \mathrm{Jac}(K)$.

Note that this is not a priori obvious, as to define $\mathbb{E}^{\infty}_{n,e}$ one first goes to analytic topology. The result is a consequence of Theorem 1 in view of the following two facts:

(i) The map ξ_Z^{-1} is given by a correspondence.

[†]The corresponding cosimplicial manifold and its connection to the unipotent fundamental group already appear in Cartier **[2]**.

(ii) The algebraic cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ are defined over K.

Theorem 2 is due to Darmon, Rotger, and Sols **[7]** in the case $n = 2$. For each n, it associates to the extension $\mathbb{E}^{\infty}_{n,e}$ a family of rational points on the Jacobian parametrized by $CH_{n-1}(X_0^{2n-2})$.

It would be very interesting to investigate whether the families

$$\{\xi_Z^{-1}(\mathbb{E}^{\infty}_{n,e}) \ : \ Z \in CH_{n-1}(X_0^{2n-2})\} \subset \mathrm{Jac}(K) \tag{4}$$

contain non-torsion points (if $\mathrm{Jac}(K)$ has nonzero rank). One should keep in mind that for different values of n these families arise from different parts of the weight filtration of the mixed Hodge structure on $\pi_1(X-\{\infty\}, e)$, and are parametrized by algebraic cycles on different powers of X_0. Of course, a motivation for going deeper in the weight filtration is the possibility of using algebraic cycles on higher powers of the curve.

We close this introduction with a few words on the structure of the paper. We recall some background material in Sections 2 and 3. Nothing in these two sections is original. Sections 4-10 build towards the proof of Theorem 1. In Sections 4 and 5 we define a section s_F of the quotient map

$$L_n(X-\{\infty\}, e) \to \frac{L_n}{L_{n-1}}(X-\{\infty\}, e) \cong (H^1)^{\otimes n}$$

(over $\mathbb{C}$) that is compatible with the Hodge filtration (Lemma 5.7.1). To define this section we use some ideas of Pulte **[28]** and Darmon-Rotger-Sols **[7]**, together with the fact that the space of closed iterated integrals on a manifold M is calculated by the degree zero cohomology of the reduced bar construction on any subcomplex of the complex of smooth differential forms on M that calculates the cohomology of M. The composition

$$(H^1)^{\otimes n} \xrightarrow{s_F} L_n(X-\{\infty\}, e) \to \frac{L_n}{L_{n-2}}(X-\{\infty\}, e) \tag{5}$$

turns out to be defined over $\mathbb{R}$. In section 6 we use the map (5) to calculate the extension $\mathbb{E}^{\infty}_{n,e}$ as an element of $\mathrm{J}\underline{\mathrm{Hom}}((H^1)^{\otimes n}, (H^1)^{\otimes n-1})$ (via the isomorphism of Carlson). We shall see as a consequence that

$$\mathbb{E}^{\infty}_{n,e} = \sum_{i=1}^{n-1} (H^1)^{\otimes i-1} \otimes \mathbb{E}^{\infty}_{2,e} \otimes (H^1)^{\otimes n-1-i} \tag{6}$$

(Corollary 6.6.2). In section 7 we define the algebraic cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ and realize them as boundaries of some topological chains. After a brief review of Griffiths' Abel-Jacobi maps we restate Theorem 1 in Section 8. In Section 9 we consider $n = 2$ case. Here we prove a lemma that enables us to move the point ∞ if necessary (Lemma 9.1.1). With this lemma in hand, the $n = 2$ case of Theorem 1 follows from Darmon-Rotger-Sols' proof of (2). In Section 10 we use the material of the previous sections, in particular the reduction formula (6) together with the $n = 2$ case to verify the general case of Theorem 1. Section 12 contains the proof of Theorem 2.

In another article we shall give an application of the contents of this paper to periods. (See the author's thesis **[14]**.)

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2. Recollections from Hodge theory

In this section we briefly recall a few basic definitions and facts about mixed Hodge structures.

2.1. Unless otherwise stated, by a (pure or mixed) Hodge structure we mean one that is over $\mathbb{Z}$. We use the standard notations $F^{\cdot}$ and $W_{\cdot}$ for the Hodge and weight filtrations. We denote the category of mixed (resp. pure) Hodge structures by **MHS** (resp. **HS**). We will often denote a mixed Hodge structure by a capital English letter, and then decorate it with the subscript $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$ to refer to its corresponding $\mathbb{K}$-module. For example, if H is a mixed Hodge structure, by $H_{\mathbb{Z}}$, $H_{\mathbb{Q}}$, and $H_{\mathbb{C}}$ we refer to the corresponding $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ modules. For each integer n, we denote by $\mathbb{Z}(-n)$ the unique Hodge structure of weight $2n$ with the underlying abelian group $\mathbb{Z}$. For any mixed Hodge structure A, as usual $A(n) := A \otimes \mathbb{Z}(n)$. Given mixed Hodge structures A and B, we denote their internal hom by $\underline{\mathrm{Hom}}(A, B)$; it is a mixed Hodge structure defined as follows: Its underlying abelian group is $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}})$, and the filtrations are given by

$$W_n \mathrm{Hom}_{\mathbb{Q}}(A_{\mathbb{Q}}, B_{\mathbb{Q}}) = \{f : A_{\mathbb{Q}} \to B_{\mathbb{Q}} \mid f(W_l A_{\mathbb{Q}}) \subset W_{n+l} B_{\mathbb{Q}} \ \text{ for all } l\}$$

and

$$F^p \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}) = \{f : A_{\mathbb{C}} \to B_{\mathbb{C}} \mid f(F^l A_{\mathbb{C}}) \subset F^{p+l} B_{\mathbb{C}} \ \text{ for all } l\}.$$

If A and B are pure of weights a and b, $\underline{\mathrm{Hom}}(A, B)$ is pure of weight $b - a$. The dual to a mixed Hodge structure A is $A^{\vee} := \underline{\mathrm{Hom}}(A, \mathbb{Z}(0))$. We adopt the convention $A^{\otimes n} := (A^{\otimes -n})^{\vee}$ for n negative.

2.2. Carlson Jacobians. As a generalization of Griffiths' intermediate Jacobians of a variety, given a mixed Hodge structure A, Carlson **[1]** defined its n^{th} Jacobian by

$$J^n(A) := \frac{A_{\mathbb{C}}}{F^n A_{\mathbb{C}} + A_{\mathbb{Z}}},$$

where by $A_{\mathbb{Z}}$ we obviously mean its image in $A_{\mathbb{C}}$. It is easy to see that for n bigger than half the highest weight of A, the natural map

$$A_{\mathbb{R}} := A_{\mathbb{Z}} \otimes \mathbb{R} \to \frac{A_{\mathbb{C}}}{F^n A_{\mathbb{C}}} \tag{7}$$

(given by the inclusion $A_{\mathbb{R}} \subset A_{\mathbb{C}}$) is injective, whence $J^n(A)$ is the quotient of a complex vector space by a discrete subgroup. It is easy to see that in general J^n is a functor from **MHS** to the category of abelian groups that respects direct sums.

Of special interest to us is the case of the "middle Jacobian" $JA := J^n A$ of a pure Hodge structure A of weight $2n - 1$ (possibly negative). It is easy to see that in this case, the map (7) is an isomorphism, and hence induces an isomorphism of real tori

$$\frac{A_{\mathbb{R}}}{A_{\mathbb{Z}}} \cong JA. \tag{8}$$

We record, for future reference, a few easy statements in the following lemma. The proofs are straightforward and are omitted.

LEMMA 2.2.1. Let A, B and C be mixed Hodge structures.

(a) If $B_{\mathbb{Z}}$ is free, the canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}} \otimes C_{\mathbb{Z}}) \cong \mathrm{Hom}_{\mathbb{Z}}\left(A_{\mathbb{Z}} \otimes B_{\mathbb{Z}}^{\vee}, C_{\mathbb{Z}}\right)$ induces an isomorphism $\underline{\mathrm{Hom}}(A, B \otimes C) \cong \underline{\mathrm{Hom}}\left(A \otimes B^{\vee}, C\right)$.

(b) The canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}}) \otimes C_{\mathbb{Z}} \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}} \otimes C_{\mathbb{Z}})$ induces an isomorphism $\underline{\mathrm{Hom}}(A, B) \otimes C \cong \underline{\mathrm{Hom}}(A, B \otimes C)$.

(c) $J^n A(-p) = J^{n-p} A$

(d) If A is pure of odd weight, $JA(-p) = JA$.
(e) $J^n\underline{\mathrm{Hom}}(A(-p), B) = J^{n+p}\underline{\mathrm{Hom}}(A, B)$.
(f) If A and B are pure of opposite parity weights, then $J\underline{\mathrm{Hom}}(A(-p), B) = J\underline{\mathrm{Hom}}(A, B)$.

2.3. Carlson's theorem on classifying extensions in MHS. Let A and B be mixed Hodge structures. By $\mathrm{Ext}(A, B)$ we mean the group of extensions of A by B in the category **MHS**. Suppose the highest weight of B is less than the lowest weight of A. Carlson **[1]** shows that there is a functorial isomorphism

$$\mathrm{Ext}(A, B) \cong J^0\underline{\mathrm{Hom}}(A, B).$$

Given an extension $\mathbb{E}$ given by a short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0\,,$$

one way to describe the corresponding element in the Jacobian is as follows: Choose a Hodge section σ_F of $E_{\mathbb{C}} \to A_{\mathbb{C}}$, and an integral retraction $\rho_{\mathbb{Z}}$ of $B_{\mathbb{C}} \to E_{\mathbb{C}}$. The extension $\mathbb{E}$ corresponds to the class of $\rho_{\mathbb{Z}} \circ \sigma_F$. (By a Hodge section we mean a section that is compatible with the Hodge filtrations, and by integral we mean a map that is induced by a map between the underlying $\mathbb{Z}$-modules.)

In the interest of simplifying the notation, we shall identify $\mathrm{Ext}(A, B)$ and $J^0\underline{\mathrm{Hom}}(A, B)$ via the isomorphism of Carlson.

3. Hodge theory of π_1- Recollections from the general theory

3.1. Review of the reduced bar construction. In this paragraph, we briefly review certain aspects of the reduced bar construction on a differential graded algebra. The construction is due to K.T. Chen, and the reader can refer to **[4]** and **[21]** for references. We only discuss a special case that is of interest to us. Throughout this paragraph $\mathbb{K}$ is a field of characteristic 0.

By a differential graded algebra over $\mathbb{K}$ we mean one that is concentrated in degree ≥ 0. More precisely, this is a graded $\mathbb{K}$-algebra $A^{\cdot} = \bigoplus_{n\geq 0} A^n$, equipped with a differential d of degree 1 (so that one has a complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

of $\mathbb{K}$-vector spaces) such that the graded Leibniz rule holds, i.e.

$$d(ab) = (da)b + (-1)^{\deg(a)}a(db)$$

for homogeneous elements $a, b \in A^{\cdot}$, where deg is the degree. Moreover, we say $A^{\cdot}$ is commutative if

$$ab = (-1)^{\deg(a)\deg(b)}ba$$

for all homogeneous a, b.

Note that $\mathbb{K}$ itself can be thought of as a differential graded algebra over $\mathbb{K}$ in an obvious way. Suppose $A^{\cdot} = \bigoplus_{n\geq 0} A^n$ is a differential graded algebra over $\mathbb{K}$, with the differential denoted by d. Denote the positive degree part by A^+. Let $\epsilon : A^{\cdot} \to \mathbb{K}$ be an augmentation (i.e. a morphism of differential graded algebras in to $\mathbb{K}$). For any integers r, s ($r \geq 0$), let $T^{-r,s}(A^{\cdot})$ be the degree s part of $(A^+)^{\otimes r}$, i.e. the $\mathbb{K}$-span of all terms of the form

$$a_1 \otimes \cdots \otimes a_r, \tag{9}$$

where $a_i \in A^+$ and $\sum \deg a_i = s$. (By convention, $(A^+)^{\otimes 0} = \mathbb{K}$.) It is customary to use the notation

$$[a_1|\cdots|a_r]$$

for the element (9). The $T^{-r,s}(A^\cdot)$ form a second quadrant bicomplex $T^{\cdot,\cdot}(A^\cdot)$, with $T^{-r,s}(A^\cdot)$ being the $(-r,s)$ bidegree component, and anti-commuting differentials both of degree 1 defined below. Here $Ja = (-1)^{\deg a} a$ for any homogeneous element $a \in A^\cdot$.

- The horizontal differential d_h:

$$d_h([a_1|\cdots|a_r]) = \sum_{i=1}^{r-1} (-1)^{i+1} [Ja_1|\cdots|Ja_{i-1}|(Ja_i)a_{i+1}|a_{i+2}|\cdots|a_r]$$

- The vertical differential d_v:

$$d_v([a_1|\cdots|a_r]) = \sum_{i=1}^{r} (-1)^{i} [Ja_1|\cdots|Ja_{i-1}|da_i|a_{i+1}|\cdots|a_r].$$

The formulas for the differentials are particularly important for us when all the a_i are of degree 1. In this case the formulas simplify to

$$d_h[a_1|\cdots|a_r] = -\sum_i [a_1|\cdots|a_i a_{i+1}|\cdots|a_r] \tag{10}$$

and

$$d_v[a_1|\cdots|a_r] = -\sum_i [a_1|\cdots|da_i|\cdots|a_r]. \tag{11}$$

The associated total complex $\mathrm{Tot}(T^{\cdot,\cdot}(A^\cdot))$ is concentrated in non-negative degrees, and its degree zero part is $\bigoplus_{s\geq 0} T^{-s,s}(A^\cdot) = \bigoplus_{s\geq 0} (A^1)^{\otimes s}$. The reduced bar construction $\overline{B}(A^\cdot,\epsilon) = \bigoplus_{n\geq 0} \overline{B}^n(A^\cdot,\epsilon)$ of $A^\cdot$ relative to ϵ is by definition a certain quotient of $\mathrm{Tot}(T^{\cdot,\cdot}(A^\cdot))$. See **[21]** for the quotienting relations. If $A^0 = \mathbb{K}$, the relations become trivial and $\overline{B}(A^\cdot,\epsilon)$ is then simply $\mathrm{Tot}(T^{\cdot,\cdot}(A^\cdot))$.

The image of $[a_1|\ldots|a_r]$ in the reduced bar construction is denoted by $(a_1|\ldots|a_r)$. From now on we drop the augmentation ϵ from our notation for $\overline{B}$ if it will not lead to any confusion.

The reduced bar construction is naturally filtered by tensor length: Let

$$\mathcal{T}_n = \bigoplus_{r\leq n} \left(T^{-r,s}(A^\cdot)\right).$$

The filtration $\{\mathcal{T}_n\}$ of the double complex $(T^{\cdot,\cdot}(A^\cdot))$ induces a filtration $\{\mathcal{B}_n\}$ on the reduced bar construction. We denote the filtration induced on the cohomology of $\overline{B}(A^\cdot)$ also by $\{\mathcal{B}_n\}$.

The reduced bar construction is functorial. In particular, if $A^\cdot$ and $\tilde{A}^\cdot$ are differential graded $\mathbb{K}$-algebras, and $\epsilon : A^\cdot \to \mathbb{K}$ and $\tilde{\epsilon} : \tilde{A}^\cdot \to \mathbb{K}$ are augmentations, a morphism $f : A^\cdot \to \tilde{A}^\cdot$ of differential graded algebras satisfying $\tilde{\epsilon}\circ f = \epsilon$ induces a morphisms of complexes $\overline{B}(A^\cdot) \to \overline{B}(\tilde{A}^\cdot)$ compatible with the filtrations $\{\mathcal{B}_n\}$. Moreover, if f is a quasi-isomorphism, then the induced maps between the reduced bar constructions and the $\mathcal{B}_n$ are also quasi-isomorphisms ([**21**, Corollary (1.2.3)] and [**22**, Lemma 7.1]).

If $A^\cdot$ is commutative, then $\overline{B}(A^\cdot)$ is in fact a commutative differential graded algebra, with multiplication given by the so called shuffle product. For degree zero elements, the multiplication is given by the formula†

$$(a_1|\cdots|a_r)\cdot(a_{r+1}|\cdots|a_{r+s}) = \sum_{\text{(r,s) shuffles } \sigma} (a_{\sigma(1)}|\cdots|a_{\sigma(r+s)}).$$

†Recall that $\sigma \in S_{r+s}$ is an (r,s) shuffle if

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(r) \quad \text{and} \quad \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s).$$

In particular, when $A^{\cdot}$ is commutative, $H^0\overline{B}(A^{\cdot})$ is also a commutative algebra. If $f : A^{\cdot} \to B^{\cdot}$ is a morphism of commutative differential graded algebras, then the induced map between the reduced bar constructions respects the multiplications.

3.2. Let G be a finitely generated group and $\mathbb{K}$ a field of characteristic zero. The Malcev or (pro)-unipotent completion of G over $\mathbb{K}$ is a pro-unipotent algebraic group $G^{un}_{\mathbb{K}}$ over $\mathbb{K}$, together with a homomorphism $G \to G^{un}_{\mathbb{K}}(\mathbb{K})$, such that for any pro-unipotent group U over $\mathbb{K}$ and any homomorphism $G \to U(\mathbb{K})$, there is a unique morphism $G^{un}_{\mathbb{K}} \to U$ of group schemes over $\mathbb{K}$ making the obvious diagram commute. It follows immediately that the image of G is dense in $G^{un}_{\mathbb{K}}$. The group $G^{un}_{\mathbb{K}}$ can be defined explicitly as $\mathrm{Spec}(\mathcal{O}_{G^{un}_{\mathbb{K}}})$, where

$$\mathcal{O}_{G^{un}_{\mathbb{K}}} = \varinjlim \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee},$$

and I is the augmentation ideal. One can think of

$$\left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee}$$

as the space of $\mathbb{K}$-valued functions on G which (after being extended linearly to $\mathbb{K}[G]$) vanish on I^{m+1}.

3.3. Chen's theory of iterated integrals and the description of $\mathcal{O}(\pi_1{}^{un}_{\mathbb{C}})$. We review some results of K.T Chen in this paragraph. For details and proofs, see **[3]**, **[4]** and **[5]**. Throughout this paragraph $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

As a generalization of the notion of a manifold, Chen in **[5]** defines the notion of a differentiable space. He associates to each differentiable space a commutative differential graded algebra of $\mathbb{K}$-valued differential forms. The degree 0 forms are, as expected, "differentiable" functions, and the multiplication on them is simply point-wise multiplication of functions. The closed 0-forms are the locally constant functions.

Let U be a path-connected (smooth) manifold, $e \in U$, and Ω_e be the (smooth) loop space at e. Let $E^{\cdot}_{\mathbb{K}}(U)$ be the complex of $\mathbb{K}$-valued differential forms on U. The loop space Ω_e is naturally made into a differentiable space. For $\omega_1, \ldots, \omega_r \in E^{\cdot}_{\mathbb{K}}(U)$ of positive degree, Chen defines a $\mathbb{K}$-valued differential form of degree $-r + \sum \deg(\omega_i)$ on Ω_e denoted by $\int \omega_1 \ldots \omega_r$. A $\mathbb{K}$-valued iterated integral of degree d is by definition a linear combination of the d-forms of the form $\int \omega_1 \ldots \omega_r$. In the case that $\omega_1, \ldots, \omega_r$ are all 1-forms on U, the zero form, i.e. function, $\int \omega_1 \ldots \omega_r$ on the loop space is defined by

$$\Big(\gamma : [0,1] \to U\Big) \mapsto \int\limits_{0 \le t_1 \le \ldots \le t_r \le 1} f_1(t_1)dt_1 \cdots f_r(t_r)dt_r,$$

where $f_i(t)dt = \gamma^*(\omega_i)$. If $r = 0$, the "empty" iterated integral is defined to be the constant function 1. The value of $\int \omega_1 \ldots \omega_r$ on γ is denoted by $\int_{\gamma} \omega_1 \ldots \omega_r$. It is clear that for $r = 1$, this coincides with the usual integral.

Following **[4]**, we denote the space of $\mathbb{K}$-valued iterated integral of degree d by ${A'_{\mathbb{K}}}^{d}$. The space $A'_{\mathbb{K}} := \oplus {A'_{\mathbb{K}}}^{d}$ is a sub-complex of the complex of $\mathbb{K}$-valued differential forms on the loop space Ω_e. It is also closed under multiplication (and hence is a sub-differential graded algebra). For degree 0 iterated integrals, this is thanks to the so-called shuffle product property given by the formula

$$\int_\gamma \omega_1 \cdots \omega_r \int_\gamma \omega_{r+1} \cdots \omega_{r+s} = \sum_{\text{(r,s) shuffles } \sigma} \int_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(r+s)}, \tag{12}$$

where γ is a loop at e.

An element of $A'^{d}_{\mathbb{K}}$ that can be expressed as a linear combination of $\int \omega_1 \ldots \omega_r$ with $r \leq m$ is said to be of length $\leq m$. The elements of $A'_{\mathbb{K}}$ of length $\leq m$ form a subcomplex $A'_{\mathbb{K}}(m)$. The complex $A'_{\mathbb{K}}$ is naturally filtered by length. Since $A'_{\mathbb{K}}$ is concentrated in degree ≥ 0, one has

$$H^0(A'_{\mathbb{K}}(m)) \subset H^0(A'_{\mathbb{K}}),$$

and the $\{H^0(A'_{\mathbb{K}}(m))\}$ is a filtration on $H^0(A'_{\mathbb{K}})$.

From now on, by an iterated integral we mean one of degree zero. The following formula describes how iterated integrals behave relative to composition of paths. Here α and β are loops at e.

$$\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{i=0}^{r} \int_\alpha \omega_1 \cdots \omega_i \int_\beta \omega_{i+1} \cdots \omega_r \tag{13}$$

One can show that iterated integrals also satisfy the following relations (as functions on Ω_e). Here f is a (smooth) function on U.

$$\begin{aligned}
\int (df)\omega_2 \cdots \omega_r &= \int (f\omega_2) \cdots \omega_r - f(e) \int \omega_2 \cdots \omega_r \\
\int \omega_1 \cdots \omega_{i-1}(df)\omega_{i+1} \cdots \omega_r &= \int \omega_1 \cdots \omega_{i-1}(f\omega_{i+1}) \cdots \omega_r - \int \omega_1 \cdots (f\omega_{i-1})\omega_{i+1} \cdots \omega_r \\
\int \omega_1 \cdots \omega_{r-1}(df) &= f(e) \int \omega_1 \cdots \omega_{r-1} - \int \omega_1 \cdots (f\omega_{r-1})
\end{aligned} \tag{14}$$

An iterated integral induces a function on $G = \pi_1(U, e)$ if and only if it is locally constant on the loop space if and only if it is closed (as an element of the complex $A'_{\mathbb{K}}$). It follows from (13) that a closed iterated integral of length $\leq m$ vanishes on $I^{m+1} \subset \mathbb{K}[G]$, so that one has a natural inclusion

$$H^0(A'_{\mathbb{K}}(m)) \subset \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^\vee.$$

The main theorem of [**3**] (Theorem 5.3) asserts that indeed

$$H^0(A'_{\mathbb{K}}(m)) = \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^\vee.$$

The algebraic structure of $H^0(A'_{\mathbb{K}}(m))$ can be described using the reduced bar construction on the complex $E^{\cdot}_{\mathbb{K}}(U)$ of smooth $\mathbb{K}$-valued differential forms on U, augmented by "evaluation at e". One has a natural map of differential graded algebras $\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \to A'_{\mathbb{K}}$ given by integration

$$(\omega_1|\cdots|\omega_r) \mapsto \int \omega_1 \cdots \omega_r.$$

This map† induces an isomorphism $H^0\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \to H^0(A'_{\mathbb{K}})$ strictly compatible with the length filtrations, i.e. we have a natural isomorphism

$$\mathcal{B}_m H^0\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \xrightarrow{\int} H^0(A'_{\mathbb{K}}(m)) = \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee}.$$

REMARK. If U is (the associated complex manifold to) a smooth complex variety, and $U = Y\backslash D$ where Y is smooth projective and D is a normal crossing divisor, one can replace $E^{\cdot}_{\mathbb{C}}(U)$ by the complex $E^{\cdot}(Y \log D)$ of smooth differential forms on Y with at most logarithmic singularity along D (since the reduced bar construction respects quasi-isomorphisms).

3.4. Mixed Hodge structure on π_1 of a smooth complex variety. Let U be a smooth variety over $\mathbb{C}$, $e \in U(\mathbb{C})$, $G = \pi_1(U, e)$, where with abuse of notation we denote a smooth complex variety and its associated complex manifold by the same symbol. Here we briefly recall Hain's mixed Hodge structure on the integral lattice

$$\left(\frac{\mathbb{Z}[G]}{I^{m+1}}\right)^{\vee},$$

which we denote by $L_m = L_m(U, e)$. For details and proofs, see **[21]**.

Let $U = Y \setminus D$, where Y is a smooth projective variety and D is a normal crossing divisor. In view of the isomorphism

$$\mathcal{B}_m H^0\overline{B}(E^{\cdot}(Y \log D)) \xrightarrow{\int} \left(\frac{\mathbb{C}[G]}{I^{m+1}}\right)^{\vee} = (L_m)_{\mathbb{C}}$$

the weight and Hodge filtrations on L_m are described as follows:

- The weight filtration: $W_n(L_m)_{\mathbb{C}}$ is the space of those closed iterated integrals that can be expressed as a sum of (not necessarily closed) iterated integrals of the form $\int \omega_1 \ldots \omega_r$, with $r \leq m$ and $\omega_i \in E^1(Y \log D)$, such that at most $n - r$ of the ω_i are not smooth along D. One can prove that this filtration is indeed defined over $\mathbb{Q}$. It is easy to see that $W_n(L_m) \subset L_n$.

- The Hodge filtration: $F^p(L_m)_{\mathbb{C}}$ is the space of those closed iterated integrals that can be expressed as a sum of (not necessarily closed) iterated integrals of the form $\int \omega_1 \ldots \omega_r$, where $r \leq m$ and $\omega_i \in E^1(Y \log D)$, such that at least p of the ω_i are of type (1,0).

Note that the L_m form a direct system of mixed Hodge structures.

REMARK. (1) One can show that L_m only depends on the pair (U, e), and not on the embedding of U as $Y \setminus D$. As in the case of mixed Hodge structure on cohomology, to explicitly describe the Hodge and weight filtrations on L_m one usually embeds U as $Y \setminus D$ as above.
(2) $L_m(U, e)$ is functorial in (U, e).

4. Construction of certain elements in the bar construction

In this section, given an augmented differential graded algebra satisfying certain properties, we give a procedure that constructs elements is $H^0\overline{B}$ with prescribed highest length terms. This construction will be used several times in Section 5.

We assume that $A^{\cdot}$ is an augmented differential graded algebra, and that

(i) $d(A^1) = (A^1)^2$,

†The relations by which one mods out $\mathrm{Tot}(T^{\cdot,\cdot}(E^{\cdot}_{\mathbb{K}}(U))$ to get $\overline{B}(E^{\cdot}_{\mathbb{K}}(U))$ are defined exactly based on relations (14) satisfied by iterated integrals, so that the map just described is well-defined.

(ii) for each pair (a, b) of elements of A^1, $s(a, b) \in A^1$ is such that $d(s(a, b)) = -ab$.

Let $a_1, \ldots, a_n \in A^1$ be closed. Our goal is to give a closed element of $\overline{B}^0(A^\cdot)$ of the form

$$(a_1|\cdots|a_n) \ + \ \text{lower length terms.}$$

For this, it suffices to construct a closed element of $\oplus T^{-r,r}(A^\cdot)$ of the form

$$[a_1|\cdots|a_n] \ + \ \text{lower length terms.}$$

Set $\lambda_n = [a_1|\cdots|a_n]$. Then $d_v(\lambda_n) = 0$, and $d_h(\lambda_n) \in T^{-n+1,n}$. The idea is to define, for each $r = n-1, \ldots, 1$, an element $\lambda_r \in T^{-r,r}$ such that $d_v(\lambda_r) = -d_h(\lambda_{r+1})$. The element

$$\lambda_n + \lambda_{n-1} + \cdots + \lambda_1$$

will then be closed.

For $r = n-1, \ldots, 1$, define λ_r to be the sum of all simple tensors in $T^{-r,r}$ of the form

$$[...|...|...|... \ ... \ ...|...], \tag{15}$$

where each block is formed by (possibly 0) successions of $s(\ , \ )$, and such that when we remove the symbols "|" and "$s(\ , \ )$", we are left with

$$[a_1 \ a_2 \ \cdots \ a_n]. \tag{16}$$

For example,

$$\lambda_{n-1} = \sum_{i=1}^{n-1} [a_1|\cdots|s(a_i, a_{i+1})|\cdots|a_n],$$

and

$$\begin{aligned}
\lambda_{n-2} \quad = & \sum_{1\leq i<j-1\leq n-2} [a_1|\cdots|s(a_i, a_{i+1})|\cdots|s(a_j, a_{j+1})|\cdots|a_n] \\
+ & \sum_{i=1}^{n-2} [a_1|\cdots|s(s(a_i, a_{i+1}), a_{i+2})|\cdots|a_n] \\
+ & \sum_{i=1}^{n-2} [a_1|\cdots|s(a_i, s(a_{i+1}, a_{i+2}))|\cdots|a_n].
\end{aligned}$$

There will be much more variety for λ_{n-3}:

$$\begin{aligned}
\lambda_{n-3} = & \sum [a_1|\cdots|s(a_i,a_{i+1})|\cdots|s(a_j,a_{j+1})|\cdots|s(a_k,a_{k+1})|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(a_i,a_{i+1})|\cdots|s(s(a_j,a_{j+1}),a_{j+2})|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(a_i,a_{i+1})|\cdots|s(a_j,s(a_{j+1},a_{j+2}))|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(s(a_i,a_{i+1}),a_{i+2})|\cdots|s(a_j,a_{j+1})|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(a_i,s(a_{i+1},a_{i+2}))|\cdots|s(a_j,a_{j+1})|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(s(s(a_i,a_{i+1}),a_{i+2}),a_{i+3})|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(s(a_i,s(a_{i+1},a_{i+2})),a_{i+3})|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(a_i,s(s(a_{i+1},a_{i+2}),a_{i+3}))|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(a_i,s(a_{i+1},s(a_{i+2},a_{i+3})))|\cdots|a_n] \\
+ & \sum [a_1|\cdots|s(s(a_i,a_{i+1}),s(a_{i+2},a_{i+3}))|\cdots|a_n].
\end{aligned}$$

Note that in every summand of λ_r, there are exactly $n-r$ occurrences of s.

LEMMA 4.0.1. The element $\lambda_n + \cdots + \lambda_1$ is closed.

PROOF. Note that $d_v(\lambda_n) = d_h(\lambda_1) = 0$. It remains to check that for each r, $-d_h(\lambda_{r+1}) = d_v(\lambda_r)$. But in view of the formulas (10) and (11), both $-d_h(\lambda_{r+1})$ and $d_v(\lambda_r)$ are the sum of all simple tensors in $T^{-r,r+1}$ of the form (15) where each block is formed by (possibly 0) successions of $s(\ ,\ )$, and such that when we remove the symbols "|" and "$s(\ ,\ )$", we are left with (16). That each a_i is closed is important to make sure $d_v(\lambda_r)$ is equal to the aforementioned sum. □

REMARK. It is easy to see that if $s : A^1 \times A^1 \to A^1$ is bilinear, then the above construction gives a linear map $(A^1_{\text{closed}})^{\otimes n} \to \mathcal{B}_n H^0\overline{B}(A)$.

5. Hodge theory of π_1- The case of a punctured curve

From here until the end of the paper, X is a smooth (connected) projective curve over $\mathbb{C}$ of genus $g \geq 1$, and $\infty, e \in X(\mathbb{C})$ are distinct points. Our main objective in this section is to construct a map (see Lemma 5.7.1) which will play a crucial role later on.

5.1. Let $S \subset X(\mathbb{C})$ be of finite cardinality $|S| \geq 1$, $U = X - S$, and $e \in U(\mathbb{C})$. Let $G = \pi_1(U, e)$ and $L_m = L_m(U, e)$. Our goal in this paragraph is to study $(L_m)_\mathbb{C}$ more closely.

It is well-known that in this case there are holomorphic differential forms α_i ($1 \leq i \leq 2g + |S| - 1$) on U whose classes form a basis of $H^1_{dR}(U)$. We can, and will, take these such that $\alpha_1, \ldots, \alpha_g$ are of first kind (i.e. holomorphic on X), $\alpha_{g+1}, \ldots, \alpha_{2g}$ are of second kind (i.e. meromorphic on X with zero residue along S), and $\alpha_{2g+1}, \ldots, \alpha_{2g+|S|-1}$ are of third kind with simple poles at points in S. Let $R^\cdot$ be the sub-object of $E^\cdot_\mathbb{C}(U)$ given by $R^0 = \mathbb{C}$, $R^1 = \sum_{i=1}^{2g+|S|-1} \alpha_i \mathbb{C}$, and $R^2 = 0$. The inclusion map $R^\cdot \to E^\cdot_\mathbb{C}(U)$ is a quasi-isomorphism, so that in particular

$$\mathcal{B}_m H^0\overline{B}(R^\cdot) \cong \mathcal{B}_m H^0\overline{B}(E^\cdot_\mathbb{C}(U)) \qquad \text{and} \qquad H^0\overline{B}(R^\cdot) \cong H^0\overline{B}(E^\cdot_\mathbb{C}(U)).$$

It is easy to see that $H^0\overline{B}(R^\cdot)$, as a vector space, is the (underlying vector space of the) tensor algebra on R^1, and the multiplication is the shuffle product. In other words, $H^0\overline{B}(R^\cdot)$ is the shuffle algebra on the letters α_i ($1 \le i \le 2g + |S| - 1$). The filtration $\mathcal{B}_\cdot$ is the tensor length filtration. The following description of L_m is now immediate.

PROPOSITION 5.1.1. The integration map $H^0\overline{B}(R^\cdot) \to \varinjlim \left(\frac{\mathbb{C}[G]}{I^{m+1}}\right)^\vee$ which maps

$$[\alpha_{i_1}|\cdots|\alpha_{i_r}] \mapsto \int \alpha_{i_1}\cdots\alpha_{i_r}$$

is an isomorphism, which maps $\mathcal{B}_m$ onto $(L_m)_\mathbb{C}$. In particular, any complex valued function on G that (after extending linearly to $\mathbb{C}[G]$) vanishes on I^{m+1} is given by a unique (linear combination of) iterated integral(s) of length $\le m$ in the forms α_i.

5.2. From now on, let $S = \{\infty\}$. (Thus $U = X - \{\infty\}$ and $L_n = L_n(X - \{\infty\}, e)$.) The complex $F^1E^\cdot(X \log\infty)$ is exact in degree 2. For each $a, a' \in E^1(X\log\infty)$, let $s(a, a') \in F^1E^1(X\log\infty)$ be such that $d(s(a, a')) = -a \wedge a'$. If $a \wedge a' = 0$, we specifically take $s(a, a') = 0$.

The differential graded algebra $E^\cdot(X\log\infty)$ meets the conditions of Section 4, and hence for $\omega_1, \ldots, \omega_n$ closed smooth 1-forms on X, the construction given in that section gives us a closed element of $\overline{B}^0E^\cdot(X\log\infty)$ of the form

$$(\omega_1|\cdots|\omega_n) + \text{lower length terms},$$

and thus a closed iterated integral on $X - \{\infty\}$ of the form

$$\int \omega_1\cdots\omega_n + \text{lower length terms}, \tag{17}$$

where all the 1-forms involved are in $E^1(X\log\infty)$. Moreover, by construction, in each term of length r above there are $n - r$ occurrences of s, and hence at most $n - r$ forms with a pole at ∞. In view of the description of the weight filtration given in Paragraph 3.4, this implies the following lemma.

LEMMA 5.2.1. Given closed smooth 1-forms $\omega_1, \ldots, \omega_n$ on X, there is an element of $W_n(L_n)_\mathbb{C}$ of the form (17).

5.3. The Weight Filtration of L_m: We now show that the weight filtration on L_m coincides with the length filtration.

PROPOSITION 5.3.1. For $n \le m$, $W_nL_m = L_n$.

PROOF. It is enough to show $W_nL_n = L_n$ for all n, for then, if $n \le m$, we see in view of $W_nL_m \subset L_n$ that $W_nL_m = L_n$. We argue by induction on n. This is trivial for $n = 0$. Suppose $W_{n-1}L_{n-1} = L_{n-1}$. In view of Proposition 5.1.1, it suffices to show that

$$\int \alpha_{j_1}\cdots\alpha_{j_n} \in W_n(L_n)_\mathbb{C}.$$

For each i, let $\omega_i \in E^1_\mathbb{C}(X)$ be such that $\alpha_{j_i} = \omega_i + df_i$ on U, where f_i is a smooth function on U; this can be done because inclusion of U in X gives an isomorphism on the level of H^1. Thanks to the relations (14) satisfied by iterated integrals, we have

$$\int \alpha_{j_1}\cdots\alpha_{j_n} = \int \omega_1\cdots\omega_n + \text{lower length terms}.$$

In view of Lemma 5.2.1 we can write

$$\int \alpha_{j_1}\cdots\alpha_{j_n} = \begin{aligned}&\Big(\text{an element of } W_n(L_n)_{\mathbb{C}} \text{ of the form}\\ &\int \omega_1\cdots\omega_n + \text{lower length terms}\Big)\\ &+\int \text{terms of length} \leq n-1.\end{aligned}$$

The left hand side and the first integral on the right are both closed, so that the second integral on the right also has to be closed, hence in $(L_{n-1})_{\mathbb{C}}$, and by the induction hypothesis in $W_{n-1}(L_{n-1})_{\mathbb{C}} \subset W_n(L_n)_{\mathbb{C}}$. The desired conclusion follows. □

5.4. In this paragraph we review some facts from group theory and then apply them to our setting. Let Γ be a finitely generated group, $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$, and I be the augmentation ideal in $\mathbb{K}[\Gamma]$. Let $\Gamma^{ab} := \frac{\Gamma}{[\Gamma,\Gamma]}$. It is well-known that

$$\frac{I}{I^2} \to \Gamma^{ab} \otimes \mathbb{K} \qquad [\gamma - 1] \mapsto [\gamma] \tag{18}$$

is an isomorphism. For $n > 1$ however, the quotients $\frac{I^n}{I^{n+1}}$ become increasingly more complicated in general. (See Stallings [**29**].) On the other hand, if Γ is free, these quotients are easy to describe: One has an isomorphism

$$\frac{I^n}{I^{n+1}} \to \left(\frac{I}{I^2}\right)^{\otimes n} \tag{19}$$

given by

$$[(\gamma_1 - 1)\cdots(\gamma_n - 1)] \mapsto [\gamma_1 - 1] \otimes \cdots \otimes [\gamma_n - 1].$$

Let Γ be free. Then $\frac{I}{I^2}$, and hence $\frac{I^n}{I^{n+1}}$ for every n, is a free $\mathbb{K}$-module. (Of course, this is only interesting when $\mathbb{K} = \mathbb{Z}$.) One has for each n an obvious exact sequence (of $\mathbb{K}$-modules)

$$0 \longrightarrow \frac{I^n}{I^{n+1}} \longrightarrow \frac{\mathbb{K}[\Gamma]}{I^{n+1}} \longrightarrow \frac{\mathbb{K}[\Gamma]}{I^n} \longrightarrow 0.$$

We see by induction that each $\dfrac{\mathbb{K}[\Gamma]}{I^n}$ is free, and hence dualizing the previous sequence we get exact

$$0 \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^n}\right)^{\vee} \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^{n+1}}\right)^{\vee} \longrightarrow \left(\frac{I^n}{I^{n+1}}\right)^{\vee} \longrightarrow 0.$$

Via

$$\left(\frac{I^n}{I^{n+1}}\right)^{\vee} \overset{(19)}{\simeq} \left(\left(\frac{I}{I^2}\right)^{\otimes n}\right)^{\vee} \overset{(18)}{\simeq} \left(\left(\Gamma^{ab} \otimes \mathbb{K}\right)^{\otimes n}\right)^{\vee},$$

we get a short exact sequence

$$0 \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^n}\right)^{\vee} \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^{n+1}}\right)^{\vee} \xrightarrow{q_{\mathbb{K}}} \left((\Gamma^{ab} \otimes \mathbb{K})^{\otimes n}\right)^{\vee} \longrightarrow 0. \tag{20}$$

Unwinding definitions, it is easy to see that $q_{\mathbb{K}}$ sends $f \in \left(\dfrac{\mathbb{K}[\Gamma]}{I^{n+1}}\right)^{\vee}$ to the map

$$[\gamma_1] \otimes \cdots \otimes [\gamma_n] \;\mapsto\; f\left([(\gamma_1 - 1)\cdots(\gamma_n - 1)]\right).$$

It is clear that (20) is compatible with extending $\mathbb{K}$.

We apply this to the group $G = \pi_1(U, e)$. In view of the definition of $(L_n)_{\mathbb{K}}$, the isomorphism $G^{ab} \otimes \mathbb{K} \simeq H_1(U, \mathbb{K})$ given by $[\gamma] \mapsto [\gamma]$, and

$$\left(H_1(U, \mathbb{K})^{\otimes n}\right)^{\vee} \cong \left(H_1(U, \mathbb{K})^{\vee}\right)^{\otimes n} \cong \left(H^1(U)_{\mathbb{K}}\right)^{\otimes n},$$

the sequence (20) reads

$$0 \longrightarrow (L_{n-1})_{\mathbb{K}} \xrightarrow{\text{inclusion}} (L_n)_{\mathbb{K}} \xrightarrow{q_{\mathbb{K}}} \left(H^1(U)_{\mathbb{K}}\right)^{\otimes n} \longrightarrow 0. \tag{21}$$

Compatibility with extending $\mathbb{K}$ implies the maps in this sequence when $\mathbb{K} = \mathbb{C}$ are defined over $\mathbb{Z}$ (i.e. take integral lattices to integral lattices, and hence rationals to rationals), and the sequence when $\mathbb{K} = \mathbb{Z}$ (resp. $\mathbb{K} = \mathbb{Q}$) is the restriction of the sequence for $\mathbb{K} = \mathbb{C}$ to integral (resp. rational) lattices. In particular, these restrictions are exact.

The inclusion $U \subset X$ gives an isomorphism $H^1(X) \to H^1(U)$. We will always identify the two Hodge structures via this map, and from now on simply write H^1 for $H^1(U) = H^1(X)$. Unwinding definitions, in view of

$$\int_{(\gamma_1-1)\cdots(\gamma_n-1)} \omega_1 \cdots \omega_n + \text{lower length terms} = \prod_{i=1}^{n} \int_{\gamma_i} \omega_i, \tag{22}$$

we see that the map $q_{\mathbb{C}}$ sends

$$\int \omega_1 \cdots \omega_n + \text{lower length terms} \ \mapsto [\omega_1] \otimes \cdots \otimes [\omega_n], \tag{23}$$

where the integral on the left is closed, each ω_i is a closed smooth 1-form on U, and $[\omega_i]$ denotes the cohomology class of ω_i. Note that (22) is a consequence of (13).

It is clear from the description of the weight filtration on L_n given in Proposition 5.3.1 that the map $q_{\mathbb{C}}$ is compatible with the weight filtrations. It is well-known that $q_{\mathbb{C}}$ is also compatible with the Hodge filtrations, so that the map

$$\overline{q} : \frac{L_n}{L_{n-1}} \to (H^1)^{\otimes n}$$

induced by $q_{\mathbb{C}}$ is an isomorphism of (mixed) Hodge structures. We won't however need to invoke this fact. Over the next three paragraphs, we will construct a particular section s_F of $q_{\mathbb{C}}$ that respects the Hodge filtrations (Lemma 5.7.1). This map enjoys some nice properties and will play an important role later on. Existence of a section of $q_{\mathbb{C}}$ respecting the Hodge filtrations in particular implies that $\overline{q}$ respects the Hodge filtrations, and hence is an isomorphism (Corollary 5.8.1).

5.5. In this paragraph, we review some basic facts about Green functions. For the proofs and further details, see [**25**].

Let φ be a real non-exact smooth form of type (1,1) on X, D be a nonzero divisor on X, and $\mathrm{supp}(D)$ be the support of D. Then φ is exact on $X - \mathrm{supp}(D)$. Indeed, one can prove that there is a unique (smooth) function $g_{D,\varphi} : X - \mathrm{supp}(D) \to \mathbb{R}$, called the Green function for φ relative to D, satisfying the following properties:

(1) If D is represented by a meromorphic function f on an open set (in analytic topology) V of X, then the function $V - \mathrm{supp}(D) \to \mathbb{R}$ defined by[†]

$$P \mapsto g_{D,\varphi}(P) + (\int_X \varphi)\ \log|f(P)|^2$$

extends smoothly to V.

(2) $dd^c g_{D,\varphi} = (\deg D)\varphi$ on $X - \mathrm{supp}(D)$, where $d^c = \frac{1}{4\pi i}(\partial - \bar\partial)$ with the ∂, $\bar\partial$ the usual operators.

(3) $\int_X g_{D,\varphi}\varphi = 0$.

Take $D = \infty$. It follows from (1) that locally near the point ∞, with a chart taken such that ∞ corresponds to $z = 0$, the function $g_{\infty,\varphi}$ looks like

$$-(\int_X \varphi)\ \log z\bar z + \text{a smooth function.}$$

It follows that $\partial g_{\infty,\varphi}$ near ∞ (again with $z = 0$ corresponding to the point ∞) is of the form

$$-(\int_X \varphi)\ \frac{dz}{z} + \text{a smooth 1-form,}$$

so that $\partial g_{\infty,\varphi}$ is in $E^1(X \log \infty)$. By condition (2), $d(\frac{1}{2\pi i}\partial g_{\infty,\varphi}) = \varphi$ on U. To sum up, given a a non-exact real two-form φ on X, we have a specific 1-form $\frac{1}{2\pi i}\partial g_{\infty,\varphi}$ of type (1,0) in $E^1(X \log \infty)$ with residue $-\frac{1}{2\pi i}\int_X \varphi$ at ∞ whose d is φ on U.

5.6. Throughout this paragraph, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{H}^1_{\mathbb{K}}(X)$ be the space of $\mathbb{K}$-valued harmonic 1-forms on X. One has a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^1_{\mathbb{R}}(X) & \cong & H^1_{\mathbb{R}} \\ \cap & & \cap \\ \mathcal{H}^1_{\mathbb{C}}(X) & \cong & H^1_{\mathbb{C}}. \end{array}$$

Via the horizontal isomorphisms we get a pure real Hodge structure $\mathcal{H}^1(X)$ of weight one with $\mathbb{K}$-vector space $\mathcal{H}^1_{\mathbb{K}}(X)$. The subspace $F^1\mathcal{H}^1_{\mathbb{C}}(X)$ is the space of holomorphic 1-forms on X. Let $\wedge : \mathcal{H}^1_{\mathbb{K}} \otimes \mathcal{H}^1_{\mathbb{K}} \to E^2_{\mathbb{K}}(X)$ be the "wedge product" map, i.e. given by $\wedge(\omega_1 \otimes \omega_2) = \omega_1 \wedge \omega_2$. The following lemma combines some ideas of Pulte **[28]** and Darmon, Rotger and Sols **[7]**.

LEMMA 5.6.1. There is a $\mathbb{C}$-linear map

$$\nu : \mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X) \to E^1(X \log \infty)$$

such that

(i) for each $w \in \mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X)$, $d(\nu(w)) = -\wedge(w)$ on U,

(ii) ν respects the Hodge filtration F,

(iii) for each $w \in \mathcal{H}^1_{\mathbb{R}}(X) \otimes \mathcal{H}^1_{\mathbb{R}}(X)$, there is a smooth real 1-form $\nu_{\mathbb{R}} = \nu_{\mathbb{R}}(w)$ on U such that $\nu(w) - \nu_{\mathbb{R}}$ is exact on U,

[†]The appearance of the extra factor $\int_X \varphi$ compared to Lang comes from the fact that φ is not normalized here.

(iv) for every $w \in \mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X)$, the residue of $\nu(w)$ at ∞ is $\dfrac{1}{2\pi i}\int\limits_X \wedge(w)$.

PROOF. The cup product $H^1 \otimes H^1 \overset{\smile}{\to} H^2(X)$ is a morphism of Hodge structures. Let K be its kernel. Ignoring the rational structures, we can think of K as a sub-Hodge structure of the real Hodge structure $H^1 \otimes H^1$. Let $\mathcal{K}$ be its copy in $\mathcal{H}^1(X) \otimes \mathcal{H}^1(X)$. Thus $\mathcal{K}_{\mathbb{K}}$ consists of those $w \in \mathcal{H}^1_{\mathbb{K}}(X) \otimes \mathcal{H}^1_{\mathbb{K}}(X)$ for which $\wedge(w) \in E^2_{\mathbb{K}}(X)$ is exact. One has a short exact sequence of real Hodge structures

$$0 \longrightarrow \mathcal{K} \overset{\text{inclusion}}{\longrightarrow} \mathcal{H}^1(X) \otimes \mathcal{H}^1(X) \cong H^1 \otimes H^1 \overset{\smile}{\longrightarrow} H^2(X) \longrightarrow 0.$$

The category of pure real Hodge structures is semi-simple, so that there is

$$\phi \in \mathcal{H}^1_{\mathbb{R}}(X) \otimes \mathcal{H}^1_{\mathbb{R}}(X) \ \cap\ F^1(\mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X))$$

giving rise to a decomposition of $\mathcal{H}^1(X) \otimes \mathcal{H}^1(X)$ as an internal direct sum

$$\mathcal{H}^1(X) \otimes \mathcal{H}^1(X) = \mathcal{K} \oplus \mathcal{L},$$

where $\mathcal{L}$ is the one dimensional sub-object of $\mathcal{H}^1(X) \otimes \mathcal{H}^1(X)$ generated by ϕ[†]. Because of the linear nature of the requirements, it suffices to define ν on $\mathcal{K}_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{C}}$ satisfying (i)-(iv).

Definition of ν on $\mathcal{K}_{\mathbb{C}}$: This part is due to Pulte [**28**]. The operator d on X is strict with respect to the Hodge filtration, so that one can choose

$$\nu' : \mathcal{K}_{\mathbb{C}} \to E^1_{\mathbb{C}}(X)$$

respecting the Hodge filtration such that $d\nu'(w) = -\wedge(w)$ on X. Now recall that one has a decomposition $E^1_{\mathbb{K}}(X) = \mathcal{H}^1_{\mathbb{K}}(X) \oplus \mathcal{H}^1_{\mathbb{K}}(X)^{\perp}$, where $\mathcal{H}^1_{\mathbb{K}}(X)^{\perp}$ is the space of $\mathbb{K}$-valued 1-forms orthogonal to $\mathcal{H}^1_{\mathbb{K}}(X)$ with respect to the inner product defined using the Hodge $*$ operator. Recall also that the projections $E^1_{\mathbb{C}}(X) \to \mathcal{H}^1_{\mathbb{C}}(X)$ and $E^1_{\mathbb{C}}(X) \to \mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$ preserve type. Define ν to be the composition of ν' and the latter projection. Since harmonic forms are closed, we have $d\nu(w) = d\nu'(w) = -\wedge(w)$. Note that condition (iv) holds trivially. We claim that ν satisfies property (iii) as well. Let $w \in \mathcal{K}_{\mathbb{R}}$. Then $\wedge(w)$ is exact and real, so that there is $\nu'_{\mathbb{R}} \in E^1_{\mathbb{R}}(X)$ such that $d\nu'_{\mathbb{R}} = -\wedge(w)$. Let $\nu_{\mathbb{R}}$ be the component of $\nu'_{\mathbb{R}}$ in $\mathcal{H}^1_{\mathbb{R}}(X)^{\perp}$. Then $d\nu_{\mathbb{R}} = d\nu'_{\mathbb{R}} = -\wedge(w)$, so that $\nu(w) - \nu_{\mathbb{R}} \in \mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$ is closed. The desired conclusion follows from the general fact that a closed element of $\mathcal{H}^1_{\mathbb{K}}(X)^{\perp}$ is necessarily exact. Note that on the subspace $\mathcal{K}_{\mathbb{C}}$ the requirements of the lemma hold on all of X, not just U.

Definition of ν on $\mathcal{L}_{\mathbb{C}}$: Define ν on the subspace $\mathcal{L}_{\mathbb{C}} = \mathbb{C}\phi$ by $\nu(\phi) = -\frac{1}{2\pi i}\partial g_{\infty,\wedge(\phi)}$. Conditions (i), (ii) and (iv) hold by Paragraph 5.5. As for condition (iii), note that $-d^c g_{\infty,\wedge(\phi)}$ is real, and

$$-\frac{1}{2\pi i}\partial g_{\infty,\wedge(\phi)} + d_c g_{\infty,\wedge(\phi)} = -\frac{1}{4\pi i} d g_{\infty,\wedge(\phi)}.$$

□

If the point ∞ is not clear from the context, we will write ν_∞ instead of ν. Note that the map ν is not natural; it depends on the choices of ϕ and ν'.

[†]We could have instead worked over $\mathbb{Q}$ here, as the Mumford-Tate group of X is reductive. But this would not result in any major simplification.

5.7. In this paragraph, we use Lemma 5.6.1 to construct a section s_F of $q_{\mathbb{C}} : (L_n)_{\mathbb{C}} \to (H^1_{\mathbb{C}})^{\otimes n}$ that is compatible with the Hodge filtrations, and also such that its composition with $(L_n)_{\mathbb{C}} \to \left(\frac{L_n}{L_{n-2}}\right)_{\mathbb{C}}$ is defined over $\mathbb{R}$. This map is of crucial importance in the later parts of the paper.

By exactness of $F^1E^{\cdot}(X\log\infty)$ in degree 2, one can (non-uniquely) extend the map ν of the previous paragraph to a map

$$\tilde{\nu} : E^1(X\log\infty) \otimes E^1(X\log\infty) \to E^1(X\log\infty)$$

respecting the Hodge filtrations and satisfying $d(\tilde{\nu}(w)) = -\wedge(w)$ for every $w \in E^1(X\log\infty) \otimes E^1(X\log\infty)$. The differential graded algebra $E^{\cdot}(X\log\infty)$ with the data of $s(a,a') = \tilde{\nu}(a \otimes a')$ for each $a, a' \in E^1(X\log\infty)$ satisfies the conditions of Section 4, and hence in particular for $\omega_1, \dots, \omega_n \in \mathcal{H}^1_{\mathbb{C}}(X)$, we have a closed iterated integral on U of the form

$$\int \omega_1 \cdots \omega_n + \sum_{i=1}^{n-1} \omega_1 \cdots \nu(\omega_i \otimes \omega_{i+1}) \cdots \omega_n + \text{terms of length at most } n-2. \tag{24}$$

(See the construction of Section 4.) In view of $(H^1_{\mathbb{C}})^{\otimes n} \cong (\mathcal{H}^1_{\mathbb{C}})^{\otimes n}$, we define the map $s_F : (H^1_{\mathbb{C}})^{\otimes n} \to (L_n)_{\mathbb{C}}$ by

$$[\omega_1] \otimes \cdots \otimes [\omega_n] \mapsto \text{the iterated integral described above,}$$

where $\omega_i \in \mathcal{H}^1_{\mathbb{C}}(X)$ and $[\omega_i]$ denotes the cohomology class of ω_i. This is well-defined and linear (see the final remark of Section 4), and in view of (23) it is a section of $q_{\mathbb{C}}$ (of Paragraph 5.4). Also, it is apparent from the construction of Section 4 that since $\tilde{\nu}$ preserves the Hodge filtration F, so does s_F. That s_F respects the weight filtration (over $\mathbb{C}$) is obvious from $W_n(L_n)_{\mathbb{C}} = (L_n)_{\mathbb{C}}$. We have proved parts (i)-(iii) of the following lemma.

LEMMA 5.7.1. There is a $\mathbb{C}$-linear map $s_F : (H^1_{\mathbb{C}})^{\otimes n} \to (L_n)_{\mathbb{C}}$ that satisfies the following properties:

(i) Given $\omega_1, \dots, \omega_n \in \mathcal{H}^1_{\mathbb{C}}(X)$, $s_F([\omega_1] \otimes \cdots \otimes [\omega_n])$ is of the form (24).
(ii) s_F is a section of $q_{\mathbb{C}} : (L_n)_{\mathbb{C}} \to (H^1_{\mathbb{C}})^{\otimes n}$.
(iii) s_F respects the Hodge and weight filtrations.
(iv) The composition

$$\mathfrak{s}_F : \ (H^1_{\mathbb{C}})^{\otimes n} \xrightarrow{s_F} (L_n)_{\mathbb{C}} \overset{\text{quotient}}{\longrightarrow} (\frac{L_n}{L_{n-2}})_{\mathbb{C}}$$

is defined over $\mathbb{R}$.

PROOF. (of (iv)) We must show that if $w \in (H^1_{\mathbb{R}})^{\otimes n}$, then

$$\mathfrak{s}_F(w) \in (\frac{L_n}{L_{n-2}})_{\mathbb{R}} \subset (\frac{L_n}{L_{n-2}})_{\mathbb{C}},$$

or equivalently, $s_F(w) \in (L_n)_{\mathbb{R}} + (L_{n-2})_{\mathbb{C}}$. It suffices to consider $w = [\omega_1] \otimes \ldots \otimes [\omega_n]$, where the $\omega_i \in \mathcal{H}^1_{\mathbb{R}}(X)$. In view of Lemma 5.6.1(iii) and the relations (14) satisfied by iterated integrals, we have

$$s_F(w) = \int \omega_1 \cdots \omega_n + \sum_{i=1}^{n-1} \omega_1 \cdots \nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1}) \cdots \omega_n + \text{terms of length} \leq n-2.$$

Applying the construction of Section 4 to the differential graded algebra $E^{\cdot}_{\mathbb{R}}(U)$ with $s(-,-)$ chosen such that $s(\omega_i, \omega_{i+1}) = \nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1})$, we get a closed element of $\overline{B}^0(E^{\cdot}_{\mathbb{R}}(U))$ of the form

$$(\omega_1|\cdots|\omega_n) + \sum_{i=1}^{n-1}(\omega_1|\cdots|\nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1})|\cdots|\omega_n) + \text{terms of length} \leq n-2,$$

and hence an element of $(L_n)_{\mathbb{R}}$ of the form

$$\int \omega_1\cdots\omega_n + \sum_{i=1}^{n-1}\omega_1\cdots\nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1})\cdots\omega_n + \text{terms of length} \leq n-2.$$

This differs from $s_F(w)$ by an element of $(L_{n-2})_{\mathbb{C}}$, giving the desired conclusion. □

5.8. Let $\overline{q}_{\mathbb{C}}$ be the isomorphism of vector spaces

$$\left(\frac{L_n}{L_{n-1}}\right)_{\mathbb{C}} \to (H^1_{\mathbb{C}})^{\otimes n}$$

induced by $q_{\mathbb{C}}$. Let $\overline{s}_F$ be the composition

$$(H^1_{\mathbb{C}})^{\otimes n} \overset{s_F}{\to} (L_n)_{\mathbb{C}} \overset{\text{quotient}}{\to} \left(\frac{L_n}{L_{n-1}}\right)_{\mathbb{C}}.$$

Then $\overline{s}_F$ is the inverse of $\overline{q}_{\mathbb{C}}$. By the discussion of Paragraph 5.4, $\overline{q}_{\mathbb{C}}$ restricts to an isomorphism of the integral lattices. It follows that the same is true for $\overline{s}_F$. Moreover, $\overline{s}_F$ is compatible with the Hodge and weight filtrations (because so is s_F), and hence is a morphism of mixed Hodge structures. In view of strictness of morphisms in **MHS** with respect to the Hodge filtration, $\overline{q}_{\mathbb{C}}$ is also compatible with the Hodge filtration. The following statement follows. (Compatibility of $\overline{q}_{\mathbb{C}}$ with the weight filtration is obvious.)

COROLLARY 5.8.1. The map $q_{\mathbb{C}}$ induces an isomorphism of mixed Hodge structures

$$\overline{q} : \frac{L_n}{L_{n-1}} \to (H^1)^{\otimes n}.$$

In the interest of keeping the notation simple, here we did not incorporate n in the notation for $\overline{q}$. When there is a possibility of confusion, we will instead use the decorated notation $\overline{q}_n$ for the isomorphism given in Corollary 5.8.1.

REMARK. (1) Note that in particular this says even though the mixed Hodge structure L_m may depend on the base point e, the quotient $Gr^W_n L_m = \frac{L_n}{L_{n-1}}$ does not. In fact, it does not even depend on the point ∞ we removed from X. It is true in general that for any smooth connected complex variety the quotients $\frac{L_n}{L_{n-1}}$ are independent of the base point. See (3.22) Remark (iii) of [**23**].
(2) It follows from the above that the map $q_{\mathbb{C}}$ is also compatible with the Hodge filtration, and that (21) is a short exact sequence of mixed Hodge structures.
(3) We should clarify that Corollary 5.8.1 is not a new result. For instance, it can be deduced from the ideas behind Remark (iii) of Paragraph (3.22) of [**23**]. Here we included a proof as it was easy to do so with the section s_F at hand, and in the interest of making the paper more self-contained.

6. The extension $\mathbb{E}^{\infty}_{n,p}$

6.1. Let A be a mixed Hodge structure with torsion-free $A_{\mathbb{Z}}$. The kernel of the surjective map

$$\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z})$$

induced by the natural quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})$. Putting this together with

$$\mathrm{Hom}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}),$$

we obtain

$$\frac{\mathrm{Hom}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R})}{\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})} \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}).$$

Now suppose A is pure of odd weight. Then so is $A^{\vee}$, and

$$JA^{\vee} \overset{(8)}{\cong} \frac{\mathrm{Hom}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R})}{\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})} \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}).$$

Unwinding definitions, we see that given $f : A_{\mathbb{C}} \to \mathbb{C}$ defined over $\mathbb{R}$, the class of f in $JA^{\vee}$ corresponds under the identification to the composition

$$A_{\mathbb{Z}} \overset{\text{inclusion}}{\to} A_{\mathbb{R}} \overset{f}{\to} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \tag{25}$$

in $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z})$.

6.2. Let $H_1 := (H^1)^{\vee}$. We identify $(H_1)_{\mathbb{Z}}$ with $H_1(X, \mathbb{Z})$ (the singular homology). One has an isomorphism of Hodge structures $H^1(1) \cong H_1$ given by Poincare duality

$$\mathrm{PD} : H^1(1) \xrightarrow{\simeq} H_1, \qquad [\omega] \mapsto \int_X [\omega] \wedge -,$$

where ω is a smooth closed 1-form on X. This gives for each positive n an isomorphism

$$\mathrm{PD}^{\otimes n} : (H^1)^{\otimes n}(n) \longrightarrow H_1^{\otimes n} \cong (H^1)^{\otimes -n},$$

given by

$$[\omega_1] \otimes \cdots \otimes [\omega_n] \mapsto \mathrm{PD}([\omega_1]) \otimes \cdots \otimes \mathrm{PD}([\omega_n]) = \left([\omega_1'] \otimes \cdots \otimes [\omega_n'] \mapsto \prod_i \int_X [\omega_i] \wedge [\omega_i'] \right).$$

We have

$$\begin{aligned} \mathrm{Ext}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right) &\overset{\text{Carlson (Par. 2.3)}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right) \\ &\overset{\text{Lemma 2.2.1(a)}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n} \otimes (H^1)^{\otimes 1-n}, \mathbb{Z}(0)\right) \\ &\overset{\mathrm{PD}^{\otimes n-1}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n} \otimes (H^1)^{\otimes n-1}(n-1), \mathbb{Z}(0)\right) \\ &\overset{\text{Lemma 2.2.1(f)}}{\cong} J((H^1)^{\otimes 2n-1})^{\vee}. \end{aligned} \tag{26}$$

Let Ψ be the composition isomorphism

$$\mathrm{Ext}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right) \longrightarrow J((H^1)^{\otimes 2n-1})^{\vee}.$$

We denote by Φ the isomorphism

$$J((H^1)^{\otimes 2n-1})^\vee \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left((H^1_{\mathbb{Z}})^{\otimes 2n-1}, \mathbb{R}/\mathbb{Z}\right)$$

given by Paragraph 6.1. (To make the notation slightly simpler we did not include n as a part of the symbol for the maps Φ and Ψ. This should not cause any confusion as n will be clear from the context.)

6.3. Definition of $\mathbb{E}^\infty_{n,e}$. Let $n \geq 2$. In this paragraph, we use $\frac{L_n}{L_{n-2}}$ to define an element

$$\mathbb{E}^\infty_{n,e} \in \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1}).$$

It follows from Proposition 5.3.1 that the weight filtration on $\frac{L_n}{L_{n-2}}$ is given by

$$W_{n-2} = 0, \qquad W_{n-1} = \frac{L_{n-1}}{L_{n-2}}, \qquad \text{and} \quad W_n = \frac{L_n}{L_{n-2}}.$$

The filtration gives rise to the exact sequence

$$0 \longrightarrow \frac{L_{n-1}}{L_{n-2}} \xrightarrow{\iota} \frac{L_n}{L_{n-2}} \xrightarrow{\text{quotient}} \frac{L_n}{L_{n-1}} \longrightarrow 0,$$

where ι is the inclusion map. Using the isomorphism of Corollary 5.8.1 to replace $\frac{L_{n-1}}{L_{n-2}}$ (resp. $\frac{L_n}{L_{n-1}}$) by $(H^1)^{\otimes n-1}$ (resp. $(H^1)^{\otimes n}$), we get the exact sequence

$$0 \longrightarrow (H^1)^{\otimes n-1} \xrightarrow{i} \frac{L_n}{L_{n-2}} \xrightarrow{q} (H^1)^{\otimes n} \longrightarrow 0. \tag{27}$$

Here $i = \iota \overline{q}^{\,-1}$, and q is the composition

$$\frac{L_n}{L_{n-2}} \xrightarrow{\text{quotient}} \frac{L_n}{L_{n-1}} \xrightarrow{\overline{q}} (H^1)^{\otimes n}.$$

Let $\mathbb{E}^\infty_{n,e} \in \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1})$ be the extension defined by the sequence (27).

REMARK. One can deduce from a theorem of Pulte **[28]** that the map

$$X(\mathbb{C}) - \{\infty\} \to \mathrm{Ext}((H^1)^{\otimes 2}, H^1)$$

defined by $e \mapsto \mathbb{E}^\infty_{2,e}$ is injective.

Our goal in the remainder of this section is to describe the images of $\mathbb{E}^\infty_{n,e}$ under Ψ and $\Phi \circ \Psi$. To this end, in view of Paragraph 6.1 and Paragraph 2.3, we will define an integral retraction of i and a Hodge section of q defined over $\mathbb{R}$. (See (27).)

6.4. An integral retraction of i. In this paragraph, we define an integral retraction $r_{\mathbb{Z}}$ of i, i.e. a linear map

$$r_{\mathbb{Z}} : (\frac{L_n}{L_{n-2}})_{\mathbb{C}} \longrightarrow (H^1_{\mathbb{C}})^{\otimes n-1}$$

defined over $\mathbb{Z}$, that is left inverse to i.

Choose $\beta_1, \ldots, \beta_{2g} \in \pi_1(U, e)$ such that the $[\beta_j] \in H_1(X, \mathbb{Z})$ form a basis. To define an element of $(H^1_{\mathbb{C}})^{\otimes n-1}$, it suffices to specify how it pairs with the elements $[\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}]$ of $H_1(X, \mathbb{Z})^{\otimes n-1}$.

Moreover, an element of $(H^1_{\mathbb{C}})^{\otimes n-1}$ is in $(H^1_{\mathbb{Z}})^{\otimes n-1}$ if and only if it produces integer values when pairing with the $[\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}]$. Given an element

$$f = \int \sum_{i \le n} w_i \; + \; (L_{n-2})_{\mathbb{C}} \; \in \; (\frac{L_n}{L_{n-2}})_{\mathbb{C}},$$

where w_i is a sum of terms of length i and the iterated integral is closed, set $r_{\mathbb{Z}}(f)$ to be the element of $(H^1_{\mathbb{C}})^{\otimes n-1}$ satisfying

$$[\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}](r_{\mathbb{Z}}(f)) = \int\limits_{(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)} \sum_{i \le n} w_i. \tag{28}$$

Note that

$$\int\limits_{(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)} \sum_{i \le n} w_i = \int\limits_{(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)} w_n + w_{n-1}.$$

Since $(L_{n-2})_{\mathbb{C}}$ vanishes on I^{n-1}, $r_{\mathbb{Z}}$ is well-defined. Moreover, $r_{\mathbb{Z}}$ is defined over $\mathbb{Z}$, for if $f \in (\frac{L_n}{L_{n-2}})_{\mathbb{Z}}$, the iterated integral $\int \sum w_i$ can be chosen to be integer-valued on $\pi_1(U, e)$, and hence (28) is an integer. Finally, we check that $r_{\mathbb{Z}}$ is a retraction of i. In view of Lemma 5.2.1 and the formula (23) for $q_{\mathbb{C}}$, if $\omega_1, \ldots, \omega_{n-1}$ are smooth closed 1-forms on X, $i([\omega_1] \otimes \ldots \otimes [\omega_{n-1}])$ is of the form

$$\int \omega_1 \cdots \omega_{n-1} + \text{lower length terms} \quad \text{mod } (L_{n-2})_{\mathbb{C}},$$

where the iterated integral is closed. We have

$$\begin{aligned} [\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}] \; \left(r_{\mathbb{Z}} \circ i([\omega_1] \otimes \cdots \otimes [\omega_{n-1}]) \right) &= \int\limits_{(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)} \omega_1 \cdots \omega_{n-1} \\ &= \int\limits_{\beta_{j_1}} \omega_1 \cdots \int\limits_{\beta_{j_{n-1}}} \omega_{n-1}, \end{aligned}$$

which is the same as

$$[\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}] \left([\omega_1] \otimes \cdots \otimes [\omega_{n-1}] \right),$$

as desired.

REMARK. The retraction $r_{\mathbb{Z}}$ is by no means natural, as it depends on the choice of the β_j.

6.5. Description of the images of $\mathbb{E}^{\infty}_{n,e}$ under Ψ and $\Phi \circ \Psi$.

PROPOSITION 6.5.1. The map $\mathfrak{s}_F$ (defined in Lemma 5.7.1(iv)) is a section of $q : (\frac{L_n}{L_{n-2}})_{\mathbb{C}} \longrightarrow (H^1_{\mathbb{C}})^{\otimes n}$ defined over $\mathbb{R}$ that respects the Hodge and weight filtrations.

PROOF. This is immediate from Lemma 5.7.1 (ii), (iii) and (iv). □

PROPOSITION 6.5.2. (a) $\Psi(\mathbb{E}^{\infty}_{n,e})$ is the class of the map that given $c \in (H^1_{\mathbb{C}})^{\otimes n}$, $d \in (H^1_{\mathbb{C}})^{\otimes n-1}$, it sends $c \otimes d$ to $PD^{\otimes n-1}(d)(r_{\mathbb{Z}} \circ \mathfrak{s}_F(c))$. More explicitly, if $\beta_j \in \pi_1(U, e)$ $(1 \le j \le 2g)$ are such that $\{[\beta_j]\}$ is a basis of $H_1(X, \mathbb{Z})$, and $\omega_1, \ldots, \omega_n \in \mathcal{H}^1_{\mathbb{C}}(X)$, $\Psi(\mathbb{E}^{\infty}_{n,e})$ is the class of the map that sends

$$[\omega_1] \otimes \cdots \otimes [\omega_n] \otimes (PD^{\otimes n-1})^{-1}([\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}])$$

to

$$\int\limits_{(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)} \omega_1\cdots\omega_n + \sum_i \omega_1\cdots\nu(\omega_i\otimes\omega_{i+1})\cdots\omega_n.$$

(b) $\Phi\circ\Psi(\mathbb{E}^\infty_{n,e})$ is the map that given $c\in(H^1_{\mathbb{Z}})^{\otimes n}$, $d\in(H^1_{\mathbb{Z}})^{\otimes n-1}$, it sends $c\otimes d$ to $\mathrm{PD}^{\otimes n-1}(d)(r_{\mathbb{Z}}\circ\mathfrak{s}_F(c)) \mod \mathbb{Z}$. More explicitly, for $\gamma_j\in\pi_1(U,e)$ $(1\le j\le n-1)$, and $\omega_1,\dots,\omega_n\in\mathcal{H}^1_{\mathbb{R}}(X)$ with integral periods, $\Phi\circ\Psi(\mathbb{E}^\infty_{n,e})$ sends

$$[\omega_1]\otimes\cdots\otimes[\omega_n]\otimes(\mathrm{PD}^{\otimes n-1})^{-1}([\gamma_1]\otimes\cdots\otimes[\gamma_{n-1}])$$

to

$$\int\limits_{(\gamma_1-1)\cdots(\gamma_{n-1}-1)} \omega_1\cdots\omega_n + \sum_i \omega_1\cdots\nu(\omega_i\otimes\omega_{i+1})\cdots\omega_n \mod \mathbb{Z}.$$

PROOF. (a) We track $\mathbb{E}^\infty_{n,e}$ through different steps of (26). The element in $\underline{\mathrm{JHom}}\left((H^1)^{\otimes n},(H^1)^{\otimes n-1}\right)$ corresponding to $\mathbb{E}^\infty_{n,e}$ under the isomorphism of Carlson is the class of $r_{\mathbb{Z}}\circ\mathfrak{s}_F$. (See Paragraph 2.3.) That the latter goes to the described element of $J((H^1)^{\otimes 2n-1})^\vee$ is clear. For the second assertion, define $r_{\mathbb{Z}}$ using the β_j, and then the assertion follows on noting that $r_{\mathbb{Z}}\circ\mathfrak{s}_F([\omega_1]\otimes\cdots\otimes[\omega_n])$, by its definition, pairs with the element $[\beta_{j_1}]\otimes\cdots\otimes[\beta_{j_{n-1}}]\in(H_1)^{\otimes n-1}_{\mathbb{Z}}$ in the desired fashion. (See (28) and Lemma 5.7.1(i),(iv).)

(b) The section $\mathfrak{s}_F$ is defined over $\mathbb{R}$, and hence so is $r_{\mathbb{Z}}\circ\mathfrak{s}_F$. Thus the map

$$c\otimes d\mapsto \mathrm{PD}^{\otimes n-1}(d)(r_{\mathbb{Z}}\circ\mathfrak{s}_F(c))$$

of Part (a) is also defined over $\mathbb{R}$. The first assertion follows. The explicit description of Part (a) implies that (with β_j as in Part (a)) $\Phi\circ\Psi(\mathbb{E}^\infty_{n,e})$ sends

$$[\omega_1]\otimes\cdots\otimes[\omega_n]\otimes(\mathrm{PD}^{\otimes n-1})^{-1}([\beta_{j_1}]\otimes\cdots\otimes[\beta_{j_{n-1}}])$$

to

$$\int\limits_{(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)} \omega_1\cdots\omega_n + \sum_i \omega_1\cdots\nu(\omega_i\otimes\omega_{i+1})\cdots\omega_n \mod \mathbb{Z}.$$

To get the basis-independent formula, in $H_1(X,\mathbb{Z})^{\otimes n-1}$ we write

$$[\gamma_1]\otimes\cdots\otimes[\gamma_{n-1}] = \sum_{j_1,\dots,j_{n-1}} c_{j_1,\dots,j_{n-1}}[\beta_{j_1}]\otimes\cdots\otimes[\beta_{j_{n-1}}],$$

where the coefficients are all integers. In view of the isomorphisms (18) and (19), the element

$$\lambda := (\gamma_1-1)\cdots(\gamma_{n-1}-1) - \sum_{j_1,\dots,j_{n-1}} c_{j_1,\dots,j_{n-1}}(\beta_{j_1}-1)\cdots(\beta_{j_{n-1}}-1)\in I^{n-1},$$

where $I\in\mathbb{Z}[\pi_1(U,e)]$ is the augmentation ideal, actually belongs to I^n. Thus

$$\int\limits_\lambda \omega_1\cdots\omega_n + \sum_i \omega_1\cdots\nu(\omega_i\otimes\omega_{i+1})\cdots\omega_n = \int\limits_\lambda \omega_1\cdots\omega_n \in\mathbb{Z},$$

as $\lambda\in I^n$ and the ω_i have integer periods. This gives the desired conclusion. □

REMARK. (1) The use of a basis in Part (a) of the proposition is just to make the map well-defined.

(2) Let K be the kernel of the cup product $H^1 \otimes H^1 \to H^2(X)$. The map $\Phi \circ \Psi(\mathbb{E}^\infty_{n,e})$ can be thought of as an analog of the pointed harmonic volume

$$I_e \in \mathrm{Hom}(K_\mathbb{Z} \otimes H^1_\mathbb{Z}, \mathbb{R}/\mathbb{Z})$$

of B. Harris **[24]**. Pulte **[28]** showed that I_e corresponds under the isomorphisms

$$\mathrm{Ext}(K, H^1) \overset{\text{Carlson}}{\cong} J\underline{\mathrm{Hom}}(K, H^1) \overset{\text{Poincare duality}}{\cong} J\underline{\mathrm{Hom}}(K \otimes H^1, \mathbb{Z}(0)) \cong \mathrm{Hom}(K_\mathbb{Z} \otimes H^1_\mathbb{Z}, \mathbb{R}/\mathbb{Z})$$

to the extension

(29) $$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{L_1}{L_0}(X,e) & \longrightarrow & \dfrac{L_2}{L_0}(X,e) & \longrightarrow & \dfrac{L_2}{L_1}(X,e) & \longrightarrow & 0. \\ & & \wr\| & & & & \wr\| & & \\ & & H^1 & & & & K & & \end{array}$$

6.6. In this paragraph we will use Proposition 6.5.2 to express the extension $\mathbb{E}^\infty_{n,e}$ in terms of $\mathbb{E}^\infty_{2,e}$. For each $0 < i < n$, let

$$(H^1)^{\otimes i-1} \otimes \mathbb{E}^\infty_{2,e} \otimes (H^1)^{\otimes n-1-i} \in \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1})$$

be the extension obtained by tensoring $\mathbb{E}^\infty_{2,e} \in \mathrm{Ext}((H^1)^{\otimes 2}, H^1)$ on the left with $(H^1)^{\otimes i-1}$ and on the right with $(H^1)^{\otimes n-1-i}$:

(30)

$$0 \longrightarrow (H^1)^{\otimes i-1} \otimes H^1 \otimes (H^1)^{\otimes n-1-i} \overset{1\otimes i \otimes 1}{\longrightarrow} (H^1)^{\otimes i-1} \otimes \frac{L_2}{L_0} \otimes (H^1)^{\otimes n-1-i} \overset{1\otimes q \otimes 1}{\longrightarrow} (H^1)^{\otimes i-1} \otimes (H^1)^{\otimes 2} \otimes (H^1)^{\otimes n-1-i} \longrightarrow 0.$$

PROPOSITION 6.6.1. Given $\omega_1, \dots, \omega_n \in \mathcal{H}^1_\mathbb{R}(X)$ with integral periods and $\gamma_j \in \pi_1(U, e)$ $(1 \le j \le n-1)$, the map

$$\Phi \circ \Psi((H^1)^{\otimes i-1} \otimes \mathbb{E}^\infty_{2,e} \otimes (H^1)^{\otimes n-1-i}) : (H^1_\mathbb{Z})^{\otimes 2n-1} \longrightarrow \mathbb{R}/\mathbb{Z}$$

sends

$$[\omega_1] \otimes \cdots \otimes [\omega_n] \otimes (PD^{\otimes n-1})^{-1}([\gamma_1] \otimes \cdots \otimes [\gamma_{n-1}])$$

to

$$\prod_{\ell=1}^{i-1} \int_{\gamma_\ell} \omega_\ell \int_{\gamma_i} (\omega_i \omega_{i+1} + \nu(\omega_i \otimes \omega_{i+1})) \prod_{\ell=i+1}^{n-1} \int_{\gamma_\ell} \omega_{\ell+1} \quad \mathrm{mod}\ \mathbb{Z}.$$

PROOF. The map

$$1 \otimes \mathfrak{s}_F \otimes 1 : (H^1)^{\otimes i-1} \otimes (H^1)^{\otimes 2} \otimes (H^1)^{\otimes n-1-i} \to (H^1)^{\otimes i-1} \otimes \frac{L_2}{L_0} \otimes (H^1)^{\otimes n-1-i}$$

is a Hodge section of the map $1 \otimes q \otimes 1$ in (30). The map

$$1 \otimes r_\mathbb{Z} \otimes 1 : (H^1)^{\otimes i-1} \otimes \frac{L_2}{L_0} \otimes (H^1)^{\otimes n-1-i} \to (H^1)^{\otimes i-1} \otimes H^1 \otimes (H^1)^{\otimes n-1-i}$$

is an integral retraction of $1 \otimes i \otimes 1$ in (30). Thus the extension $(H^1)^{\otimes i-1} \otimes \mathbb{E}^\infty_{2,e} \otimes (H^1)^{\otimes n-1-i}$ as an element of $J\underline{\mathrm{Hom}}((H^1)^{\otimes n}, (H^1)^{\otimes n-1})$ is represented by the map

$$1 \otimes (r_\mathbb{Z} \circ \mathfrak{s}_F) \otimes 1 : (H^1)^{\otimes i-1} \otimes (H^1)^{\otimes 2} \otimes (H^1)^{\otimes n-1-i} \to (H^1)^{\otimes i-1} \otimes H^1 \otimes (H^1)^{\otimes n-1-i}.$$

It follows that $\Psi((H^1)^{\otimes i-1} \otimes \mathbb{E}^\infty_{2,e} \otimes (H^1)^{\otimes n-1-i})$ is represented by the map that sends

(31) $$[\omega_1] \otimes \cdots \otimes [\omega_n] \otimes (PD^{\otimes n-1})^{-1}([\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}])$$

(with $\{\beta_j\}_{j=1}^{2g} \subset \pi_1(U, e)$ used to define $r_{\mathbb{Z}}$) to

$$([\beta_{j_1}] \otimes \cdots \otimes [\beta_{j_{n-1}}])([\omega_1] \otimes \cdots \otimes [\omega_{i-1}] \otimes (r_{\mathbb{Z}} \circ \mathfrak{s}_F)([\omega_i] \otimes [\omega_{i+1}]) \otimes [\omega_{i+2}] \otimes \cdots \otimes [\omega_n]$$

$$= \prod_{\ell=1}^{i-1} \int_{\beta_{j_\ell}} \omega_\ell \int_{\beta_{j_i}} (\omega_i\omega_{i+1} + \nu(\omega_i \otimes \omega_{i+1})) \prod_{\ell=i+1}^{n-1} \int_{\beta_{j_\ell}} \omega_{\ell+1}. \tag{32}$$

This representative map is defined over $\mathbb{R}$ (as $r_{\mathbb{Z}} \circ \mathfrak{s}_F$ is defined over $\mathbb{R}$), thus it can be used to calculate $\Phi \circ \Psi((H^1)^{\otimes i-1} \otimes \mathbb{E}^{\infty}_{2,e} \otimes (H^1)^{\otimes n-1-i})$: the latter map sends (31) to (32) mod $\mathbb{Z}$. A similar argument to the one for Proposition 6.5.2(b) can now be used to verify the basis independent formula. □

COROLLARY 6.6.2. We have

$$\mathbb{E}^{\infty}_{n,e} = \sum_{i=1}^{n-1} (H^1)^{\otimes i-1} \otimes \mathbb{E}^{\infty}_{2,e} \otimes (H^1)^{\otimes n-1-i}. \tag{33}$$

PROOF. For ω_i $(1 \le i \le n)$ harmonic forms and $\gamma_j \in \pi_1(U, e)$ $(1 \le j \le n-1)$, we have

$$\sum_{i=1}^{n-1} \prod_{\ell=1}^{i-1} \int_{\gamma_\ell} \omega_\ell \int_{\gamma_i} (\omega_i\omega_{i+1} + \nu(\omega_i \otimes \omega_{i+1})) \prod_{\ell=i+1}^{n-1} \int_{\gamma_\ell} \omega_{\ell+1}$$

$$= \sum_{i=1}^{n-1} \prod_{\ell=1}^{i-1} \int_{\gamma_\ell} \omega_\ell \int_{\gamma_i} \omega_i\omega_{i+1} \prod_{\ell=i+1}^{n-1} \int_{\gamma_\ell} \omega_{\ell+1} + \sum_{i=1}^{n-1} \prod_{\ell=1}^{i-1} \int_{\gamma_\ell} \omega_\ell \int_{\gamma_i} \nu(\omega_i \otimes \omega_{i+1}) \prod_{\ell=i+1}^{n-1} \int_{\gamma_\ell} \omega_{\ell+1}$$

$$\overset{(13)}{=} \int_{(\gamma_1-1)\cdots(\gamma_{n-1}-1)} \omega_1 \cdots \omega_n + \sum_{i=1}^{n-1} \int_{(\gamma_1-1)\cdots(\gamma_{n-1}-1)} \omega_1 \cdots \omega_{i-1}\nu(\omega_i \otimes \omega_{i+1})\omega_{i+2} \cdots \omega_n.$$

Thus in view of Propositions 6.6.1 and 6.5.2(b), the two sides of (33) coincide after applying $\Phi \circ \Psi$. □

7. Algebraic cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$

7.1. Notation. Given a variety Y over a field K, $\mathcal{Z}_i(Y)$ (resp. $\mathcal{Z}^i(Y)$) denotes the group of algebraic cycles of dimension (resp. codimension) i (i.e. the free abelian group on the set of irreducible closed subsets of Y of dimension (resp. codimension) i). The Chow group $CH_i(Y)$ (resp. $CH^i(Y)$) is $\mathcal{Z}_i(Y)$ (resp. $\mathcal{Z}^i(Y)$) modulo rational equivalence. As usual, $\mathcal{Z}(Y) := \bigoplus \mathcal{Z}^i(Y)$ and $CH(Y) := \bigoplus CH^i(Y)$. Notation-wise, we do not distinguish between an algebraic cycle and its class in the corresponding Chow group. Given Y and Y' of dimensions d and d', the group of degree zero correspondences from Y to Y' is $Cor(Y, Y') := \mathcal{Z}_d(Y \times Y')$. If $f : Y \to Y'$ is a morphism, the graph of f is denoted by Γ_f; it is an element of $Cor(Y, Y')$. We use the standard notation (lower star) for push-forwards along morphisms. Given algebraic cycles $Z \in \mathcal{Z}_i(Y)$ and $Z' \in \mathcal{Z}_j(Y')$, $Z \times Z' \in \mathcal{Z}_{i+j}(Y \times Y')$ denotes the Cartesian product. Given $Z \in \mathcal{Z}_i(Y \times Y')$, ${}^tZ$ is the transpose of Z; it is an element of $\mathcal{Z}_i(Y' \times Y)$. Finally, if Y is a smooth variety over a subfield of $\mathbb{C}$, $\mathcal{Z}^{hom}_i(Y)$ (resp. $CH^{hom}_i(Y)$) refers to the subgroup of null-homologous cycles in $\mathcal{Z}_i(Y)$ (resp. $CH_i(Y)$).

7.2. A construction of Gross and Schoen. In this paragraph, we recall a construction of Gross and Shoen **[19]**. Let m be a positive integer. By convention, we set $X^0 = \operatorname{Spec} \mathbb{C}$. For (possibly empty) $T \subset \{1, \ldots, m\}$, let $p_T : X^m \to X^{|T|}$ be the projection map onto the coordinates in T, and

$q_T : X^{|T|} \to X^m$ be the embedding that is a right inverse to p_T and fills the coordinates that are not in T by e. For instance, if $m = 3$ and $T = \{2, 3\}$,

$$(x_1, x_2, x_3) \overset{p_T}{\mapsto} (x_2, x_3) \qquad \text{and} \qquad (x_1, x_2) \overset{q_T}{\mapsto} (e, x_1, x_2).$$

In general, the composition $q_T \circ p_T : X^m \to X^m$ is the morphism that keeps the T coordinates unchanged, and replaces the rest by e. Let

$$P_e = \sum_T (-1)^{|T^c|} \Gamma_{q_T \circ p_T} \in \mathrm{Cor}(X^m, X^m),$$

where T^c denotes the complement of T. For the proof of the following result, see [**19**].

THEOREM 7.2.1. If $i < m$, the map $(P_e)^h_* : H_i(X^m) \to H_i(X^m)$ induced by P_e on homology is zero.

Let $(P_e)_*$ be the push forward map $\mathcal{Z}(X^m) \to \mathcal{Z}(X^m)$ defined by the correspondence P_e. Then

$$(P_e)_* = \sum_T (-1)^{|T^c|} (q_T \circ p_T)_*.$$

In view of commutativity of the diagram

$$\begin{array}{ccc} \mathcal{Z}_i(X^m) & \xrightarrow{(P_e)_*} & \mathcal{Z}_i(X^m) \\ \downarrow & & \downarrow \\ H_{2i}(X^m, \mathbb{C}) & \xrightarrow{(P_e)^h_*} & H_{2i}(X^m, \mathbb{C}), \end{array}$$

where the vertical maps are class maps, it follows from the previous theorem that if $2i < m$, then

$$(P_e)_*(\mathcal{Z}_i(X^m)) \subset \mathcal{Z}_i^{\mathrm{hom}}(X^m).$$

This gives a way of constructing null-homologous cycles.

Example. For $m \geq 2$, denote by $\Delta^{(m)}(X)$ the diagonal copy of X in X^m, i.e.

$$\{(x, x, \ldots, x) : x \in X\} \in \mathcal{Z}_1(X^m).$$

For $m \geq 3$, by the previous observation, the *modified diagonal cycle* $(P_e)_*(\Delta^{(m)}(X))$ is null-homologous. As it is pointed out in [**19**], this cycle has zero Abel-Jacobi image if $m > 3$. On the other hand, if $m = 3$, this cycle, which was first defined by Gross and Kudla in [**18**] and then studied more by Gross and Schoen in [**19**], is well-known to be interesting. It is easy to see from its definition that

$$\begin{aligned} (P_e)_*(\Delta^{(3)}(X)) &= \Delta^{(3)}(X) - \{(e, x, x) : x \in X\} - \{(x, e, x) : x \in X\} - \{(x, x, e) : x \in X\} \\ &\quad + \{(e, e, x) : x \in X\} + \{(e, x, e) : x \in X\} + \{(x, e, e) : x \in X\}. \end{aligned}$$

We denote this cycle by $\Delta_{GKS,e}$, the modified diagonal cycle of Gross, Kudla, and Schoen.

Note that

$$(P_e)_*(\Delta^{(2)}(X)) = \Delta^{(2)}(X) - \{e\} \times X - X \times \{e\},$$

which is homologically nontrivial.

7.3. Let $n \geq 2$. In this paragraph, we use the construction of Gross and Schoen to define a null-homologous cycle $\Delta_{n,e} \in \mathcal{Z}_{n-1}(X^{2n-1})$, which will play a crucial role in the remainder of the paper. We use the notation of Paragraph 7.2 with $m = 2n-1$.

For $0 < i < n$, let $\delta_i : X^{n-1} \to X^n$ be the embedding

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_i, x_i, \ldots, x_{n-1}).$$

Then ${}^t\Gamma_{\delta_i} \in \mathcal{Z}_{n-1}(X^{2n-1})$, and thus $(P_e)_*({}^t\Gamma_{\delta_i})$ is null-homologous. We define

$$\Delta_{n,e} := (P_e)_*\left(\sum_i (-1)^{i-1}\, {}^t\Gamma_{\delta_i}\right) = \sum_i (-1)^{i-1}(P_e)_*({}^t\Gamma_{\delta_i}) \in \mathcal{Z}^{\text{hom}}_{n-1}(X^{2n-1}).$$

It is clear from the definition that $\Delta_{2,e}$ is simply the modified diagonal cycle $\Delta_{KGS,e}$ of Gross, Kudla, and Schoen in X^3.

7.4. In this paragraph, we realize the cycle $\Delta_{n,e}$ as the boundary of a chain. This will be useful later when we study the image of $\Delta_{n,e}$ under the Abel-Jacobi map.

Let Λ_n be the closed subvariety

$$\{(x_1, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}) : x_i \in X\}$$

of X^{2n-1}, where each x_i $(i > 1)$ is appearing in exactly two coordinates. It is a copy of X^{n-1} embedded in X^{2n-1} via the map

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}),$$

and can also be thought of as an element of $\mathcal{Z}_{n-1}(X^{2n-1})$. It is easy to see that

$$(P_e)_*(\Lambda_n) = \Delta_{2,e} \times \overbrace{(P_e)_*(\Delta^{(2)}(X)) \times \cdots \times (P_e)_*(\Delta^{(2)}(X))}^{n-2 \text{ factors}}. \tag{34}$$

Let $\partial^{-1}(\Delta_{2,e})$ be an integral 3-chain in X^3 whose boundary is $\Delta_{2,e}$. (See for instance the proof of Lemma 2.3 of [7] for such a 3-chain.) Then $(P_e)_*(\Lambda_n)$ is the boundary of

$$\partial^{-1}(\Delta_{2,e}) \times \overbrace{(P_e)_*(\Delta^{(2)}(X)) \times \cdots \times (P_e)_*(\Delta^{(2)}(X))}^{n-2 \text{ factors}} =: \partial^{-1}(P_e)_*(\Lambda_n).$$

It is clear that each ${}^t\Gamma_{\delta_i}$ is a copy of Λ_n. Specifically, ${}^t\Gamma_{\delta_i} = (\sigma_i)_*(\Lambda_n)$ where σ_i is the automorphism of X^{2n-1} that sends $(x_1, \ldots, x_{2n-1})$ to

$$(x_4, x_6, \ldots, x_{2i}, x_1, x_2, x_{2i+2}, x_{2i+4}, \ldots, x_{2n-2}, x_5, x_7, \ldots, x_{2i+1}, x_3, x_{2i+3}, x_{2i+5}, \ldots, x_{2n-1}).$$

LEMMA 7.4.1. $(P_e)_*$ and $(\sigma_i)_*$ commute (as maps $\mathcal{Z}(X^{2n-1}) \to \mathcal{Z}(X^{2n-1})$).

PROOF. With abuse of notation, suppose σ_i is the permutation of $1, 2, \ldots, 2n-1$ such that

$$\sigma_i(x_1, \ldots, x_{2n-1}) = (x_{\sigma_i^{-1}(1)}, x_{\sigma_i^{-1}(2)}, \ldots, x_{\sigma_i^{-1}(2n-1)}).$$

Then for each subset T of $\{1,2,\ldots,2n-1\}$, $q_T\circ p_T\circ\sigma_i=\sigma_i\circ q_{\sigma_i^{-1}T}\circ p_{\sigma_i^{-1}T}$. We have

$$\begin{aligned}(P_e)_*\circ(\sigma_i)_* &= \left(\sum_T(-1)^{|T^c|}(q_T\circ p_T)_*\right)(\sigma_i)_*\\ &= \sum_T(-1)^{|T^c|}(q_T\circ p_T\circ\sigma_i)_*\\ &= \sum_T(-1)^{|T^c|}(\sigma_i\circ q_{\sigma_i^{-1}T}\circ p_{\sigma_i^{-1}T})_*\\ &= (\sigma_i)_*\left(\sum_T(-1)^{|T^c|}(q_{\sigma_i^{-1}T}\circ p_{\sigma_i^{-1}T})_*\right)\\ &= (\sigma_i)_*\circ(P_e)_*.\end{aligned}$$

□

It follows from the lemma that

(35) $$(\sigma_i)_*\left((P_e)_*(\Lambda_n)\right)=(P_e)_*\left({}^t\Gamma_{\delta_i}\right),$$

and hence

$$\partial\left((\sigma_i)_*(\partial^{-1}(P_e)_*(\Lambda_n))\right)=(P_e)_*\left({}^t\Gamma_{\delta_i}\right).$$

REMARK. In view of (34), (35) and definition of $\Delta_{n,e}$, if $\Delta_{2,e}$ is torsion in $CH_1^{hom}(X^3)$, then so is $\Delta_{n,e}$ in $CH_{n-1}^{hom}(X^{2n-1})$ for every n.

7.5. In this paragraph, we define another family of null-homologous cycles that will be used later on. Let $n\geq 2$. Given $y\in X(\mathbb{C})$, for $0<i<n$, let $Z^y_{n,i}\in\mathcal{Z}_{n-1}(X^{2n-1})$ be

$$\{(x_1,\ldots,x_{i-1},x_i,x_i,x_{i+1},\ldots,x_{n-1},x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_{n-1}):x_1,\ldots,x_{n-1}\in X\}.$$

Here each x_j appears in exactly two coordinates. There are different ways of thinking about this cycle. For instance,

$$Z^y_{n,i}=(\pi_{n+i,y})_*{}^t\Gamma_{\delta_i},$$

where $\pi_{n+i,y}$ is the map $X^{2n-1}\to X^{2n-1}$ that replaces the $(n+i)$-th coordinate by y, and keeps the other coordinates unchanged.

It is clear that the cycle $Z^\infty_{n,i}-Z^e_{n,i}$ is null-homologous. For future reference, here we explicitly define a chain whose boundary is $Z^\infty_{n,i}-Z^e_{n,i}$. Choose a path γ_e^∞ in X from e to ∞, and let

$$C^\infty_{n,e}:=\Delta^{(2)}(X)^{n-1}\times\gamma_e^\infty=\{(x_1,x_1,\ldots,x_{n-1},x_{n-1},\gamma_e^\infty(t)):x_i\in X,t\in[0,1]\}.$$

One clearly has

$$\partial C^\infty_{n,e}=\Delta^{(2)}(X)^{n-1}\times\{\infty\}-\Delta^{(2)}(X)^{n-1}\times\{e\}.$$

For $0<i<n$, let τ_i be the automorphism of X^{2n-1} that maps $(x_1,\ldots,x_{2n-1})$ to

$$(x_1,x_3,\ldots,x_{2(i-1)-1},x_{2i-1},x_{2i},x_{2(i+1)-1},\ldots,x_{2(n-1)-1},x_2,x_4,\ldots,x_{2(i-1)},x_{2n-1},x_{2(i+1)},\ldots,x_{2(n-1)}),$$

which is designed so that

$$(\tau_i)_*\left(\Delta^{(2)}(X)^{n-1}\times\{y\}\right)=Z^y_{n,i}$$

for every y. Then

(36) $$\partial(\tau_i)_*(C^\infty_{n,e})=Z^\infty_{n,i}-Z^e_{n,i}.$$

We put together all the $Z^{\infty}_{n,i} - Z^{e}_{n,i}$ and define

$$Z^{\infty}_{n,e} := \sum_{i=1}^{n-1} (-1)^{i-1}(Z^{\infty}_{n,i} - Z^{e}_{n,i}) \ \in \mathcal{Z}^{\hom}_{n-1}(X^{2n-1}).$$

7.6. While we worked over $\mathbb{C}$ in this section, it is clear that the constructions of $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ remain valid over any field K that can be embedded into $\mathbb{C}$. More precisely, if X_0 is a geometrically connected smooth projective curve over K, and $e, \infty \in X_0(K)$, the above constructions give null-homologous cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ in $\mathcal{Z}_{n-1}(X_0^{2n-1})$ (or in $\mathrm{CH}_{n-1}(X_0^{2n-1})$).

8. Statement of the main theorem

Our goal in this section is to state the main result of the paper, which expresses the extension $\mathbb{E}^{\infty}_{n,e}$ in terms of the Abel-Jacobi images of the cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$.

8.1. Review of Griffiths' Abel-Jacobi maps. Let Y be a smooth projective variety over $\mathbb{C}$. The n-th Abel-Jacobi map associated to Y is the map[†]

$$\mathrm{AJ} : \mathcal{Z}^{\hom}_{n}(Y) \to JH^{2n+1}(Y)^{\vee}$$

defined as follows. First note that the restriction map $\left(H^{2n+1}_{\mathbb{C}}(Y)\right)^{\vee} \to \left(F^{n+1}H^{2n+1}(Y)\right)^{\vee}$ gives an isomorphism

$$JH^{2n+1}(Y)^{\vee} \cong \frac{\left(F^{n+1}H^{2n+1}(Y)\right)^{\vee}}{H_{2n+1}(Y,\mathbb{Z})},$$

where an element of $H_{2n+1}(Y,\mathbb{Z})$ is considered as an element of $\left(F^{n+1}H^{2n+1}(Y)\right)^{\vee}$ via integration. Thus we can equivalently define AJ as a map into

$$\frac{\left(F^{n+1}H^{2n+1}(Y)\right)^{\vee}}{H_{2n+1}(Y,\mathbb{Z})}.$$

Given a null-homologous n-dimensional cycle Z on Y, there is an integral chain C such that $\partial C = Z$. Given $\varphi \in F^{n+1}H^{2n+1}(Y)$, take a representative $\omega \in F^{n+1}E^{2n+1}_{\mathbb{C}}(Y)$, and set

$$\int_C \varphi = \int_C \omega.$$

One can show that this is independent of the choice of ω. Then

$$\mathrm{AJ}(Z) \in \frac{\left(F^{n+1}H^{2n+1}(Y)\right)^{\vee}}{H_{2n+1}(Y,\mathbb{Z})}$$

is defined to be the class of the map

$$\varphi \mapsto \int_C \varphi.$$

The ambiguity in having to choose C is resolved by modding out by $H_{2n+1}(Y,\mathbb{Z})$. If one insists on having $\mathrm{AJ}(Z) \in JH^{2n+1}(Y)^{\vee}$, it is the class of any map $H^{2n+1}_{\mathbb{C}}(Y) \to \mathbb{C}$ whose restriction to $F^{n+1}H^{2n+1}(Y)$ is the map $\int_C$ above.

[†]That our notation for this map does not incorporate Y or n should not lead to any confusion.

One can show that AJ factors through $\mathrm{CH}_n^{\mathrm{hom}}(Y)$. The induced map

$$\mathrm{CH}_n^{\mathrm{hom}}(Y) \to JH^{2n+1}(Y)^\vee$$

is also called Abel-Jacobi, and with abuse of notation we denote it by AJ as well.

8.2. Notation. We adopt the following notation for the Kunneth decomposition of cohomology. Given manifolds M and N, we think of $H^i(M) \otimes H^j(N)$ (singular or de Rham cohomology) as a subspace of $H^{i+j}(M \times N)$. Given $c \in H^i(M)$, $d \in H^j(N)$, the element $c \otimes d$ of $H^{i+j}(M \times N)$ is $pr_1^*(c) \wedge pr_2^*(d)$, where pr_i the the projection of $M \times N$ onto its i^{th} factor. We adopt a similar notation for differential forms: given ω and ϕ differential forms on M and N, we refer to the differential form $pr_1^*(\omega) \wedge pr_2^*(\phi)$ on $M \times N$ by $\omega \otimes \phi$. Similar notation is used for more than two factors.

8.3. For $n \geq 1$, let h_n be the composition of the Abel-Jacobi map

$$\mathrm{CH}_{n-1}^{\mathrm{hom}}(X^{2n-1}) \longrightarrow JH^{2n-1}(X^{2n-1})^\vee$$

with the map

$$JH^{2n-1}(X^{2n-1})^\vee \longrightarrow J((H^1)^{\otimes 2n-1})^\vee$$

induced by the Kunneth inclusion $(H^1)^{\otimes 2n-1} \subset H^{2n-1}(X^{2n-1})$. It is easy to see from definitions that if $Z \in \mathcal{Z}_{n-1}^{\mathrm{hom}}(X^{2n-1})$ and C is an integral chain in X^{2n-1} whose boundary is Z, $h_n(Z)$ is the class of the map that, given harmonic 1-forms $\omega_1, \ldots, \omega_{2n-1}$ on X, it sends

$$[\omega_1] \otimes \cdots \otimes [\omega_{2n-1}] \mapsto \int_C \omega_1 \otimes \cdots \otimes \omega_{2n-1}. \tag{37}$$

Note that h_1 is just the "classical" Abel-Jacobi map $\mathrm{CH}_0^{\mathrm{hom}}(X) \to J(H^1)^\vee$.

If Z and C are as above, since the map (37) is defined over $\mathbb{R}$,

$$\Phi(h_n(Z)) : (H^1_{\mathbb{Z}})^{\otimes 2n-1} \to \mathbb{R}/\mathbb{Z}$$

is the map that, given harmonic forms $\omega_1, \ldots, \omega_{2n-1}$ on X with integral periods, it maps

$$[\omega_1] \otimes \cdots \otimes [\omega_{2n-1}] \mapsto \int_C \omega_1 \otimes \cdots \otimes \omega_{2n-1} \mod \mathbb{Z}.$$

(See Paragraph 6.1 and Paragraph 6.2.)

8.4. Now we are ready to state the main result.

THEOREM 8.4.1. Let $n \geq 2$. We have

$$\Psi(\mathbb{E}_{n,e}^\infty) = (-1)^{\frac{n(n-1)}{2}} h_n\left(\Delta_{n,e} - Z_{n,e}^\infty\right). \tag{38}$$

When $n = 2$, a slightly weaker version of this is due to Darmon, Rotger, and Sols [7]. (See the next section.)

9. $n = 2$ case of Theorem 8.4.1 - A formula of Darmon et al revisited

9.1. Independence of $-\Psi(\mathbb{E}^{\infty}_{2,e}) + h_2(Z^{\infty}_{2,e})$ from ∞.

LEMMA 9.1.1. The element $-\Psi(\mathbb{E}^{\infty}_{2,e}) + h_2(Z^{\infty}_{2,e})$ is independent of the point $\infty \neq e$, i.e. if $\infty_1, \infty_2 \neq e$, then

$$-\Psi(\mathbb{E}^{\infty_1}_{2,e}) + h_2(Z^{\infty_1}_{2,e}) = -\Psi(e^{\infty_2}_{2,e}) + h_2(Z^{\infty_2}_{2,e}).$$

PROOF. Let $\infty_1, \infty_2 \neq e$ be distinct. After passing to $\mathrm{Hom}\left((H^1_{\mathbb{Z}})^{\otimes 3}, \mathbb{R}/\mathbb{Z}\right)$ via Φ, in view of Proposition 6.5.2(b), we need to show that if ω, ρ, η are harmonic forms with integral periods on X, and $\gamma_\eta \in \pi_1(X - \{\infty_1, \infty_2\}, e)$ is such that its homology class in $H_1(X, \mathbb{Z})$ is $PD([\eta])$, then

$$-\int_{\gamma_\eta} \omega\rho + \nu_{\infty_1}(\omega \otimes \rho) \ + \int_X \omega \wedge \rho \int_e^{\infty_1} \eta \overset{\mathbb{Z}}{\equiv} -\int_{\gamma_\eta} \omega\rho + \nu_{\infty_2}(\omega \otimes \rho) \ + \int_X \omega \wedge \rho \int_e^{\infty_2} \eta,$$

or equivalently

$$-\int_{\gamma_\eta} \nu_{\infty_1}(\omega \otimes \rho) - \nu_{\infty_2}(\omega \otimes \rho) \ + \int_X \omega \wedge \rho \int_{\infty_2}^{\infty_1} \eta \in \mathbb{Z}, \tag{39}$$

where the integrals of η are over any path in X with the specified end points. Fix ω and ρ. For brevity we write ν_i for $\nu_{\infty_i}(\omega \otimes \rho)$. Note that if $\omega \wedge \rho$ is exact on X, then the statement clearly holds, as then $\nu_i \in \mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$ and $\nu_1 - \nu_2$, being a closed element of $\mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$, is exact, so that the number above is simply zero. (See the proof of Lemma 5.6.1.) So we may assume $\omega \wedge \rho$ is not exact on X. Then the 1-form $\nu_1 - \nu_2$ satisfies the following properties:

- (i) It is meromorphic on X, holomorphic on $X - \{\infty_1, \infty_2\}$, with logarithmic poles at ∞_1 and ∞_2 with residues $\frac{a}{2\pi i}$ and $-\frac{a}{2\pi i}$ respectively for some integer $a \neq 0$.
- (ii) Its cohomology class in $H^1(X - \{\infty_1, \infty_2\})$ is real, i.e. it can be written on $X - \{\infty_1, \infty_2\}$ as the sum of an exact form and a real closed form.

Indeed, (i) follows from that both ν_1 and ν_2 are of type (1,0), and $d\nu_1 = d\nu_2 = -\omega \wedge \rho$ on $X - \{\infty_1, \infty_2\}$, so that $\nu_1 - \nu_2$ is holomorphic on $X - \{\infty_1, \infty_2\}$. For the behavior at ∞_i, note that $\nu_i \in E^1(X \log \infty_i)$. The statement about the residues is immediate from Lemma 5.6.1(iv) ($a = \int_X \omega \wedge \rho$). Statement (ii) follows from that each form ν_i can be written as a real form on $X - \{\infty_i\}$ plus an exact form on the same space. (See Lemma 5.6.1(iii).)

The statement (39) now follows from the following lemma. □

LEMMA 9.1.2. Let $\infty_1, \infty_2 \neq e$, and ζ be any 1-form satisfying conditions (i) and (ii) above. Then for any harmonic 1-form η on X with integral periods,

$$-\int_{\gamma_\eta} \zeta \ + a \int_{\infty_2}^{\infty_1} \eta \in \mathbb{Z},$$

where $\gamma_\eta \in \pi_1(X - \{\infty_1, \infty_2\}, e)$ satisfies $PD([\eta]) = [\gamma_\eta]$.

PROOF. First note that the integral $\int_X \zeta \wedge \eta$ converges for any $\eta \in \mathcal{H}^1_{\mathbb{C}}(X)$, as the integral of $\frac{dz d\bar{z}}{z}$ converges on the unit disk in $\mathbb{C}$. Thus one gets a map $h : H^1_{\mathbb{C}} \to \mathbb{C}$ given by $[\eta] \mapsto \int_X \zeta \wedge \eta$. We claim that this map takes integer values on $H^1_{\mathbb{Z}}$. Note that since h vanishes on F^1H^1, by the remark in

Paragraph 6.1, it suffices to show that it is defined over $\mathbb{R}$. Suppose $\eta \in \mathcal{H}^1_{\mathbb{R}}(X)$ has integer periods. The claim is established if we show $h([\eta])$ is real. We may assume that the map

$$\int \eta : H_1(X, \mathbb{Z}) \to \mathbb{Z} \tag{40}$$

is surjective, and that $\gamma_\eta \in \pi_1(X - \{\infty_1, \infty_2\}, e)$ (Poincare dual to $[\eta]$ in $H_1(X, \mathbb{Z})$) has a simple representative loop, which we also denote by γ_η. One can show that there is a Riemann surface $\tilde{X}$, a covering projection $\pi : \tilde{X} \to X$, and a deck transformation T of π such that

- $\pi^*\eta = df$ for a real function f on $\tilde{X}$.
- $fT - f$ is the constant function 1.
- There is a lift $\tilde{\gamma}_\eta$ of γ_η, and a submanifold with boundary $X^{(0)}$ of $\tilde{X}$ such that $\partial X^{(0)} = T\tilde{\gamma}_\eta - \tilde{\gamma}_\eta$, and the restriction of π to $X^{(0)} - \partial X^{(0)}$ is an isomorphism of Riemann surfaces onto $X - \gamma_\eta$.[†]

Now let for each i, D_i be an open disk around ∞_i in X, small enough so that $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\overline{D_i} \cap \gamma_\eta = \emptyset$ (bar denoting closure). Denote by $\tilde{\infty}_i$ and $\tilde{D}_i$ the lift of ∞_i and D_i in $X^{(0)}$. Then we have

$$\begin{aligned}
\int_{X - D_1 \cup D_2} \zeta \wedge \eta = \int_{X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2} \pi^*\zeta \wedge \pi^*\eta &= \int_{X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2} -df \wedge \pi^*\zeta \\
&= \int_{X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2} -d(f\pi^*\zeta) \\
&= - \int_{\partial(X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2)} f\pi^*\zeta \\
&= \int_{\tilde{\gamma}_\eta - T\tilde{\gamma}_\eta + \partial\tilde{D}_1 + \partial\tilde{D}_2} f\pi^*\zeta \\
&= \int_{\tilde{\gamma}_\eta - T\tilde{\gamma}_\eta} f\pi^*\zeta + \int_{\partial\tilde{D}_1 + \partial\tilde{D}_2} f\pi^*\zeta.
\end{aligned}$$

It follows that

$$\int_{X - D_1 \cup D_2} \zeta \wedge \eta = - \int_{\gamma_\eta} \zeta + \int_{\partial\tilde{D}_1 + \partial\tilde{D}_2} f\pi^*\zeta \, .$$

We would like to know what happens as $D_i \to \{\infty_i\}$. Write

$$\int_{\partial\tilde{D}_i} f\pi^*\zeta = \int_{\partial\tilde{D}_i} f(\tilde{\infty}_i)\pi^*\zeta + \int_{\partial\tilde{D}_i} (f - f(\tilde{\infty}_i))\, \pi^*\zeta.$$

[†]Such a covering projection is obtained by taking a copy $X^{(i)}$ of X for each integer i, "cutting" the $X^{(i)}$ along γ_η, and then gluing $X^{(i)}$ to $X^{(i+1)}$ appropriately along γ_η. The deck transformation simply sends a point in $X^{(i)}$ to its counterpart in $X^{(i+1)}$.

Since ζ is holomorphic on $\tilde{D}_i - \tilde{\infty}_i$ with a pole of order 1 at ∞_i, and $f - f(\tilde{\infty}_i)$ is smooth and vanishes at $\tilde{\infty}_i$, the second term goes to zero as $D_i \to \{\infty_i\}$. The first term is equal to $2\pi i f(\tilde{\infty}_i)\mathrm{res}_{\infty_i}(\zeta)$. Thus

$$\int_X \zeta \wedge \eta = -\int_{\gamma_\eta} \zeta \; + \; a(f(\tilde{\infty}_1) - f(\tilde{\infty}_2)). \tag{41}$$

The second term on the right is real as a and f are real. The first term is also real because the cohomology class of ζ in $H^1(X - \{\infty_1, \infty_2\})$ is real. Thus the claim is established.

Now it is easy to conclude the lemma. Let η be as described in the statement. Without loss of generality we may assume that (40) is surjective, and that γ_η has a simple representative loop. Then we know (41), and hence

$$\int_X \zeta \wedge \eta \overset{\mathbb{Z}}{\equiv} -\int_{\gamma_\eta} \zeta \; + \; a\int_{\infty_2}^{\infty_1} \eta.$$

The left hand side (which is $h(\eta)$) is an integer. □

9.2. When $n = 2$, Theorem 8.4.1 asserts that

$$\Psi(\mathbb{E}^{\infty}_{2,e}) = h_2(-\Delta_{2,e} + Z^{\infty}_{2,e}). \tag{42}$$

This is a slightly stronger version of a theorem of Darmon, Rotger, and Sols [**7**, Theorem 2.5]. Their result can be stated as to assert that, for every Hodge class ξ of $(H^1)^{\otimes 2}$, one has

$$\xi^{-1}\left(\Psi(\mathbb{E}^{\infty}_{2,e})\right) = \xi^{-1}\left(h_2(-\Delta_{2,e} + Z^{\infty}_{2,e})\right), \tag{43}$$

where $\xi^{-1} : J((H^1)^{\otimes 3})^\vee \to J(H^1)^\vee$ is the map that sends $[f] \mapsto [f(\xi \otimes -)]$ for any $f \in \left((H^1_{\mathbb{C}})^{\otimes 3}\right)^\vee$. (This is well-defined because ξ is a Hodge class.)

Let $\{\beta_j\}_j \subset \pi_1(U, e)$ be such that $\{[\beta_j]\}_j$ forms a basis of $H_1(X, \mathbb{Z})$. For each j, let η_j be the harmonic form on X such that $PD([\eta_j]) = [\beta_j]$. In view of our description of $\Psi(\mathbb{E}^{\infty}_{2,e})$ given in Proposition 6.5.2, (43) is equivalent to that if $\xi = \sum [\omega_i] \otimes [\rho_i]$ with ω_i and ρ_i harmonic forms on X with integral periods, then the map $H^1_{\mathbb{C}} \to \mathbb{C}$ given by

$$[\eta_j] \mapsto \int_{\partial^{-1}\Delta_{2,e}} \sum_i \omega_i \otimes \rho_i \otimes \eta_j \; + \; \left(\int_{\beta_j} \sum \omega_i \rho_i + \nu(\xi)\right) \; - \int_{\Delta^{(2)}(X)} \xi \int_{\gamma_e^\infty} \eta_j$$

agrees on $F^1H^1_{\mathbb{C}}$ with an element of $(H_1)_{\mathbb{Z}}$ (so that it represents zero in $J(H^1)^\vee$). The latter is what Darmon, Rotger and Sols show in [**7**].

The argument given in [**7**] combined with Lemma 9.1.1 indeed implies (42). To see this, note that (42) is equivalent to that, for every $\xi = [\omega] \otimes [\rho] \in (H^1_{\mathbb{Z}})^{\otimes 2}$, where the ω and ρ are harmonic forms on X with integral periods, the map $H^1_{\mathbb{C}} \to \mathbb{C}$ defined by

$$[\eta_j] \mapsto \int_{\partial^{-1}\Delta_{2,e}} \omega \otimes \rho \otimes \eta_j \; + \; \left(\int_{\beta_j} \omega\rho + \nu(\xi)\right) \; - \int_{\Delta^{(2)}(X)} \xi \int_{\gamma_e^\infty} \eta_j$$

agrees on $F^1H^1_{\mathbb{C}}$ with an element of $(H_1)_{\mathbb{Z}}$. This is exactly Theorem 2.5 of [**7**], except that here ξ is not necessarily a Hodge class, but rather merely an integral class. However, the argument in [**7**] works just as well here too, as long as one can replace the point ∞ by a point at which certain technical conditions† hold. Lemma 9.1.1 allows one to do this‡.

9.3. We close this section by noting that applying the map Φ to (42), we see that, if ω, ρ, η are harmonic forms on X with integral periods, and $\gamma_\eta \in \pi_1(U, e)$ is such that $\mathrm{PD}([\eta]) = [\gamma_\eta]$ in homology of X, then

$$\int_{\partial^{-1}\Delta_{2,e}} \omega \otimes \rho \otimes \eta \overset{\mathbb{Z}}{\equiv} -\int_{\gamma_\eta} (\omega\rho + \nu(\omega \otimes \rho)) \; + \int_X \omega \wedge \rho \int_e^\infty \eta. \tag{44}$$

(See Proposition 6.5.2(b).)

10. Proof of the general case of Theorem 8.4.1

Our goal here is to use the contents of the previous sections to prove Theorem 8.4.1 in $n \geq 3$ case. In view of Corollary 6.6.2, it suffices to show that for $0 < i < n$,

$$\Psi\left((H^1)^{\otimes i-1} \otimes \mathbb{E}^\infty_{2,e} \otimes (H^1)^{\otimes n-1-i} \right) = (-1)^{\frac{n(n-1)}{2}+i-1} h_n \left((P_e)_*({}^t\Gamma_{\delta_i}) - (Z^\infty_{n,i} - Z^e_{n,i}) \right). \tag{45}$$

(For the definition of Γ_{δ_i}, $Z^\infty_{n,i}$, and other algebraic cycles, chains, and permutations that appear in the calculations below see Section 7.) We will show that the two sides of (45) have equal images under Φ. Let $\omega_1, \dots, \omega_n$ and $\eta_1, \dots, \eta_{n-1}$ be harmonic forms on X with integral periods, and for each j, $\gamma_j \in \pi_1(U, e)$ be such that $[\gamma_j] = \mathrm{PD}([\eta_j])$ in $H_1(X, \mathbb{Z})$. All equalities below take place in $\mathbb{R}/\mathbb{Z}$. We use the notation $[\cdots|\cdots]$ for $\cdots \otimes \cdots$, and for brevity denote $\frac{n(n-1)}{2}$ by m.

We have

$$\begin{aligned}
\Phi\left(h_n\left((P_e)_*({}^t\Gamma_{\delta_i}) \right)\right)([\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]) &= \int_{(\sigma_i)_*(\partial^{-1}(P_e)_*(\Lambda_n))} [\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}] \\
&= \int_{\partial^{-1}(P_e)_*(\Lambda_n)} (\sigma_i)^*([\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]).
\end{aligned}$$

Recalling how σ_i permutes the coordinates of X^{2n-1}, we see this is

$$\begin{aligned}
&= (-1)^{n+i+m} \int_{\partial^{-1}(P_e)_*(\Lambda_n)} [\omega_i|\omega_{i+1}|\eta_i|\omega_1|\eta_1|\cdots|\omega_{i-1}|\eta_{i-1}|\omega_{i+2}|\eta_{i+1}|\cdots|\omega_n|\eta_{n-1}] \\
&= (-1)^{n+i+m} \int_{(\partial^{-1}\Delta_{2,e})\times((P_e)_*\Delta^{(2)}(X))^{n-2}} [\omega_i|\omega_{i+1}|\eta_i|\omega_1|\eta_1|\cdots|\omega_{i-1}|\eta_{i-1}|\omega_{i+2}|\eta_{i+1}|\cdots|\omega_n|\eta_{n-1}] \\
&= (-1)^{n+i+m} \int_{\partial^{-1}\Delta_{2,e}} [\omega_i|\omega_{i+1}|\eta_i] \prod_{j=1}^{i-1} \int_{(P_e)_*\Delta^{(2)}(X)} [\omega_j|\eta_j] \prod_{j=i+2}^{n} \int_{(P_e)_*\Delta^{(2)}(X)} [\omega_j|\eta_{j-1}] \\
&= (-1)^{n+i+m} \int_{\partial^{-1}\Delta_{2,e}} [\omega_i|\omega_{i+1}|\eta_i] \prod_{j=1}^{i-1} \int_{\Delta^{(2)}(X)} [\omega_j|\eta_j] \prod_{j=i+2}^{n} \int_{\Delta^{(2)}(X)} [\omega_j|\eta_{j-1}],
\end{aligned}$$

† on the "positioning" of ∞ relative to $\partial^{-1}\Delta_{2,e}$

‡ In [7], a similar task is performed by Lemma 1.3, which asserts that our Lemma 9.1.1 holds after applying ξ^{-1}.

as the other summands in $(P_e)_*\Delta^{(2)}(X)$ do not contribute to the integrals. In view of (44), the last expression is

$$= (-1)^{n+i+m}\left(-\int_{\gamma_i}\omega_i\omega_{i+1}+\nu([\omega_i|\omega_{i+1}]) + \int_X \omega_i\wedge\omega_{i+1}\int_e^\infty \eta_i\right)\prod_{j=1}^{i-1}\int_X\omega_j\wedge\eta_j \prod_{j=i+2}^{n}\int_X\omega_j\wedge\eta_{j-1}$$

$$= (-1)^{i+m}\left(-\int_{\gamma_i}\omega_i\omega_{i+1}+\nu([\omega_i|\omega_{i+1}]) + \int_X \omega_i\wedge\omega_{i+1}\int_e^\infty \eta_i\right)\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j \prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j.$$

Thus

$$(-1)^{m+i-1}\Phi\left(h_n\left((P_e)_*({}^t\Gamma_{\delta_i})\right)\right)([\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]) = (I) - (II), \tag{46}$$

where

$$(I) = \left(\int_{\gamma_i}\omega_i\omega_{i+1}+\nu([\omega_i|\omega_{i+1}])\right)\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j \prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j$$

and

$$(II) = \int_X \omega_i\wedge\omega_{i+1}\int_e^\infty\eta_i\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j \prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j.$$

By Proposition 6.6.1,

$$(I) = \Phi\left(\Psi\left((H^1)^{\otimes i-1}\otimes\mathbb{E}^\infty_{2,e}\otimes(H^1)^{\otimes n-1-i}\right)\right)([\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]). \tag{47}$$

On the other hand, in view of (36),

$$\begin{aligned}\Phi\left(h_n(Z^\infty_{n,i}-Z^e_{n,i})\right)([\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]) &= \int_{(\tau_i)_*(C^\infty_{n,e})}[\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]\\ &= \int_{C^\infty_{n,e}}(\tau_i)^*[\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}],\end{aligned}$$

which, in view of the definition of $C^\infty_{n,e}$ and on recalling how τ_i permutes the coordinates of X^{2n-1}, is

$$= (-1)^{m+n+i}\int_X\omega_i\wedge\omega_{i+1}\int_{\gamma_e^\infty}\eta_i\prod_{j=1}^{i-1}\int_X\omega_j\wedge\eta_j\prod_{j=i+2}^{n}\int_X\omega_j\wedge\eta_{j-1}$$

$$= (-1)^{m+i}\int_X\omega_i\wedge\omega_{i+1}\int_{\gamma_e^\infty}\eta_i\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j.$$

Thus

$$\Phi\left(h_n(Z^\infty_{n,i}-Z^e_{n,i})\right)([\omega_1|\cdots|\omega_n|\eta_1|\cdots|\eta_{n-1}]) = (-1)^{m+i}(II). \tag{48}$$

Combining (46), (47), and (48) we obtain (45).

REMARK. Let

$$\Delta'_{n,e} := \sum_{i=1}^{n-1}(-1)^{i-1}(\sigma_i)_*\left(\Delta_{2,e}\times\overbrace{\Delta^{(2)}(X)\times\cdots\times\Delta^{(2)}(X)}^{n-2\text{ factors}}\right).$$

(Compare with $\Delta_{n,e} = \sum_{i=1}^{n-1}(-1)^{i-1}(\sigma_i)_* \left(\Delta_{2,e}\times \left((P_e)_*\Delta^{(2)}(X) \right)^{n-2}\right)$.) It is easy to see that $h_n(\Delta'_{n,e}) = h_n(\Delta_{n,e})$: if ω and η are harmonic forms on X, the integrals of $\omega \otimes \eta$ over the diagonal $\Delta^{(2)}(X)$ and the modified diagonal $(P_e)_*\Delta^{(2)}(X)$ are the same. Thus in the formula (38), one can replace $\Delta_{n,e}$ by $\Delta'_{n,e}$.

11. Two corollaries of Theorem 8.4.1

In this section we give two corollaries of Theorem 8.4.1. First we establish a lemma.

LEMMA 11.0.1. The map

$$CH_0^{hom}(X) \to J((H^1)^{\otimes 2n-1})^\vee \quad \infty - e \mapsto h_n(Z^\infty_{n,e}) \tag{49}$$

is injective.

PROOF. It is clear from the definition of $Z^\infty_{n,e}$ that (49) is a (well-defined) group map. Now suppose

$$\sum_j h_n(Z^{\infty_j}_{n,e}) = 0.$$

We will show that $\sum_j(\infty_j - e)$ is zero in $CH_0^{hom}(X)$. Let η be a harmonic 1-form on X with integral periods. In view of the isomorphisms

$$CH_0^{hom}(X) \overset{AJ=h_1}{\cong} J(H^1)^\vee \cong Hom(H^1_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}),$$

it suffices to show that

$$\sum_j \int_e^{\infty_j} \eta \in \mathbb{Z}.$$

We may assume that $\int \eta : H_1(X, \mathbb{Z}) \to \mathbb{Z}$ is surjective. Let ω be a harmonic 1-form with integral periods such that $\int_X \omega \wedge \eta = 1$. We shall use the notation as in Paragraph 7.5 and write

$$Z^{\infty_j}_{n,e} = \sum_i (-1)^{i-1}(Z^{\infty_j}_{n,i} - Z^e_{n,i}).$$

On recalling the definition of the cycles involved in the equation above, one easily sees that in $\mathbb{R}/\mathbb{Z}$,

$$\begin{aligned} \Phi(h_n(Z^{\infty_j}_{n,e}))(\omega \otimes \eta^{\otimes n} \otimes \omega^{\otimes n-2}) &= \Phi(h_n(Z^{\infty_j}_{n,1} - Z^e_{n,1}))(\omega \otimes \eta^{\otimes n} \otimes \omega^{\otimes n-2}) \\ &= (-1)^{\frac{(n-3)(n-2)}{2}} \int_e^{\infty_j} \eta. \end{aligned}$$

The result follows from that $\sum_j \Phi(h_n(Z^{\infty_j}_{n,e})) = 0$. □

We now give two consequences of Theorem 8.4.1. The first is in the spirit of Corollary 5.4 of Pulte **[28]**.

COROLLARY 11.0.2. The function

$$X(\mathbb{C}) - \{e\} \to Ext((H^1)^{\otimes n}, (H^1)^{\otimes n-1}) \quad \infty \mapsto \mathbb{E}^\infty_{n,e}$$

is injective.

PROOF. Let $\infty_1, \infty_2 \in X(\mathbb{C}) - \{e\}$. By Theorem 8.4.1,

$$(-1)^{\frac{n(n-1)}{2}} \Psi(\mathbb{E}_{n,e}^{\infty_1} - \mathbb{E}_{n,e}^{\infty_2}) = h_n(Z_{n,e}^{\infty_2} - Z_{n,e}^{\infty_1}) = h_n(Z_{n,\infty_1}^{\infty_2}).$$

The result follows from the previous lemma. □

COROLLARY 11.0.3. Suppose X has genus 1, or that X is hyperelliptic and e is a ramification point of X. Then $\mathbb{E}_{n,e}^{\infty}$ is torsion if and only if $\infty - e$ is torsion in $CH_0^{hom}(X)$.

PROOF. By results of Gross and Schoen [**19**, Propositions 4.5 and 4.8], $\Delta_{2,e}$ is torsion in each of these cases. It follows that $\Delta_{n,e}$ is torsion for all n. (See the remark at the end of Paragraph 7.4.) Thus by Theorem 8.4.1, $\mathbb{E}_{n,e}^{\infty}$ is torsion if and only if $h_n(Z_{n,e}^{\infty})$ is torsion. The desired conclusion follows from Lemma 11.0.1. □

12. $\mathbb{E}_{n,e}^{\infty}$ and rational points on the Jacobian

In this final section we assume that X, e, ∞ are defined over a subfield $K \subset \mathbb{C}$. We show that one can associate to the extension $\mathbb{E}_{n,e}^{\infty}$ a family of K-rational points on the Jacobian of X. This generalizes Theorem 1 and Corollary 2 of [**7**]. Our approach follows the ideas leading to those results, and generally speaking, is in line with Darmon's philosophy of trying to construct rational points on Jacobian varieties using higher dimensional varieties.

12.1. Recollection: Maps between intermediate Jacobians induces by correspondences. Let Y (resp. Y') be a smooth projective variety over $\mathbb{C}$ of dimension d (resp. d') over $\mathbb{C}$. Suppose $l \le d + d'$. One has natural isomorphisms

$$\begin{aligned} H^{2l}(Y \times Y')^{\vee} &\cong \left(\bigoplus_r H^r(Y) \otimes H^{2l-r}(Y') \right)^{\vee} \\ &\cong \bigoplus_r \underline{\mathrm{Hom}} \left(H^r(Y), H^{2l-r}(Y')^{\vee} \right) \\ &\overset{\text{Poincare duality}}{\cong} \bigoplus_r \underline{\mathrm{Hom}} \left(H^{2d-r}(Y)^{\vee}, H^{2l-r}(Y')^{\vee} \right)(d). \end{aligned}$$

Let $Z \in CH_l(Y \times Y')$. Then the class $cl(Z)$ of Z is a Hodge class in

$$H^{2l}(Y \times Y')^{\vee},$$

which is given by integration over Z (or more precisely, the smooth locus of Z) if Z is an irreducible closed subset. In view of the isomorphisms above, $cl(Z)$ decomposes as a sum of Hodge classes in

$$\underline{\mathrm{Hom}} \left(H^{2d-r}(Y)^{\vee}, H^{2l-r}(Y')^{\vee} \right).$$

It follows that for each r, $cl(Z)$ gives a morphism of Hodge structures

$$H^{2d-r}(Y)^{\vee}(l-d) \to H^{2l-r}(Y')^{\vee}. \tag{50}$$

If r is odd, this induces a map

$$JH^{2d-r}(Y)^{\vee} = JH^{2d-r}(Y)^{\vee}(l-d) \to JH^{2l-r}(Y')^{\vee}. \tag{51}$$

With abuse of notation we denote the maps (50) and (51) also by $cl(Z)$.

Let $m \le d$. The push-forward map

$$Z_* : CH_m(Y) \to CH_{m+l-d}(Y')$$

restricts to a map

$$Z_*: CH_m^{hom}(Y) \to CH_{m+l-d}^{hom}(Y').$$

One has a commutative diagram

$$\begin{array}{ccc} CH_m^{hom}(Y) & \xrightarrow{AJ} & JH^{2m+1}(Y)^\vee \\ \downarrow{\scriptstyle Z_*} & & \downarrow{\scriptstyle cl(Z)} \\ CH_{m+l-d}^{hom}(Y') & \xrightarrow{AJ} & JH^{2m+2l-2d+1}(Y')^\vee \end{array}$$

(functoriality of Abel-Jacobi maps with respect to correspondences).

12.2. Fix a subfield $K \subset \mathbb{C}$. We assume that the curve X and the points e, ∞ are defined over K. More precisely, suppose $X = X_0 \times_K \operatorname{Spec}(\mathbb{C})$, where X_0 is a projective curve over K, and that $e, \infty \in X_0(K)$. Let $\mathrm{Jac} = \mathrm{Jac}(X_0)$ be the Jacobian of X_0. Throughout, we identify

$$\mathrm{Jac}(\mathbb{C}) = CH_0^{hom}(X) \overset{AJ}{\cong} J(H^1)^\vee.$$

Thus in particular, $\mathrm{Jac}(K)$ is identified as a subgroup of $J(H^1)^\vee$. For a Hodge class ξ in $(H^1)^{\otimes 2n-2}$, let

$$\xi^{-1}: J((H^1)^{\otimes 2n-1})^\vee \to J(H^1)^\vee$$

be the map $[f] \mapsto [f(\xi \otimes -)]$. For an algebraic cycle $Z \in CH_{n-1}(X_0^{2n-2})$, we denote by ξ_Z the $(H^1)^{\otimes 2n-2}$ Kunneth component of

$$cl(Z) \in H_{\mathbb{C}}^{2n-2}(X^{2n-2})^\vee \overset{\text{Poincare duality}}{\cong} H_{\mathbb{C}}^{2n-2}(X^{2n-2}).$$

THEOREM 12.2.1. *Let $Z \in CH_{n-1}(X_0^{2n-2})$. Then*

$$\xi_Z^{-1}(\Psi(\mathbb{E}_{n,e}^\infty)) \in \mathrm{Jac}(K).$$

We should point out that this is not a priori obvious, as to get the extension $\mathbb{E}_{n,e}^\infty$ one first extends the scalars to $\mathbb{C}$. Note that varying Z, we get a family of points in $\mathrm{Jac}(K)$ associated to $\mathbb{E}_{n,e}^\infty$ parametrized by $CH_{n-1}(X_0^{2n-2})$. When $n = 2$, the result is due to Darmon, Rotger, and Sols [**7**, Corollary 2].

With abuse of notation, we denote the compositions

$$CH_{n-1}^{hom}(X_0^{2n-1}) \overset{\text{natural map}}{\to} CH_{n-1}^{hom}(X^{2n-1}) \overset{AJ}{\to} JH^{2n-1}(X^{2n-1})^\vee$$

and

$$CH_{n-1}^{hom}(X_0^{2n-1}) \overset{\text{natural map}}{\to} CH_{n-1}^{hom}(X^{2n-1}) \overset{h_n}{\to} J((H^1)^{\otimes 2n-1})^\vee$$

by AJ and h_n respectively. In view of Theorem 8.4.1 and the fact that both $\Delta_{n,e}$ and $Z_{n,e}^\infty$ are defined over K (see Paragraph 7.6), Theorem 12.2.1 follows immediately from the following lemma.

LEMMA 12.2.2. *Let $Z \in CH_{n-1}(X_0^{2n-2})$. Then the image of the composition*

$$CH_{n-1}^{hom}(X_0^{2n-1}) \overset{h_n}{\to} J((H^1)^{\otimes 2n-1})^\vee \overset{\xi_Z^{-1}}{\to} J(H^1)^\vee$$

lies in the subgroup $\mathrm{Jac}(K)$.

PROOF. Denote the diagonal of X_0 by $\Delta(X_0)$. Let $Z' \in CH_n(X_0^{2n})$ be such that its class in

$$H^{2n}(X^{2n})^\vee$$

is the $((H^1)^{\otimes 2n})^\vee$ Kunneth component of

$$cl(Z \times \Delta(X_0)) \in H^{2n}(X^{2n})^\vee.$$

Such Z' can be explicitly constructed using the fact that the Kunneth components of the class of the diagonal $\Delta(X_0)$ are algebraic. We will show that the diagram

$$\begin{array}{ccc} CH^{hom}_{n-1}(X_0^{2n-1}) & \xrightarrow{h_n} & J((H^1)^{\otimes 2n-1})^\vee \\ \downarrow Z'_* & & \downarrow \xi_Z^{-1} \\ CH^{hom}_0(X_0) & \xrightarrow{h_1 = AJ} & J(H^1)^\vee \end{array} \tag{52}$$

commutes. This will prove the assertion, as the subgroup $Jac(K)$ of $J(H^1)^\vee$ is precisely the image of $CH_0^{hom}(X_0)$.

By functoriality of the Abel-Jacobi maps with respect to correspondences, one has a commutative diagram

$$\begin{array}{ccc} CH^{hom}_{n-1}(X_0^{2n-1}) & \xrightarrow{AJ} & JH^{2n-1}(X^{2n-1})^\vee \\ \downarrow Z'_* & & \downarrow cl(Z') \\ CH^{hom}_0(X_0) & \xrightarrow{AJ} & J(H^1)^\vee. \end{array} \tag{53}$$

Thus to establish commutativity of (52), it suffices to verify the commutativity of

$$\begin{array}{ccc} H^{2n-1}_{\mathbb{C}}(X^{2n-1})^\vee & \xrightarrow[\text{projection}]{\text{natural}} & ((H^1_{\mathbb{C}})^{\otimes 2n-1})^\vee \, , \\ \downarrow cl(Z') & \swarrow \xi_Z^{-1} & \\ (H^1_{\mathbb{C}})^\vee & & \end{array} \tag{54}$$

where with abuse of notation ξ_Z^{-1} denotes the map $f \mapsto f(\xi_Z \otimes -)$. Note that since

$$cl(Z') \in ((H^1_{\mathbb{C}})^{\otimes 2n})^\vee \subset H^{2n}_{\mathbb{C}}(X^{2n})^\vee,$$

we only need to verify commutativity on the direct summand

$$((H^1_{\mathbb{C}})^{\otimes 2n-1})^\vee \subset H^{2n-1}_{\mathbb{C}}(X^{2n-1})^\vee.$$

Let $f \in ((H^1_{\mathbb{C}})^{\otimes 2n-1})^\vee$. Suppose f is the Poincare dual of $\alpha \in H^{2n-1}_{\mathbb{C}}(X^{2n-1})$, i.e.

$$f(-) = \int_{X^{2n-1}} \alpha \wedge -.$$

Then α lies in the Kunneth component $(H^1_{\mathbb{C}})^{\otimes 2n-1}$. Let $\beta \in H^1_{\mathbb{C}}$. Unwinding definitions, in view of the fact that $cl(Z')$ is the $((H^1)^{\otimes 2n})^\vee$ component of $cl(Z \times \Delta(X_0))$, we have

$$cl(Z')(f)(\beta) = cl(Z')(\alpha \otimes \beta) = cl(Z \times \Delta(X_0))(\alpha \otimes \beta).$$

Let
$$\alpha = \sum_i \alpha_1^{(i)} \otimes \cdots \otimes \alpha_{2n-1}^{(i)}.$$
Then
$$\begin{aligned}
cl(Z')(f)(\beta) &= \sum_i cl(Z \times \Delta(X_0))(\alpha_1^{(i)} \otimes \cdots \otimes \alpha_{2n-1}^{(i)} \otimes \beta) \\
&= \sum_i cl(Z)(\alpha_1^{(i)} \otimes \cdots \otimes \alpha_{2n-2}^{(i)}) \int_X \alpha_{2n-1}^{(i)} \wedge \beta \\
&= \sum_i \int_{X^{2n-2}} \xi_Z \wedge (\alpha_1^{(i)} \otimes \cdots \otimes \alpha_{2n-2}^{(i)}) \int_X \alpha_{2n-1}^{(i)} \wedge \beta \\
&= \sum_i \int_{X^{2n-1}} \left(\xi_Z \wedge (\alpha_1^{(i)} \otimes \cdots \otimes \alpha_{2n-2}^{(i)})\right) \otimes (\alpha_{2n-1}^{(i)} \wedge \beta) \\
&= \int_{X^{2n-1}} \alpha \wedge (\xi_Z \otimes \beta) \\
&= f(\xi_Z \otimes \beta).
\end{aligned}$$
Thus $cl(Z')(f) = \xi_Z^{-1}(f)$ as desired. □

REMARK. It was pointed out to me by Darmon that the idea of constructing points on the Jacobian of X_0 using Hodge classes in $H^2(X^2)$ first arose in the work **[32]** of W. Yuan, S. Zhang, and W. Zhang in the setting of modular curves.

References

[1] J. A. Carlson, Extensions of mixed Hodge structures, *Journées de Géometrie Algébrique d'Angers* 1979

[2] P. Cartier, Jacobiennes généralisées, monodromie unipotente et intégrales itérées, *Séminare Bourbaki*, 1987-1988, exp. n° 687, pp. 31-52

[3] K.T. Chen, Iterated integrals, fundamental groups and covering spaces, *Transactions of the American Mathematical Society*, Volume 206, 1975

[4] K.T. Chen, Reduced Bar Constructions on De Rham Complexes, *Algebra, Topology and Category Theory, Academic Press, New York (1976)* pp 19-32

[5] K.T. Chen, Iterated Integrals of Differential Forms and Loop Space Homology, *Annals of Mathematics*, Vol.97, No.2 (Mar., 1973), pp. 217-246

[6] E. Colombo, The mixed Hodge structure on the fundamental group of a hyperelliptic curve and higher cycles, *Journal of Algebraic Geometry* 11 (2002), pp. 761-790.

[7] H. Darmon, V. Rotger, and I. Sols, Iterated integrals, diagonal cycles, and rational points on elliptic curves, *Publications Mathmatiques de Besanon*, vol.2, 2012, pp. 19-46.

[8] P. Deligne, Theorie de Hodge II, *Publications Mathématiques de l'I.H.É.S.*, tome 40 (1971), p. 5-57

[9] P. Deligne, Theorie de Hodge III, *Publications Mathématiques de l'I.H.É.S.*, tome 44 (1974), p. 5-77

[10] P. Deligne (Notes by J.S. Milne), Hodge cycles on abelian varieties, In *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer-Verlog, Berlin (1982)

[11] P. Deligne and J.S. Milne, Tannakian Categories, In *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer-Verlog, Berlin (1982)

[12] P. Deligne, Le groupe fondamental de la droite projective moins trois points, *Galois Groups over* $\mathbb{Q}$, MSRI Publ. 16, pp. 79-313, Springer-Verlag (1989)

[13] P. Deligne and A. B. Goncharov, Groupes fondamentaux motiviques de Tate mixte, *Ann. Sci. de lcole Norm. Sup.* 38 (2005) 1-56

[14] P. Eskandari, *Algebraic Cycles, Fundamental Group of a Punctured Curve, and Applications in Arithmetic*, PhD thesis (2016), University of Toronto

[15] D. K. Faddeev, On the divisor class groups of some algebraic curves, *Soviet Math. Dokl.*, 2 (1961), 67-69

[16] P. Griffiths, Some results on algebraic cycles on algebraic manifolds, *Proceedings of the International Conference on Algebraic Geometry, Tata Institute*, 1968 (pp 93-191).

[17] D. M. Gordon and D. Grant, Computing the Mordell-Weil group of Jacobians of curves of genus two, *Transactions of the Americam Mathematical Society,* Volume 337, Number 2, June 1993

[18] B. H. Gross and S. Kudla, Heights and the central critical values of triple product L-functions, *Compositio Mathematica*, tome 81, n° 2 (1992), p. 143-209.

[19] B. H. Gross and C. Schoen, The modified diagoal cycle on the triple product of a pointed curve, *Annales de l'institut Fourier,* tome 45, n° 3 (1995), p. 649-679

[20] R. Gunning, Quadratic periods of hyperelliptic abelian integrals, *Problems in Analysis: A Symposium in Honor of Solomon Bochner,* editted by R. C. Gunning, Princeton University Press, 1969, p. 239-247

[21] R. Hain, The de Rham Homotopy Theory of Complex Algebraic Varieties I, *K-Theory 1* (1987), 271-324

[22] R. Hain, Iterated integrals and algebraic cycles: examples and prospects, *Cont. Trends in Alg. Geometry and Alg. Topology* S. Chern, L. Fu, R. Hain (ed.) Nankai Tracts in Math. 5, Singapore, 2002.

[23] R. Hain and S. Zucker, Unipotent variations of mixed Hodge structure, *Invent. math.* 88, 83-124 (1987)

[24] B. Harris, Harmonic volumes, *Acta Math.* 150, 1-2 (1983), 91-123

[25] S. Lang, *Introduction to Arakelov Theory,* New York: Springer-Verlag (1988)

[26] J. Morgan, The algebraic topology of smooth algebraic varieties, *Publications Mathématiques de l'I.H.É.S.,* tome 48 (1978), p. 137-204

[27] J. P. Murre, J. Nagel, and C. Peters, *Lectures on the Theory of Pure Motives,* University Lecture Series, Volume 61, American Mathematical Society, 2013

[28] M. Pulte, The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles, *Duke Math. J.* 57 (1988), 721-760

[29] John R. Stallings, Quotients of the Powers of the Augmentation Ideal in a Group Ring, in *Knots, Groups, and 3-Manifolds*, edited by L.P. Neuwirth, University of Tokyo Press, Princeton, New Jersey, 1975

[30] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry I,* Cambridge studies in advanced mathematics 76, 2002

[31] Z. Wojtkowiak, Cosimplicial objects in algebraic geometry, In *Algebraic K-theory and algebraic topology* 1993 Kluwer Academic Publishers, 287-327

[32] X. Yuan, S. Zhang, and W. Zhang, Triple product L-series and Gross-Kudla-Schoen cycles, preprint.

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