TANNAKIAN FUNDAMENTAL GROUPS OF BLENDED EXTENSIONS

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Abstract. Let A_1, A_2, A_3 be semisimple objects in a neutral tannakian category over a field of characteristic zero. Let L be an extension of A_2 by A_1, and N an extension of A_3 by A_2. Let M be a blended extension (*extension panachée*) of N by L. Under none to little very mild hypotheses, we study the unipotent radical of the tannakian fundamental group of M. Examples where our results apply include the unipotent radicals of motivic Galois groups of any mixed motive with three weights.

As an application, we give a proof of *the unipotent part* of the Hodge-Nori conjecture for 1-motives (which is now a theorem of André in the setting of Nori motives) in the setting of any tannakian category of motives where the group $Ext^1(\mathbb{1}, \mathbb{Q}(1))$ is as expected.

Contents

1. Introduction

The notion of a blended extension, invented by Grothendieck in [21], provides a natural framework to study 3-step filtrations. By definition, given a fixed extension L of A_2 by A_1 and a fixed extension N of A_3 by A_2 in an abelian category, a blended extension of N by L is a diagram of the form

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & L & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & A_3 & = & A_3 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{1}$$

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in our category with exact rows and columns. Here, with abuse of notation an extension and its middle object are denoted by the same letter, and the top row and right column are our two given fixed extensions L and N.

In this paper we shall consider blended extensions M as above in the setting of a neutral tannakian category over a field of characteristic zero. Moreover, we shall assume that A_1, A_2 and A_3 are semisimple objects. We are interested in the unipotent radical of the (tannakian) fundamental group of M. More explicitly, let $\mathfrak{u}(M)$ be the Lie algebra of the unipotent radical of the fundamental group of M; it is a canonical subobject of $\underline{End}(M) := \underline{Hom}(M, M)$ (where $\underline{Hom}$ is the internal Hom), whose image under any fiber functor ω is the Lie algebra of the unipotent radical of the fundamental group of M with respect to ω. Our subject of study is $\mathfrak{u}(M)$, and we would like to describe it, ideally as explicitly and computably as possible, in terms of the extensions that appear in (1).

The analogue of this problem for extensions, i.e. the determination of the unipotent radical of the fundamental group of an extension L of a semisimple object A_2 by a semisimple object A_1, has been studied by Bertrand [9] in the special setting of differential equations and by Hardouin ([23], [24]) and the author and Murty [18] in general.

The problem of determination of $\mathfrak{u}(M)$ for blended extensions M with semisimple A_1, A_2 and A_3 arises naturally in at least two important settings. The first is in connection to motives with 3 weights (i.e. where the associated graded with respect to the weight filtration has 3 graded components) and the realizations (e.g. Hodge or ℓ-adic realization) of such motives[1]. The weight filtration on a motive with 3 weights gives rise to a blended extension as above in the motivic or realization category. Depending on whether one considers the blended extension in a (i) tannakian category of motives, (ii) the category of mixed Hodge structures, or (iii) the category of ℓ-adic Galois representations, $\mathfrak{u}(M)$ is the Lie algebra of the unipotent radical of (i) the motivic Galois group of M, (ii) the Mumford-Tate group of M, or (iii)[2] the Zariski closure of the image of the absolute Galois group of the base field in $GL(M_\ell)$, where M_ℓ is the ℓ-adic realization of M.

The second natural setting where this problem arises is in connection to the differential Galois groups of products of three completely reducible differential operators. This second setting of the problem has been studied by Bertrand [9] and Hardouin [23] (see also the references therein). In particular, according to [23] one has a complete description of $\mathfrak{u}(M)$ in this setting.

Focusing on the setting of motives, arguably the most accessible nontrivial class of motives with three weights are those coming from Deligne's theory of 1-motives over a field [13]. In this case, the problem of determination of $\mathfrak{u}(M)$ is well-understood. Bertolin and her collaborators (in a series of papers, starting with [5] and [6], and most recently in [8]; see also the references therein) and Jossen [27] have independently studied $\mathfrak{u}(M)$ in this case, and give explicit descriptions of $\mathfrak{u}(M)$ both for the motivic Galois and Mumford-Tate groups. Jossen also does this for the ℓ-adic realization, since he proves the Mumford-Tate conjecture for 1-motive at the unipotent level.

This paper came out of the author's attempt to better understand aspects of the works of Bertolin and Jossen, and curiosity to find out how much of their results could be obtained more abstractly, without using the explicit geometric situation particular to 1-motives. In fact, analogues to some of their results can be seen in works of Bertrand and Hardouin on differential equations, hinting that some of the results should indeed hold more generally. This

[1]In the case of the ℓ-adic realization, this assumes the semisimplicity conjecture of Serre and Grothendieck.
[2]again, assuming the Grothendieck-Serre semisimplicity conjecture

connection was surely already observed by (at least) Bertrand, as it is clear from his paper [10] on self-dual blended extensions in an arbitrary tannakian category.

For convenience and to help the reader navigate through the paper with more ease, we have summarized below most of the results of the paper on the general description of $\mathfrak{u}(M)$. The reader familiar with the works cited earlier in the contexts of differential equations and 1-motives will be able to see the connections between the statements below and some of the results in those contexts. Note that the notations Hom and Ext^1 (with no subscript) refer to the Hom and Ext groups for our category $\mathbf{T}$, and $\underline{Hom}$ is the internal Hom.

Theorem A. *Let M be the blended extension* (1) *in a neutral tannakian category $\mathbf{T}$ over a field of characteristic zero. Suppose A_1, A_2, A_3 are semisimple, and that*

$$Hom(\underline{Hom}(A_3, A_2) \oplus \underline{Hom}(A_2, A_1), \underline{Hom}(A_3, A_1)) = 0. \tag{2}$$

Then we have the following:

(a) The object $\mathfrak{u}(M)$ fits in a canonical exact sequence

$$0 \longrightarrow \mathfrak{u}_{-2}(M) \longrightarrow \mathfrak{u}(M) \longrightarrow \mathfrak{u}_{-1}(M) \longrightarrow 0 \tag{3}$$

where $\mathfrak{u}_{-2}(M) := \mathfrak{u}(M) \cap \underline{Hom}(A_3, A_1)$ and $\mathfrak{u}_{-1}$ is a canonical subobject of

$$V := \underline{Hom}(A_3, A_2) \oplus \underline{Hom}(A_2, A_1).$$

Moreover, the subobjects $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_3, A_1)$ and V completely determine the subobject $\mathfrak{u}(M)$ of $\underline{End}(M)$. (See §2.6 and §2.7 for the construction of the sequence. See Proposition 2.8.1, Proposition 2.13.1 including its proof, and §2.16 for the determination of $\mathfrak{u}(M)$ from $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-1}(M)$.)

(b) (Determination of $\mathfrak{u}_{-1}(M)$) Let $\mathscr{L}$ and $\mathscr{N}$ be the elements of $Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1))$ and $Ext^1(\mathbb{1}, \underline{Hom}(A_3, A_2))$ corresponding to L and N. Consider the element

$$(\mathscr{L}, \mathscr{N}) \in Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1)) \oplus Ext^1(\mathbb{1}, \underline{Hom}(A_3, A_2)) \cong Ext^1(\mathbb{1}, V).$$

Then

$$\mathfrak{u}_{-1}(M) = \bigcap_{\substack{\phi \in End(V) \\ \phi_*(\mathscr{L}, \mathscr{N}) = 0}} \ker(\phi).$$

In particular, the subobject $\mathfrak{u}_{-1}(M)$ of V is completely determined by the pair of extensions L and N. (See §2.11 and Theorem 3.7.1.)

(c) (1st characterization of $\mathfrak{u}_{-2}(M)$.) Let $\mathscr{M}^h$ be the second row of (1)*, considered as an element of $Ext^1(\mathbb{1}, \underline{Hom}(N, A_1))$. Then $\mathfrak{u}_{-2}(M)$ is the smallest subobject of $\underline{Hom}(A_3, A_1)$ such that the pushforward*

$$\mathscr{M}^h / \mathfrak{u}_{-2}(M) \in Ext^1(\mathbb{1}, \underline{Hom}(N, A_1)/\mathfrak{u}_{-2}(M))$$

lies in the subgroup $Ext^1_{\langle L \oplus N \rangle^\otimes}(\mathbb{1}, \underline{Hom}(N, A_1)/\mathfrak{u}_{-2}(M))$ consisting of extensions that belong to the tannakian subcategory $\langle L \oplus N \rangle^\otimes$ generated by $L \oplus N$. In particular, $\mathfrak{u}_{-2}(M)$ vanishes if and only if the object M belongs to $\langle L \oplus N \rangle^\otimes$. (See Theorem 4.3.1 and Corollary 4.3.2.)

(d) (2nd characterization of $\mathfrak{u}_{-2}(M)$, part I - determination of $[\mathfrak{u}(M), \mathfrak{u}(M)]$.) The derived algebra $[\mathfrak{u}(M), \mathfrak{u}(M)]$ of $\mathfrak{u}(M)$ is completely determined by $\mathfrak{u}_{-1}(M)$. More precisely, let $\{,\}$ be the antisymmetric pairing

$$V \otimes V \to \underline{Hom}(A_3, A_1)$$

induced by composition of functions (i.e. given by $\{(f_{12},f_{23}),(g_{12},g_{23})\} = f_{12}g_{23} - g_{12}f_{23}$ *after applying a fiber functor* ω*, where* f_{ij} *and* g_{ij} *are linear maps* $\omega A_j \to \omega A_i$*). Then* $[\mathfrak{u}(M),\mathfrak{u}(M)] = \{\mathfrak{u}_{-1}(M),\mathfrak{u}_{-1}(M)\}$*.* (See Proposition 4.4.1.)

(e) (Alternative description of $[\mathfrak{u}(M),\mathfrak{u}(M)]$*.) The derived algebra* $[\mathfrak{u}(M),\mathfrak{u}(M)]$ *is the smallest subobject of* $\mathfrak{u}_{-2}(M)$ *such that the pushforward of* (3) *along the quotient map* $\mathfrak{u}_{-2}(M) \to \mathfrak{u}_{-2}(M)/[\mathfrak{u}(M),\mathfrak{u}(M)]$ *splits.* (See Proposition 4.5.1.)

(f) (2nd characterization of $\mathfrak{u}_{-2}(M)$*, part II - characterization of* $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M),\mathfrak{u}(M)]$*.) There is a canonical isomorphism (independent from any choice of a fiber functor)*

$$Hom(\frac{\mathfrak{u}_{-2}(M)}{[\mathfrak{u}(M),\mathfrak{u}(M)]}, \underline{Hom}(A_3,A_1)) \;\cong\; Ext^1_{\langle M\rangle^{\otimes}}(A_3,A_1).$$

In particular, we have $\mathfrak{u}_{-2}(M) = [\mathfrak{u}(M),\mathfrak{u}(M)]$ *if and only if there are no nontrivial extensions of* A_3 *by* A_1 *in the subcategory* $\langle M\rangle^{\otimes}$*.* (See Proposition 4.6.1 and Corollary 4.6.2. See also §3.2 for the independence of the isomorphism from the choice of a fiber functor.)

(g) With the extensions L *and* N *fixed, up to isomorphisms of blended extensions, there exists at most one blended extension* M *of* N *by* L *such that* $\mathfrak{u}_{-2}(M) = 0$*.* (See Proposition 4.7.1)

Not every assertion in the theorem requires both hypotheses of semisimplicity of the A_j and (2). See the parts of the paper relevant to each assertion for exactly what assumptions are needed (if any). In particular, part (b) only needs the hypothesis of semisimplicity of the A_j, and part (c) does not need either assumption. Condition (2) is very mild anyway[3]. For instance, it is satisfied automatically if our category $\mathbf{T}$ is filtered by weight (as are the categories of motives and mixed Hodge structures) and every weight of A_i is less than every weight of A_j whenever $i < j$. Condition (2) should also be satisfied if $\mathbf{T}$ is the category of ℓ-adic representations of the absolute Galois group of a field finitely generated over its prime field and the A_j are the ℓ-adic realizations of motives satisfying the same increasing of weights condition. Note that even if one is only interested in motivic applications, it is advantageous to work with conditions such as (2) than to assume $\mathbf{T}$ is filtered by weights, as the former is more suitable for potential applications involving ℓ-adic realizations.

I should admit that excluding part (a), which sets up the stage for the other parts, I myself find parts (b) and (hence) (d) the most satisfactory parts of the theorem. Part (e) is basic and is included mainly for comprehensiveness purposes. Parts (c) and (f), while already useful (as illustrated for the latter by the second application below), are less explicit. In the context of 1-motives, Bertolin and Philippon give more geometric interpretations of $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M),\mathfrak{u}(M)]$ (see Theorem 4.7 of [8], as well as the point R_0 in *loc. cit.*). As for part (g), one would hope that if L and N are blendable (*panachable*) and $\mathfrak{u}$ is abelian (a property than depends only on L and N), then there should always exist an M with vanishing $\mathfrak{u}_{-2}(M)$, as shown to be the case by Hardouin [23] in the setting of differential operators. In light of Bertrand's work [10] on self-dual blended extensions, one might also expect that for any blendable L and N, perhaps under some mild conditions, there should always exist an M which satisfies $\mathfrak{u}_{-2}(M) = [\mathfrak{u}(M),\mathfrak{u}(M)]$. We have left the investigation of these as well as making the characterization of $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M),\mathfrak{u}(M)]$ more explicit for the future.

We also include two applications of Theorem A in the paper:

[3]Importantly, note that unlike the results of [19] and [20] on motives with maximal unipotent radicals, here one may have

$$Hom(\underline{Hom}(A_3,A_2),\underline{Hom}(A_2,A_1)) \neq 0.$$

Application 1: *Maximality criteria for* $\mathfrak{u}(M)$ *and* $\mathfrak{u}_{-1}(M)$. The Lie algebra $\mathfrak{u}(M)$ has a trivial upper bound in $\underline{End}(M)$, namely, it is always contained in the subobject $W_{-1}\underline{End}(M)$, which by definition consists, after applying a fiber functor ω, of all the linear maps $\omega M \to \omega M$ which map $\omega A_1 \to 0$, $\omega L \to \omega A_1$, and $\omega M \to \omega L$. As a corollary of Theorem A, one obtains equivalent conditions to having $\mathfrak{u}(M) = W_{-1}\underline{End}(M)$. These conditions, recorded as Corollary 3.8.1, when applied to the settings of (i) products of three completely reducible differential operators, (ii) "graded-independent" (in the sense of [20, Definition 4.3.1]) motives with 3 weights, and (iii) 1-motives give the maximality criteria of (i) Bertrand (see [9], Theorem 2.1 when $t = 3$), (ii) the author (see Theorem 4.3.2 of [20] for $k = 3$), and Bertolin and Philippon (see Corollaries 4.5 and 4.6 of [8]).

We also obtain a more refined maximality criterion, which is one for $\mathfrak{u}_{-1}(M)$, and to our knowledge has not been considered previously in any context (not even for differential operators). A trivial upper bound for $\mathfrak{u}_{-1}(M)$ observed easily from the constructions (see §2.11) is $\mathfrak{u}(L) \oplus \mathfrak{u}(N)$, where $\mathfrak{u}(L) \subset \underline{Hom}(A_2, A_1)$ and $\mathfrak{u}(N) \subset \underline{Hom}(A_3, A_2)$ are respectively the Lie algebras of the unipotent radicals of the fundamental groups of L and N. Without much extra work, we give a refined version of the characterization of $\mathfrak{u}_{-1}(M)$ (see Theorem 3.9.1), leading to a criterion for when we have $\mathfrak{u}_{-1}(M) = \mathfrak{u}(L) \oplus \mathfrak{u}(N)$. This criterion, recorded as Corollary 3.9.2, becomes the more interesting of the criteria of Corollary 3.8.1 when the extensions L and N are totally nonsplit[4].

Application 2: *The unipotent part of the Hodge-Nori conjecture for 1-motives.* Let $\mathbb{F}$ be an algebraically closed subfield of $\mathbb{C}$. In [2] André has proved the Hodge-Nori conjecture for 1-motives over $\mathbb{F}$. More precisely, he has proved that for a Deligne 1-motive over $\mathbb{F}$, the Mumford-Tate group coincides with the motivic Galois group (of the motive attached to the 1-motive) understood in the context of Nori's tannakian category of mixed motives over $\mathbb{F}$. His proof uses a deformation argument to reduce the problem to the case of semisimple 1-motives (much like Brylinski's argument in [12] did in the setting of absolute Hodge cycles). This semisimple case follows by combining the earlier work [1] of André in the setting of motives via motivated correspondences with Arapura's result [3] on the equivalence of André's category via motivated correspondences and the tannakian subcategory of the category of Nori motives generated by semisimple objects.

André asks in [2] whether one can give a proof of the reduction to the semisimple case that does not use deformations. As an application of Theorem A (parts (a), (b), (d), (f)), in §5 we propose an approach to such a proof. More precisely, we prove that the unipotent radicals of the Mumford-Tate and motivic Galois groups of a 1-motive over $\mathbb{F}$ coincide, provided that the motivic Galois group is understood in the context of a tannakian category of motives over $\mathbb{F}$ *where every extension of* $\mathbb{1}$ *by* $\mathbb{Q}(1)$ *comes from a 1-motive.* This last condition on $Ext^1(\mathbb{1}, \mathbb{Q}(1))$ is a special case of a conjecture of Deligne [15, §2.4], which predicts that 1-motives should be closed under extensions in a good category of motives. Thus if the group $Ext^1(\mathbb{1}, \mathbb{Q}(1))$ in Nori's tannakian category of (non-effective) motives over $\mathbb{F}$ is as expected, then one gets another proof of the reduction to the semisimple case of André's theorem. Ayoub and Barbieri-Viale have proved in [4] that Deligne's aforementioned conjecture holds in the abelian category of effective Nori motives (see §8.11 therein). For our result to be applicable to the setting of Nori motives, one needs to know this for the category of non-effective Nori motives.

[4]Recall that an extension E of X by Y is called totally nonsplit if when considered as an extension of $\mathbb{1}$ by $\underline{Hom}(X, Y)$, the pushforward of E to every nonzero quotient of $\underline{Hom}(X, Y)$ is nonsplit.

In a sequel paper we shall further consider applications of Theorem A to the unipotent parts of the Hodge-Nori and Tate conjectures.

A word on conventions and notation. Our tannakian categories are always assumed to be neutral. Unless there is a chance of misinterpretation, we suppress the category from the notation for Hom and Ext groups. If needed, the category is indicated by a subscript. As already mentioned, $\underline{Hom}$ denotes the internal Hom.

The reader can consult [10, §1] or [20, §2] for a review of the background material on blended extensions. The collection of isomorphism classes of blended extensions of N by L is denoted by $Extpan(N, L)$, with the category again dropped from the notation unless there is ambiguity, in which case the intended category will be included as a subscript. With abuse of notation, we use the same symbol for an extension, its middle object, and the corresponding element in the Ext^1 group. Similarly, we use the same notation for a blended extension, it middle object, and the corresponding element in $Extpan$. If there is a possibility of confusion, we will make the intended interpretation explicit.

Finally, all actions are designed to be left actions and given an algebraic group $\mathscr{G}$ over a field $\mathbb{K}$, the category of finite dimensional representations of $\mathscr{G}$ over $\mathbb{K}$ is denoted by $\mathbf{Rep}(\mathscr{G})$.

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2. The setup and initial considerations

2.1. From here to the end of §4, we shall fix the following data:

- A tannakian category $\mathbf{T}$ over a field $\mathbb{K}$ of characteristic zero. All of what follows takes place in $\mathbf{T}$.
- Objects A_1, A_2 and A_3 of $\mathbf{T}$. For the time being, there are no conditions on these objects. In particular, they do not have to be semisimple.
- Two extensions

$$0 \longrightarrow A_1 \longrightarrow L \longrightarrow A_2 \longrightarrow 0$$

$$0 \longrightarrow A_2 \longrightarrow N \longrightarrow A_3 \longrightarrow 0.$$

- A blended extension M of N by L given by diagram (1).

Recall that for every X and Y in $\mathbf{T}$, there is a canonical isomorphism

$$Ext^1(X, Y) \cong Ext^1(\mathbb{1}, \underline{Hom}(X, Y))$$

(see for instance, [18, §3.2] for an explicit description of this isomorphism). Let

$$\mathscr{L} \in Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1)) \qquad (\text{resp. } \mathscr{N} \in Ext^1(\mathbb{1}, \underline{Hom}(A_3, A_2)))$$

be the element corresponding to L (resp. N) under the respective canonical isomorphism.

2.2. In this subsection, we recall some well known generalities in tannakian categories. Throughout the paper, unless otherwise indicated by a fiber functor we always mean a fiber functor with values in the category of finite-dimensional vector spaces over $\mathbb{K}$. Let X be an object of $\mathbf{T}$. Given any fiber functor ω for $\mathbf{T}$, we denote the fundamental group of X with respect to ω by $\mathscr{G}(X, \omega)$; in the standard notation, this is the group scheme $\underline{Aut}^{\otimes}(\omega|_{\langle X \rangle^{\otimes}})$ of the tensor automorphisms of the restriction of ω (and its extensions of scalars) to $\langle X\rangle^{\otimes}$. The functor ω gives an equivalence of categories

$$\langle X\rangle^{\otimes} \to \mathbf{Rep}(\mathcal{G}(X,\omega)) \tag{4}$$

(see for instance, [14]). Let $\mathfrak{g}(X,\omega)$ be the Lie algebra of $\mathcal{G}(X,\omega)$.

We shall identify $\mathcal{G}(X,\omega)$ as an algebraic subgroup of $GL(\omega X)$ via the natural embedding

$$\mathcal{G}(X,\omega) = \underline{Aut}^{\otimes}(\omega|_{\langle X\rangle^{\otimes}}) \hookrightarrow GL(\omega X),$$

and hence identify the Lie algebra $\mathfrak{g}(X,\omega)$ as a Lie subalgebra of the Lie algebra $End_{\mathbb{K}}(\omega X)$ of $GL(\omega X)$. We identity $\omega\underline{End}(M) = End_{\mathbb{K}}(\omega X)$, with the action of $\mathcal{G}(X,\omega)$ on $End_{\mathbb{K}}(\omega X)$ by conjugation. Via the equivalence of categories (4), considering the adjoint action of $\mathcal{G}(X,\omega)$ one obtains a canonical Lie subobject

$$\mathfrak{g}(X) \subset \underline{End}(X)$$

such that

$$\omega\mathfrak{g}(X) = \mathfrak{g}(X,\omega)$$

in $\omega\underline{End}(X) = End_{\mathbb{K}}(\omega X)$.

More generally, given any object Y of $\langle X\rangle^{\otimes}$, for any fiber functor ω set

$$\mathcal{G}(X,Y,\omega) := \ker\big(\,\mathcal{G}(X,\omega) \twoheadrightarrow \mathcal{G}(Y,\omega)\,\big),$$

where the surjective arrow is the canonical surjection given by the inclusion of $\langle Y\rangle^{\otimes}$ in $\langle X\rangle^{\otimes}$. Let $\mathfrak{g}(X,Y,\omega)$ be the Lie algebra of $\mathcal{G}(X,Y,\omega)$ (hence identified as a Lie subalgebra of $End_{\mathbb{K}}(\omega X)$). Then there exists a canonical Lie subobject

$$\mathfrak{g}(X,Y) \subset \underline{End}(X)$$

such that for *every* fiber functor ω,

$$\omega\mathfrak{g}(X,Y) = \mathfrak{g}(X,Y,\omega).$$

See for example, §2.5-2.7 of [19] for more details on this, including the independence of $\mathfrak{g}(X,Y)$ (and in particular, $\mathfrak{g}(X)$) from the choice of ω. We have a short exact sequence

$$0 \longrightarrow \mathfrak{g}(X,Y) \longrightarrow \mathfrak{g}(X) \longrightarrow \mathfrak{g}(Y) \longrightarrow 0 \tag{5}$$

which after applying any fiber functor ω, becomes the exact sequence obtained by applying the Lie algebra functor to the exact sequence

$$1 \longrightarrow \mathcal{G}(X,Y,\omega) \longrightarrow \mathcal{G}(X,\omega) \longrightarrow \mathcal{G}(Y,\omega) \longrightarrow 1.$$

2.3. Set

$$\begin{aligned}
\mathfrak{u}(L) &:= \mathfrak{g}(L, A_1 \oplus A_2)\\
\mathfrak{u}(N) &:= \mathfrak{g}(N, A_2 \oplus A_3)\\
\mathfrak{u}(L\oplus N) &:= \mathfrak{g}(L\oplus N, A_1 \oplus A_2 \oplus A_3)\\
\mathfrak{u}(M) &:= \mathfrak{g}(M, A_1 \oplus A_2 \oplus A_3).
\end{aligned}$$

If A_1, A_2 and A_3 are semisimple, for any choice of fiber functor ω, the image of $\mathfrak{u}(M)$ (resp. $\mathfrak{u}(L)$, $\mathfrak{u}(N)$ and $\mathfrak{u}(L\oplus N)$) under ω is the Lie algebra of the unipotent radical of the fundamental group of M (resp. L, N, and $L\oplus N$) with respect to ω.

The subobject $\mathfrak{u}(M)$ of $\underline{End}(M)$ determines the subgroup

$$\mathcal{U}(M,\omega) := \mathcal{G}(M, A_1 \oplus A_2 \oplus A_3, \omega)$$

of $GL(\omega M)$ for every fiber functor ω.

2.4. The Lie algebra of the fundamental group of an extension has been studied in the past by Bertrand [9] in the setting of differential modules, and by Hardouin ([24] and [25]) and the author and Kumar Murty [18] in the setting of abstract tannakian categories. These works give us a characterization of $\mathfrak{u}(L)$ as follows. (Needless to say, replacing L, A_2, A_1, and $\mathscr{L}$ respectively by N, A_3, A_2, and $\mathscr{N}$ we get the statements for $\mathfrak{u}(N)$.)

The Lie algebra object $\mathfrak{u}(L)$ is abelian. Indeed, $\mathfrak{u}(L)$ is contained in the abelian Lie subalgebra

$$\underline{Hom}(A_2, A_1) \subset \underline{End}(L),$$

the image of which under every fiber functor ω is $Hom_{\mathbb{K}}(\omega A_2, \omega A_1)$, considered as a subspace of $End_{\mathbb{K}}(\omega L)$ via functoriality of $Hom_{\mathbb{K}}$. By [18, Cor. 3.4.1] (see also its precursors [9], [24], [25]), we have:

If A_1 and A_2 are semisimple, then $\mathfrak{u}(L)$ is the smallest subobject of $\underline{Hom}(A_2, A_1)$ which satisfies the following two equivalent properties:
(1) The extension $\mathscr{L}$ is the push-forward of an element of $Ext^1(\mathbb{1}, \mathfrak{u}(L))$ under the inclusion $\mathfrak{u}(L) \subset \underline{Hom}(A_2, A_1)$.
(2) The push-forward $\mathscr{L}/\mathfrak{u}(L) \in Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1)/\mathfrak{u}(L))$ of $\mathscr{L}$ under the quotient map $\underline{Hom}(A_2, A_1) \to \underline{Hom}(A_2, A_1)/\mathfrak{u}(L)$ splits.

The equivalence of (1) and (2) is seen from the long exact sequence obtained by applying the δ-functor $Hom(\mathbb{1}, -)$ to the exact sequence

$$0 \longrightarrow \mathfrak{u}(L) \longrightarrow \underline{Hom}(A_2, A_1) \longrightarrow \underline{Hom}(A_2, A_1)/\mathfrak{u}(L) \longrightarrow 0.$$

In the more general case where A_1 and A_2 need not be semisimple one also has a characterization of $\mathfrak{u}(L)$. By [18, Theorem 3.3.1], we have:

The object $\mathfrak{u}(L)$ is the smallest subobject of $\underline{Hom}(A_2, A_1)$ which satisfies the following property: the push-forward $\mathscr{L}/\mathfrak{u}(L) \in Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1)/\mathfrak{u}(L))$ belongs to the subgroup

$$Ext^1_{\langle A_1, A_2\rangle^{\otimes}}(\mathbb{1}, \underline{Hom}(A_2, A_1)/\mathfrak{u}(L)) \subset Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1)/\mathfrak{u}(L))$$

(where $\langle A_1, A_2\rangle^{\otimes}$ is the tannakian subcategory of $\mathbf{T}$ generated by A_1 and A_2).

2.5. We shall consider the injective arrows in (1) as inclusion maps. Our blended extension gives a 3-step filtration

$$W_{-3}M := 0 \subset W_{-2}M := A_1 \subset W_{-1}M := L \subset W_0M := M$$

of M and an isomorphism

$$Gr(M) \cong A_1 \oplus A_2 \oplus A_3, \tag{6}$$

where $Gr(M)$ is the associated graded of M with respect to $W_\bullet$.

Let

$$W_{-1}\underline{End}(M) \subset \underline{End}(M)$$

be the subobject whose image under any fiber functor ω consists of the maps $f \in End_{\mathbb{K}}(\omega M)$ such that $f(\omega W_n M) \subset \omega W_{n-1} M$ for each n, i.e.

$$f(\omega M) \subset \omega L, \quad f(\omega L) \subset \omega A_1, \quad f(\omega A_1) = 0.$$

Then

$$\mathfrak{u}(M) \subset W_{-1}\underline{End}(M).$$

Indeed, for every fiber functor ω, every element of the fundamental group $\mathscr{G}(M, \omega)$ preserves the filtration $\omega W_\bullet M$ of ωM, so that $\mathscr{G}(M, \omega)$ is contained in the parabolic subgroup of $GL(\omega M)$ corresponding to this filtration. The kernel of the natural surjection of $\mathscr{G}(M, \omega)$ onto $\mathscr{G}(A_1 \oplus A_2 \oplus A_3, \omega)$ is contained in the unipotent radical of this parabolic subgroup, consisting of those elements of the parabolic subgroup that induce identity on $Gr(\omega M)$. It follows that $\omega\mathfrak{u}(M)$ is contained in the Lie algebra of this unipotent radical, which is exactly $\omega W_{-1}\underline{End}(M)$.

2.6. The bifunctoriality of $\underline{Hom}$ gives a canonical embedding

$$\underline{Hom}(A_3, A_1) \subset W_{-1}\underline{End}(M).$$

Applying any fiber functor ω, this inclusion sends a linear map $\omega A_3 \to \omega A_1$ to the composition

$$\omega M \twoheadrightarrow \omega A_3 \to \omega A_1 \hookrightarrow \omega M.$$

This embedding identifies $\omega\underline{Hom}(A_3, A_1)$ as the subspace of $\omega W_{-1}\underline{End}(M)$ consisting of linear maps $\omega M \to \omega M$ which vanish on ωL and whose image is in ωA_1. It fits in a canonical short exact sequence
(7)

$$0 \longrightarrow \underline{Hom}(A_3, A_1) \longrightarrow W_{-1}\underline{End}(M) \xrightarrow{\pi} \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2) \longrightarrow 0,$$

where the map

$$\pi : W_{-1}\underline{End}(M) \twoheadrightarrow \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$$

is obtained as follows: Bifunctoriality of $\underline{Hom}$ gives rise to two exact sequences

$$0 \longrightarrow \underline{Hom}(A_3, A_1) \longrightarrow \underline{Hom}(N, A_1) \xrightarrow{\pi_1} \underline{Hom}(A_2, A_1) \longrightarrow 0$$

and

$$0 \longrightarrow \underline{Hom}(A_3, A_1) \longrightarrow \underline{Hom}(A_3, L) \xrightarrow{\pi_2} \underline{Hom}(A_3, A_2) \longrightarrow 0.$$

We also have embeddings

$$\underline{Hom}(N, A_1) \subset W_{-1}\underline{End}(M)$$
$$\underline{Hom}(A_3, L) \subset W_{-1}\underline{End}(M)$$

the sum of which gives rise to a canonical isomorphism

$$\frac{\underline{Hom}(N, A_1) \oplus \underline{Hom}(A_3, L)}{\text{anti-diagonal copy of } \underline{Hom}(A_3, A_1)} \cong W_{-1}\underline{End}(M).$$

This makes $W_{-1}\underline{End}(M)$ the fibered coproduct of $\underline{Hom}(N, A_1)$ and $\underline{Hom}(A_3, L)$ over $\underline{Hom}(A_3, A_1)$. The map π is now obtained from the compositions

$$\underline{Hom}(N, A_1) \overset{\pi_1}{\twoheadrightarrow} \underline{Hom}(A_2, A_1) \hookrightarrow \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$$

and

$$\underline{Hom}(A_3, L) \overset{\pi_2}{\twoheadrightarrow} \underline{Hom}(A_3, A_2) \hookrightarrow \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$$

(the kernel of each of which is $\underline{Hom}(A_3, A_1)$) via the universal property of a fibered coproduct.

After applying a fiber functor ω, the map π has the following simple description: given any linear map $f \in W_{-1}End_{\mathbb{K}}(\omega M)$, write f as $f_1 + f_2$ for some

$$f_1 \in Hom_{\mathbb{K}}(\omega N, \omega A_1) \subset W_{-1}End_{\mathbb{K}}(\omega M)$$

and

$$f_2 \in Hom_{\mathbb{K}}(\omega A_3, \omega L) \subset W_{-1}End_{\mathbb{K}}(\omega M).$$

The elements f_1 and f_2 are not unique but their images under π_1 and π_2 are unique. We have $\pi(f) = (\pi_1(f_1), \pi_2(f_2))$.

Let

$$\pi_{12} : W_{-1}\underline{End}(M) \to \underline{Hom}(A_2, A_1) \quad \text{and} \quad \pi_{23} : W_{-1}\underline{End}(M) \to \underline{Hom}(A_3, A_2)$$

be the compositions of π with the projections maps to the two factors of $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$.

2.7. Set

$$\begin{aligned} \mathfrak{u}_{-2}(M) &:= \mathfrak{u}(M) \cap \underline{Hom}(A_3, A_1) \\ \mathfrak{u}_{-1}(M) &:= \pi(\mathfrak{u}(M)) \subset \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2). \end{aligned}$$

We have then a commutative diagram
(8)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{u}_{-2}(M) & \longrightarrow & \mathfrak{u}(M) & \xrightarrow{\pi|_{\mathfrak{u}(M)}} & \mathfrak{u}_{-1}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{Hom}(A_3, A_1) & \longrightarrow & W_{-1}\underline{End}(M) & \xrightarrow{\pi} & \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2) & \longrightarrow & 0 \end{array}$$

with exact rows and inclusion vertical arrows.

2.8. In general, the two subobjects $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_3, A_1)$ and $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$ may not determine the subobject $\mathfrak{u}(M)$ of $W_{-1}\underline{End}(M)$. They will however, under a mild condition that often holds in practical situations of interest.

Proposition 2.8.1. *Let ω be a fiber functor. Set $\mathcal{G} := \mathcal{G}(M, \omega)$. Suppose that there exists a reductive subgroup $\mathcal{R}$ of $\mathcal{G}$ such that*

$$Hom_{\mathcal{R}}(\omega\mathfrak{u}_{-1}(M), \omega\underline{Hom}(A_3, A_1)) \; = \; 0 \tag{9}$$

(equivalently, such that $\omega\mathfrak{u}_{-1}(M)$ and $\omega\underline{Hom}(A_3, A_1)$, when considered as representations of $\mathcal{R}$, do not have any isomorphic subquotients). Then $\mathfrak{u}(M)$ is uniquely determined inside $W_{-1}\underline{End}(M)$ by the two subobjects $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_3, A_1)$ and $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$.

Proof. We will argue that $\omega\mathfrak{u}(M)$ is determined by $\omega\mathfrak{u}_{-2}(M)$ and $\omega\mathfrak{u}_{-1}(M)$. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{u}_{-2}(M) & \longrightarrow & \mathfrak{u}(M) & \xrightarrow{\pi|_{\mathfrak{u}(M)}} & \mathfrak{u}_{-1}(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \underline{Hom}(A_3, A_1) & \longrightarrow & \pi^{-1}\mathfrak{u}_{-1}(M) & \xrightarrow{\pi|_{\pi^{-1}\mathfrak{u}_{-1}(M)}} & \mathfrak{u}_{-1}(M) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{Hom}(A_3, A_1) & \longrightarrow & W_{-1}\underline{End}(M) & \xrightarrow{\pi} & \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2) & \longrightarrow & 0, \end{array}$$

where the second row is the pullback of the third row along the inclusion of $\mathfrak{u}_{-1}M$. Given any section s of $\omega\pi|_{\pi^{-1}\mathfrak{u}_{-1}(M)}$ in the category of vector spaces that maps $\omega\mathfrak{u}_{-1}(M)$ into $\omega\mathfrak{u}(M)$, we have $\omega\mathfrak{u}(M) = \omega\mathfrak{u}_{-2}(M) + s(\omega\mathfrak{u}_{-1}(M))$ in $\omega W_{-1}\underline{End}(M)$. Since $\mathcal{R}$ is reductive, the map $\omega\pi|_{\pi^{-1}\mathfrak{u}_{-1}(M)}$ admits an $\mathcal{R}$-equivariant section s_0, which is unique thanks to (9). We will argue that s_0 maps $\omega\mathfrak{u}_{-1}(M)$ into $\omega\mathfrak{u}(M)$, hence establishing the result. Indeed, since $\mathcal{R}$ is reductive, the map $\omega\pi|_{\mathfrak{u}(M)}$ admits an $\mathcal{R}$-equivariant section, which after compositing with the inclusion $\omega\mathfrak{u}(M) \subset \omega\pi^{-1}\mathfrak{u}_{-1}(M)$ gives an $\mathcal{R}$-equivariant section of $\omega\pi|_{\pi^{-1}\mathfrak{u}_{-1}(M)}$. This section must coincide with s_0 by the uniqueness property. $\square$

2.9. Let ω be a fiber functor. Applying ω to (1) and forgetting the action of the tannakian fundamental group, we obtain a blended extension in the category of finite-dimensional vector spaces over $\mathbb{K}$. By a *splitting* of ωM we mean a choice of isomorphisms

$$\varphi_L : \omega L \xrightarrow{\simeq} \omega A_1 \oplus \omega A_2, \quad \varphi_M : \omega M \xrightarrow{\simeq} \omega A_1 \oplus \omega A_2 \oplus \omega A_3, \quad \varphi_N : \omega N \xrightarrow{\simeq} \omega A_2 \oplus \omega A_3 \tag{10}$$

of vector spaces such that replacing the objects ωL, ωN, and ωM in our blended extension by the corresponding direct sums of the ωA_j via these isomorphisms we obtain the trivial blended extension of vector spaces

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega A_1 & \longrightarrow & \omega A_1 \oplus \omega A_2 & \longrightarrow & \omega A_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega A_1 & \longrightarrow & \omega A_1 \oplus \omega A_2 \oplus \omega A_3 & \longrightarrow & \omega A_2 \oplus \omega A_3 & \longrightarrow & 0, \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \omega A_3 & = & \omega A_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{11}$$

where the injections and surjections are the canonical embeddings and projections. Thus for example, the composition

$$\omega A_1 \hookrightarrow \omega M \xrightarrow{\varphi_M} \omega A_1 \oplus \omega A_2 \oplus \omega A_3$$

(the first map being ω of the structure inclusion $A_1 \subset M$) would be the natural embedding into the first coordinate.

One can obtain a splitting of the blended extension ωM as follows: Choose a section of $\omega L \twoheadrightarrow \omega A_2$ and a section j of $\omega M \twoheadrightarrow \omega A_3$. This will give rise to isomorphisms $\omega L \simeq \omega A_1 \oplus \omega A_2$ and $\omega M \simeq \omega L \oplus \omega A_3 \simeq \omega A_1 \oplus \omega A_2 \oplus \omega A_3$. The composition $\omega A_3 \xrightarrow{j} \omega M \twoheadrightarrow \omega N$ is a section of the structure map $\omega N \twoheadrightarrow \omega A_3$. Use this section to get an isomorphism $\omega N \simeq \omega A_2 \oplus \omega A_3$. Then the three obtained isomorphisms of ωL, ωM and ωN with the corresponding sums of the ωA_j form a splitting of ωM.

It is clear that in the data of a splitting (10) of ωM, the isomorphisms φ_L and φ_N are determined by the isomorphism φ_M. In what follows, we thus may just speak of a splitting $\omega M \xrightarrow{\simeq} \omega A_1 \oplus \omega A_2 \oplus \omega A_3$.

2.10. Let ω be a fiber functor. Choose a splitting $\varphi : \omega M \xrightarrow{\simeq} \omega A_1 \oplus \omega A_2 \oplus \omega A_3$. Using φ to identity ωM with $\omega A_1 \oplus \omega A_2 \oplus \omega A_3$, we can write elements of

$$End_{\mathbb{K}}(\omega M) = End_{\mathbb{K}}(\omega A_1 \oplus \omega A_2 \oplus \omega A_3) \cong \bigoplus_{1 \leq i,j \leq 3} Hom_{\mathbb{K}}(\omega A_j, \omega A_i)$$

as 3×3 matrices $f = (f_{ij})$, where f_{ij} is the component of f in $Hom_{\mathbb{K}}(\omega A_j, \omega A_i)$. The Lie subalgebra $\omega W_{-1}\underline{End}(M)$ is then the space of all strictly upper triangular elements of $End_{\mathbb{K}}(\omega M)$. The Lie bracket of $\omega W_{-1}\underline{End}(M)$ is simply the usual Lie bracket on matrices.

The canonical embedding

$$\underline{Hom}(A_3, A_1) \hookrightarrow W_{-1}\underline{End}(M)$$

after applying ω simply places $f_{13} : \omega A_3 \to \omega A_1$ as the (13)-entry of a matrix. The canonical surjection

$$\pi = (\pi_{12}, \pi_{23}) : W_{-1}\underline{End}(M) \twoheadrightarrow \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$$

after applying ω is then simply the map

$$(f_{ij}) \mapsto (f_{12}, f_{23}).$$

The two maps $\omega\pi_{12}$ and $\omega\pi_{23}$ simply send (f_{ij}) to f_{12} and f_{23}, respectively. Note that a different choice of splitting would result to conjugation of (f_{ij}) by a unipotent upper triangular matrix in $GL(\omega A_1 \oplus \omega A_2 \oplus \omega A_3)$, and hence indeed the (12) and (23) entries would not change.

Use φ to also identity ωL and ωN respectively with $\omega A_1 \oplus \omega A_2$ and $\omega A_2 \oplus \omega A_3$. Identifying $\mathcal{G}(L, \omega)$, $\mathcal{G}(M, \omega)$ and $\mathcal{G}(N, \omega)$ as subgroups of

$$GL(\omega L) \cong GL(\omega A_1 \oplus \omega A_2), \; GL(\omega M) \cong GL(\omega A_1 \oplus \omega A_2 \oplus \omega A_3) \text{ and } GL(\omega N) \cong GL(\omega A_2 \oplus \omega A_3),$$

the canonical surjections

$$\mathcal{G}(M, \omega) \twoheadrightarrow \mathcal{G}(L, \omega) \quad \text{and} \quad \mathcal{G}(M, \omega) \twoheadrightarrow \mathcal{G}(N, \omega)$$

arising from the inclusions of $\langle L\rangle^{\otimes}$ and $\langle N\rangle^{\otimes}$ in $\langle M\rangle^{\otimes}$ are respectively simply given by

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & 0 & \sigma_{33} \end{pmatrix} \mapsto \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ 0 & \sigma_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & 0 & \sigma_{33} \end{pmatrix} \mapsto \begin{pmatrix} \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{33} \end{pmatrix}.$$

2.11. It is not difficult to see that the subobject $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$ only depends on L and N, and not on the choice of the blended extension M. Indeed, fix a fiber functor ω. Dropping ω from the notation for tannakian groups for simplicity, we have a diagram

$$\begin{array}{ccccc} \mathcal{G}(M) & \twoheadrightarrow & \mathcal{G}(L \oplus N) & \hookrightarrow & \mathcal{G}(L) \times \mathcal{G}(N) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}(A_1 \oplus A_2 \oplus A_3) & = & \mathcal{G}(A_1 \oplus A_2 \oplus A_3) & \hookrightarrow & \mathcal{G}(A_1 \oplus A_2) \times \mathcal{G}(A_2 \oplus A_3), \end{array}$$

where all the arrows are induced by restrictions to the respective subcategories. Focusing on the induced maps between the kernels of the vertical maps and passing to the Lie algebras, we obtain $\mathcal{G}(M)$-equivariant maps

$$\omega\mathfrak{u}(M) \twoheadrightarrow \omega\mathfrak{u}(L \oplus N) \hookrightarrow \omega\mathfrak{u}(L) \oplus \omega\mathfrak{u}(N)$$

and hence morphisms

$$\mathfrak{u}(M) \twoheadrightarrow \mathfrak{u}(L \oplus N) \hookrightarrow \mathfrak{u}(L) \oplus \mathfrak{u}(N), \tag{12}$$

which are in fact, independent of the choice of ω. There is a commutative diagram

$$\begin{array}{ccccc} \mathfrak{u}(M) & \twoheadrightarrow & \mathfrak{u}(L \oplus N) & \hookrightarrow & \mathfrak{u}(L) \oplus \mathfrak{u}(N) \\ \downarrow & & & & \downarrow \\ W_{-1}\underline{End}(M) & & \xrightarrow{\pi = (\pi_{12}, \pi_{23})} & & \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2). \end{array} \tag{13}$$

That this diagram commutes is seen immediately upon applying ω and taking a splitting of ωM. We thus have

$$\mathfrak{u}_{-1}(M) = \operatorname{Im}\big(\, \mathfrak{u}(L \oplus N) \hookrightarrow \mathfrak{u}(L) \oplus \mathfrak{u}(N) \,\big)\,. \tag{14}$$

In particular, we obtain:

Proposition 2.11.1. *The subobject $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$ only depends on the top horizontal and right vertical extensions L and N of* (1)*. (That is, it does not depend on the choice of M in $Extpan(N, L)$).*

Notation 2.11.2. Denote the compositions of (12) with the projections onto $\mathfrak{u}(L)$ and $\mathfrak{u}(N)$ respectively by π_L and π_N. The maps π_L and π_N are respectively induced by the inclusions of $\langle L\rangle^{\otimes}$ and $\langle N\rangle^{\otimes}$ in $\langle M\rangle^{\otimes}$.

2.12. Let ω be a fiber functor for $\mathbf{T}$ and $\mathcal{R}$ a subgroup of $\mathcal{G}(M, \omega)$. A splitting (10) is said to be $\mathcal{R}$-equivariant if the three isomorphisms (10) are $\mathcal{R}$-equivariant, with the action of $\mathcal{R}$ on the direct sums of the ωA_j being the direct sum action. A splitting (10) is $\mathcal{R}$-equivariant if and only if just the isomorphism $\omega M \to \omega A_1 \oplus \omega A_2 \oplus \omega A_3$ of the splitting is $\mathcal{R}$-equivariant. An $\mathcal{R}$-equivariant splitting of ωM exists if and only if the surjections $\omega L \twoheadrightarrow \omega A_2$ and $\omega M \twoheadrightarrow \omega A_3$ admit $\mathcal{R}$-equivariant sections, in which case an $\mathcal{R}$-equivariant splitting is obtained in the same way as described in §2.9. In particlular, if $\mathcal{R}$ is reductive, then there always exists an $\mathcal{R}$-equivariant splitting of ωM.

Lemma 2.12.1. *Let $\mathcal{R}$ be a subgroup of $\mathcal{G}(M, \omega)$ such that an $\mathcal{R}$-equivariant splitting of ωM exists. Identifying ωM with $\omega A_1 \oplus \omega A_2 \oplus \omega A_3$ via an $\mathcal{R}$-equivariant splitting φ, the map*

$$\omega\underline{Hom}(A_2, A_1) \oplus \omega\underline{Hom}(A_3, A_2) \to \omega W_{-1}\underline{End}(M) \qquad (f_{12}, f_{23}) \mapsto \begin{pmatrix} 0 & f_{12} & 0 \\ 0 & 0 & f_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

is $\mathcal{R}$-equivariant.

Proof. Let $\sigma \in \mathcal{G}(M, \omega)$. We write σ as a 3×3 matrix via the identifications $\mathcal{G}(M, \omega) \subset GL(\omega M)$ and $\omega M \cong \omega A_1 \oplus \omega A_2 \oplus \omega A_3$ given by the splitting φ. Then $\sigma = (\sigma_{ij})$ is upper triangular, and its diagonal entry σ_{jj} is the natural action of σ, as an automorphism of ω, on ωA_j. The actions of $\mathcal{G}(M, \omega)$ corresponding to $\underline{Hom}(A_2, A_1)$, $\underline{Hom}(A_3, A_2)$ and $W_{-1}\underline{End}(M)$ are given as follows: for any linear maps $f_{12} : \omega A_2 \to \omega A_1$ and $f_{23} : \omega A_3 \to \omega A_2$, we have $\sigma \cdot f_{12} = \sigma_{11} f_{12} \sigma_{22}^{-1}$ and $\sigma \cdot f_{23} = \sigma_{22} f_{23} \sigma_{33}^{-1}$. For any $f \in \omega W_{-1}\underline{End}(M)$, we have $\sigma \cdot f = (\sigma_{ij}) f (\sigma_{ij})^{-1}$. The $\mathcal{R}$-equivariance of the map in the statement of the lemma is checked by a direct computation on noting that (σ_{ij}) is diagonal if $\sigma \in \mathcal{R}$. $\square$

2.13. Combining Propositions 2.8.1 (and its proof) with Lemma 2.12.1 we obtain the following:

Proposition 2.13.1. *Let ω be a fiber functor for $\mathbf{T}$. Set $\mathcal{G} := \mathcal{G}(M, \omega)$. Suppose that there exists a reductive subgroup $\mathcal{R}$ of $\mathcal{G}$ such that condition* (9) *of Proposition 2.8.1 holds. Identifying ωM with $\omega A_1 \oplus \omega A_2 \oplus \omega A_3$ via an $\mathcal{R}$-equivariant splitting $\omega M \to \omega A_1 \oplus \omega A_2 \oplus \omega A_3$, for every (strictly upper triangular) $f = (f_{ij}) \in \omega W_{-1}\underline{End}(M)$ we have*

$$f \in \omega\mathfrak{u}(M) \qquad \Longleftrightarrow \qquad \begin{pmatrix} 0 & f_{12} & 0 \\ 0 & 0 & f_{23} \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \omega\mathfrak{u}(M).$$

Proof. Recall from the proof of Proposition 2.8.1 that we have an internal direct sum decomposition of vector spaces

$$\omega\mathfrak{u}(M) = \omega\mathfrak{u}_{-2}(M) + s(\omega\mathfrak{u}_{-1}(M))$$

in $\omega W_{-1}\underline{End}(M)$, where s is the unique $\mathcal{R}$-equivariant section of $\omega\pi|_{\pi^{-1}\mathfrak{u}_{-1}(M)}$ (notation as in the proof of Proposition 2.8.1). By the uniqueness of this section, s must be the restriction to $\omega\mathfrak{u}_{-1}(M)$ of the section of π constructed in Lemma 2.12.1. The result follows. $\square$

Remark 2.13.2. The reader should be alert that neither the result above nor Proposition 2.8.1 asserts that (even under the hypotheses of the results) $\mathfrak{u}(M)$ decomposes as a direct sum of $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-1}(M)$ in $\mathbf{T}$.

2.14. For every object X of $\mathbf{T}$ the object $\mathfrak{g}(X)$ is a Lie subobject of $\underline{End}(X)$. That is, one has a diagram

$$\begin{array}{ccc} \mathfrak{g}(X)\otimes\mathfrak{g}(X) & \xrightarrow{[\ ,\ ]} & \mathfrak{g}(X) \\ \big\downarrow & & \big\downarrow \\ \underline{End}(X)\otimes\underline{End}(X) & \xrightarrow{[\ ,\ ]} & \underline{End}(X) \end{array}$$

in $\mathbf{T}$ whose image under every ω is a similar diagram for the Lie brackets in the classical sense of the Lie algebras $\mathfrak{g}(X,\omega)$ and $End_{\mathbb{K}}(\omega X)$, making the former a Lie subalgebra of the latter. The Lie bracket

$$End_{\mathbb{K}}(\omega X)\otimes End_{\mathbb{K}}(\omega X)\to End_{\mathbb{K}}(\omega X)$$

is simply the usual Lie bracket of endomorphisms, given by

$$[f,f'] = f\circ f' - f'\circ f. \tag{15}$$

In the case of our blended extension M, the above diagram of Lie brackets restricts to a diagram

$$\begin{array}{ccc} \mathfrak{u}(M)\otimes\mathfrak{u}(M) & \xrightarrow{[\ ,\ ]} & \mathfrak{u}(M) \\ \big\downarrow & & \big\downarrow \\ W_{-1}\underline{End}(M)\otimes W_{-1}\underline{End}(M) & \xrightarrow{[\ ,\ ]} & W_{-1}\underline{End}(M). \end{array}$$

By (15), the derived Lie algebra $[W_{-1}\underline{End}(M), W_{-1}\underline{End}(M)]$ is contained in the subobject $\underline{Hom}(A_3,A_1)$ of $W_{-1}\underline{End}(M)$ (and they are equal if $A_2\neq 0$). Thus

$$[\mathfrak{u}(M),\mathfrak{u}(M)]\subset \underline{Hom}(A_3,A_1)\cap\mathfrak{u}(M)=\mathfrak{u}_{-2}(M).$$

2.15. Let us summarize our picture so far. Given our fixed data of §2.1, we have

$$0\subset[\mathfrak{u}(M),\mathfrak{u}(M)]\subset\mathfrak{u}_{-2}(M)=\mathfrak{u}(M)\cap\underline{Hom}(A_3,A_1)\subset\mathfrak{u}(M)\subset W_{-1}\underline{End}(M).$$

We also have

$$\frac{\mathfrak{u}(M)}{\mathfrak{u}_{-2}(M)}\overset{\pi}{\cong}\mathfrak{u}_{-1}(M):=\pi(\mathfrak{u}(M))\subset\underline{Hom}(A_2,A_1)\oplus\underline{Hom}(A_3,A_2),$$

where

$$\pi: W_{-1}\underline{End}(M)\twoheadrightarrow\underline{Hom}(A_2,A_1)\oplus\underline{Hom}(A_3,A_2)$$

is the canonical surjection constructed in §2.6 (with the simple description $(f_{ij})\mapsto(f_{12},f_{23})$ after applying a fiber functor and taking a splitting). Moreover, we have seen that $\mathfrak{u}_{-1}(M)$ only depends of L and N, and is contained in the subobject $\mathfrak{u}(L)\oplus\mathfrak{u}(N)$ of $\underline{Hom}(A_2,A_1)\oplus\underline{Hom}(A_3,A_2)$.

When the mild condition given in Proposition 2.8.1 holds for a fiber functor ω, then the subobjects $\mathfrak{u}_{-2}(M)$ and $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_3,A_2)$ and $\underline{Hom}(A_2,A_1)\oplus\underline{Hom}(A_3,A_2)$ uniquely determine the subobject $\mathfrak{u}(M)$ of $W_{-1}\underline{End}(M)$. In this situation, for a suitable choice of splitting $\omega M\to\omega A_1\oplus\omega A_2\oplus\omega A_3$, we also had a concrete decomposition of $\omega\mathfrak{u}(M)$ as a vector space in Proposition 2.13.1.

2.16. Our objective in this paper is to study

$$\mathfrak{u}_{-1}(M),\ \mathfrak{u}_{-2}(M),\ [\mathfrak{u}(M),\mathfrak{u}(M)] \text{ and } \mathfrak{u}_{-2}(M)/[\mathfrak{u}(M),\mathfrak{u}(M)].$$

We shall mostly (to be made clear as we go through the paper) work under the following assumption:

(C1) The objects A_1, A_2 and A_3 are semisimple.

This condition is equivalent to $\mathcal{G}(A_1 \oplus A_2 \oplus A_3, \omega)$ being reductive for any fiber functor ω, and also is equivalent to $\mathcal{U}(M,\omega)$ being the unipotent radical of $\mathcal{G}(M,\omega)$ for any such ω.

In addition, in our study of $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M),\mathfrak{u}(M)]$ (§4.6 and §4.7) we shall also assume the following condition:

(C2) We have

$$Hom(A_3 \otimes A_2, A_3 \otimes A_1) \cong Hom(A_1 \otimes A_3, A_1 \otimes A_2) \cong 0.$$

In view of the canonical isomorphisms $Hom(X,Y) \cong Hom(\mathbb{1}, \underline{Hom}(X,Y))$ and $\underline{Hom}(X,Y) \cong X^\vee \otimes Y$ in a tannakian category, (C2) is equivalent to

$$Hom(\underline{Hom}(A_2,A_1), \underline{Hom}(A_3,A_1)) \cong Hom(\underline{Hom}(A_3,A_2), \underline{Hom}(A_3,A_1)) \cong 0. \tag{16}$$

Both (C1) and (C2) are satisfied in a significant class of interesting practical situations, e.g. if $\mathbf{T}$ is a category of motives, and A_1, A_2, A_3 are pure of increasing order of weights. In general, we consider (C2) to be a very mild condition. We highlight that under (C1) and (C2) we may still have

$$Hom(\underline{Hom}(A_3,A_2), \underline{Hom}(A_2,A_1)) \neq 0.$$

(Compare with the equivalent formulation (16) of (C2).) If we were to assume that the last Hom group is zero, then the study of $\mathfrak{u}_{-1}(M)$ would considerably simplify, as it would bring us to a situation similar to the "graded-independent" case in [19] and [20]. In this simplified situation, we will always have $\mathfrak{u}_{-1}(M) = \mathfrak{u}(L) \oplus \mathfrak{u}(N)$, see Corollary 3.9.3. One of the main goals of this paper is to go beyond this simplified and limiting case.

Conditions (C1) and (C2) together guarantee that the condition of Proposition 2.8.1 holds and hence $\mathfrak{u}(M)$ is completely determined by $\mathfrak{u}_{-1}(M)$ and $\mathfrak{u}_{-2}(M)$. Indeed, referring to the notation of Proposition 2.8.1, take $\mathcal{R}$ to a Levi factor of $\mathcal{G}(M,\omega)$. Then we have

$$Hom_{\mathcal{R}}(\omega\underline{Hom}(A_j,A_{j-1}), \omega\underline{Hom}(A_3,A_1)) \cong Hom(\underline{Hom}(A_j,A_{j-1}), \underline{Hom}(A_3,A_1)) \cong 0$$

for $j = 2, 3$.

2.17. We end this section with a simple observation. Suppose $A_2 \neq 0$. Let ω be a fiber functor. Then the only Lie subalgebra of $W_{-1}End_{\mathbb{K}}(\omega M)$ that surjects onto $Hom_{\mathbb{K}}(\omega A_2, \omega A_1) \oplus Hom_{\mathbb{K}}(\omega A_3, \omega A_2)$ by $\omega\pi$ is $W_{-1}End_{\mathbb{K}}(\omega M)$. Indeed, let $\mathfrak{v}$ be a Lie subalgebra of $W_{-1}End_{\mathbb{K}}(\omega M)$ whose image under $\omega\pi$ contains (f_{12}, g_{23}) for every $f_{12} : \omega A_2 \to \omega A_1$ and $g_{23} : \omega A_3 \to \omega A_2$. Choose a splitting of ωM to write elements of $W_{-1}End_{\mathbb{K}}(\omega M)$ as strictly upper triangular matrices. By the description of $\omega\pi$ given in §2.10 and surjectivity of $\omega\pi$ on $\mathfrak{v}$, for every $f_{12} : \omega A_2 \to \omega A_1$ and $g_{23} : \omega A_3 \to \omega A_2$ there exist $f_{13}, g_{13} \in Hom_{\mathbb{K}}(\omega A_3, \omega A_1)$ such that $\mathfrak{v}$ contains

$$f := \begin{pmatrix} 0 & f_{12} & f_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g := \begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathfrak{v}$ is a Lie subalgebra, it also contains

$$[f,g] = f_{12} \circ g_{23},$$

where $f_{12} \circ g_{23}$ is considered as an element of $W_{-1}End_{\mathbb{K}}(\omega M)$ via the natural embedding of $Hom_{\mathbb{K}}(\omega A_3, \omega A_1)$ (as the (13)-entry). The claim now follows because the map

$$Hom_{\mathbb{K}}(\omega A_2, \omega A_1) \otimes Hom_{\mathbb{K}}(\omega A_3, \omega A_2) \to Hom_{\mathbb{K}}(\omega A_3, \omega A_1)$$

defined by $f_{12} \otimes g_{23} \mapsto f_{12} \circ g_{23}$ is surjective.

In view of (8), we thus obtain:

Proposition 2.17.1. *Suppose $A_2 \neq 0$. We have $\mathfrak{u}(M) = W_{-1}\underline{End}(M)$ if and only if*

$$\mathfrak{u}_{-1}(M) = \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2).$$

Remark 2.17.2. This is a generalization of Corollary 4.6 of [8].

3. The determination of $\mathfrak{u}_{-1}(M)$

Throughout this section we assume that our data of §2.1 satisfies conditions (C1) of §2.16. Our goal is to study the subobject $\mathfrak{u}_{-1}(M)$ of $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$. We shall give a description of this subobject in terms of the extensions L and N on the top row and right column of the blended extension M.

3.1. An interpretation of Ext^1 groups. In this subsection we discuss a description of Ext^1 groups in tannakian categories that will be our main tool in studying $\mathfrak{u}_{-1}(M)$. Recall that for any algebraic group $\mathcal{G}$ over $\mathbb{K}$, the category of finite dimensional representations of $\mathcal{G}$ (over $\mathbb{K}$) is denoted by $\mathbf{Rep}(\mathcal{G})$.

Proposition 3.1.1. *Let $\mathcal{G}$ be an algebraic group over $\mathbb{K}$ (a field of characteristic zero). Let $\mathcal{U}$ be the unipotent radical of $\mathcal{G}$. Let $\mathfrak{u}$ be the Lie algebra of $\mathcal{U}$, considered as an object of $\mathbf{Rep}(\mathcal{G})$ via the adjoint action. Consider the abelianization $\mathfrak{u}^{ab} := \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ also as an object of $\mathbf{Rep}(\mathcal{G})$ via the induced action. Denote the Ext^1 and Hom groups in $\mathbf{Rep}(\mathcal{G})$ by $Ext^1_{\mathcal{G}}$ and $Hom_{\mathcal{G}}$. Then for every semisimple object X of $\mathbf{Rep}(\mathcal{G})$, there is a canonical $\mathbb{K}$-linear isomorphism*

$$Ext^1_{\mathcal{G}}(\mathbb{1}, X) \to Hom_{\mathcal{G}}(\mathfrak{u}^{ab}, X) \tag{17}$$

that is functorial with respect to morphisms $X \to X'$. That is, the isomorphisms (17) *give an isomorphism of functors*

$$Ext^1_{\mathcal{G}}(\mathbb{1}, -) \to Hom_{\mathcal{G}}(\mathfrak{u}^{ab}, -)$$

on the full subcategory of $\mathbf{Rep}(\mathcal{G})$ whose objects consist of all semisimple objects of $\mathbf{Rep}(\mathcal{G})$. In particular, the isomorphism (17) *is an isomorphism of $End_{\mathcal{G}}(X)$-modules.*

This result is well known to experts (e.g. see Hain [22, §16] or Brown [11]) and follows from group cohomology for algebraic groups. In the case that $\mathcal{G}/\mathcal{U} \simeq \mathbb{G}_m$, the isomorphism (17) has been used frequently in the context of mixed Tate motives (e.g. see [17, §A.13] and its application in the same paper). Here, we will include an explicit proof for the result that uses "bare hands". The proof is included mainly because in some of the later arguments (i.e. in proofs of Lemma 3.2.1 and Proposition 3.6.1) we will use the explicit description of the isomorphism (17) that shall be given in the argument below.

We should also point out that the fact that (17) is an isomorphism of $End_{\mathcal{G}}(X)$-modules, which might have been less frequently used in other applications, will play an important role for us. The (left) action of $End_{\mathcal{G}}(X)$ on the vector space on the left hand side of (17) is given by pushforwards of extensions along endomorphisms of X, and its action on the vector space on the right hand side of (17) is given by composition. The actions of the field of scalars $\mathbb{K}$ on the two vector spaces (corresponding to the linear structures) coincide with the restrictions of the $End_{\mathcal{G}}(X)$-actions to the scalar maps. Thus the functoriality of (17) together with its additivity would imply the $End_{\mathcal{G}}(X)$-linearity and in particular, $\mathbb{K}$-linearity.

Proof of Proposition 3.1.1. In what follows, we use the same notation for a representation and its underlying vector space. Let us first make a comment about morphisms from $\mathcal{G}$ to other algebraic groups. Let $\mathcal{R}$ be a Levi factor for $\mathcal{G}$; thus $\mathcal{R}$ is a reductive subgroup of $\mathcal{G}$ and we have an identification $\mathcal{G} = \mathcal{U} \rtimes \mathcal{R}$. For any algebraic group $\mathcal{G}'$ over $\mathbb{K}$, the data of a morphism of algebraic groups $\rho : \mathcal{G} \to \mathcal{G}'$ is equivalent to the data of morphisms of algebraic groups $\rho_{\mathcal{U}} : \mathcal{U} \to \mathcal{G}'$ and $\rho_{\mathcal{R}} : \mathcal{R} \to \mathcal{G}'$ that are compatible with one another, in the sense that

$$\rho_{\mathcal{R}}(r)\rho_{\mathcal{U}}(u)\rho_{\mathcal{R}}(r)^{-1} = \rho_{\mathcal{U}}(rur^{-1}) \qquad (\forall u \in \mathcal{U}, \forall r \in \mathcal{R}).$$

The passage from ρ to $(\rho_{\mathcal{U}}, \rho_{\mathcal{R}})$ is by taking the restrictions of ρ to $\mathcal{U}$ and $\mathcal{R}$. The passage in the opposite direction is by taking ρ to be given by $\rho(ur) = \rho_{\mathcal{U}}(u)\rho_{\mathcal{R}}(r)$ for any $u \in \mathcal{U}$ and $r \in \mathcal{R}$.

We now start the construction of the isomorphism (17). Let X be a semisimple object of $\mathbf{Rep}(\mathcal{G})$. Consider an extension of $\mathbb{1}$ by X in $\mathbf{Rep}(\mathcal{G})$:

$$0 \longrightarrow X \longrightarrow E \longrightarrow \mathbb{1} \longrightarrow 0. \tag{18}$$

Choose a section s of $E \twoheadrightarrow \mathbb{1}$ as a map of vector spaces to identify $E = X \oplus \mathbb{1}$ as a vector space. Expressing the action of $\mathcal{U}$ on E (obtained by restricting the $\mathcal{G}$-action) in terms of this decomposition of E, in light of the fact that the actions of $\mathcal{U}$ on X and $\mathbb{1}$ are trivial (the former thanks to the semisimplicity assumption), we obtain a morphism

$$(\rho_{\mathcal{U}})_{12} : \mathcal{U} \to Hom_{\mathbb{K}}(\mathbb{1}, X) \cong X$$

of algebraic groups, where $Hom_{\mathbb{K}}(\mathbb{1}, X)$ and X are considered as additive algebraic groups over $\mathbb{K}$. The reader can easily check that the morphism $(\rho_{\mathcal{U}})_{12}$ is indeed independent of the choice of the section s. From this independence, we can see, in particular, that the morphism $(\rho_{\mathcal{U}})_{12}$ is $\mathcal{G}$-equivariant, where the action of $\mathcal{G}$ on $\mathcal{U}$ is by conjugation and its action on X is the natural action. Indeed, since X is abelian and semisimple, any morphism of algebraic groups $\mathcal{U} \to X$ is $\mathcal{U}$-equivariant. Thus it suffices to check $\mathcal{R}$-equivariance of $(\rho_{\mathcal{U}})_{12}$ for a Levi factor $\mathcal{R}$. Since $\mathcal{R}$ is reductive, the map $E \twoheadrightarrow \mathbb{1}$ admits an $\mathcal{R}$-equivariant section. Use such a section to calculate $(\rho_{\mathcal{U}})_{12}$. In light of the fact that our decomposition $E = X \oplus \mathbb{1}$ will be then a decomposition of $\mathcal{R}$-representations, one sees with a brief computation that compatibility of the $\mathcal{U}$-action on E with the $\mathcal{R}$-action on E translates in terms of the morphism $(\rho_{\mathcal{U}})_{12}$ to $\mathcal{R}$-equivariance of $(\rho_{\mathcal{U}})_{12}$.

The independence of $(\rho_{\mathcal{U}})_{12}$ from the choice of the section s also shows that replacing (18) with an equivalent extension does not change the morphism $(\rho_{\mathcal{U}})_{12}$. Thus we have constructed a map

$$Ext^1_{\mathcal{G}}(\mathbb{1}, X) \to Hom_{\mathcal{G}\text{-equiv}}(\mathcal{U}, X) \qquad \text{class of (28)} \mapsto (\rho_{\mathcal{U}})_{12}. \tag{19}$$

Here, $Hom_{\mathcal{G}\text{-equiv}}(\mathcal{U}, X)$ denotes the set of $\mathcal{G}$-equivariant morphisms of algebraic groups from $\mathcal{U}$ to X.

Since X is an additive algebraic group, the set $Hom_{\mathcal{G}\text{-equiv}}(\mathcal{U}, X)$ carries a natural structure of a vector space over $\mathbb{K}$. On recalling the definition of Baer summation on $Ext^1_{\mathcal{G}}(\mathbb{1}, X)$, a direct computation shows that the map (19) preserves addition. Similarly, a direct computation shows that the map (19) is functorial in X, in the sense that if X' is also a semisimple object and $\phi : X \to X'$ is a morphism in $\mathbf{Rep}(\mathcal{G})$, then (19) sends the extension $\phi_* E$ to $\phi \circ (\rho_{\mathcal{U}})_{12}$. We leave out the details of these computations.

We now show that (19) is bijective. For this, we will construct its inverse, as follows. Fix a Levi factor $\mathscr{R}$. Consider the exact sequence of $\mathscr{R}$-representations

$$0 \longrightarrow X \longrightarrow X \oplus \mathbb{1} \longrightarrow \mathbb{1} \longrightarrow 0 \tag{20}$$

with the inclusion and projection maps. Given an $\mathscr{R}$-equivariant ($=\mathscr{G}$-equivariant) morphism of algebraic groups $(\rho_{\mathscr{U}})_{12} : \mathscr{U} \to X$, we use it in the natural way to define a $\mathscr{U}$-action on $E := X \oplus \mathbb{1}$: an element $u \in \mathscr{U}$ acts on E by left multiplication by the matrix

$$\begin{pmatrix} Id_X & (\rho_{\mathscr{U}})_{12}(u) \\ 0 & Id_{\mathbb{1}} \end{pmatrix} \in GL(X \oplus \mathbb{1}).$$

This makes (20) a sequence of $\mathscr{U}$-representations. The $\mathscr{R}$-equivariance of $\rho_{\mathscr{U}}$ guarantees that the action of $\mathscr{U}$ on E is compatible with the direct sum action of $\mathscr{R}$. Thus E (resp. the sequence (20)) becomes a $\mathscr{G}$-representation (resp. an extension of $\mathscr{G}$-representations). Sending $(\rho_{\mathscr{U}})_{12}$ to the class of (20) in $Ext^1_{\mathscr{G}}(\mathbb{1}, X)$ we obtain the inverse of (19).

Thus (19) is a functorial isomorphism of additive groups. In view of the facts that $\mathscr{U}$ and X are unipotent groups, $\mathbb{K}$ has characteristic zero, and X is abelian, on passing to the Lie algebras of $\mathscr{U}$ and X the isomorphism (19) gives an additive functorial isomorphism between $Ext^1_{\mathscr{G}}(\mathbb{1}, X)$ and the space of $\mathscr{G}$-equivariant linear maps $\mathfrak{u}^{ab} \to X$, where the actions of $\mathscr{G}$ on $\mathfrak{u}^{ab}$ and X are the natural ones. □

3.2. We now apply Proposition 3.1.1 to our situation (with the data of §2.1 satisfying (C1) of §2.16). Set $\mathfrak{u}^{ab}(M) := \mathfrak{u}(M)/[\mathfrak{u}(M), \mathfrak{u}(M)]$. Fix a fiber functor ω. Thanks to condition (C1), $\mathscr{U}(M, \omega) := \mathscr{G}(M, A_1 \oplus A_2 \oplus A_3, \omega)$ is the unipotent radical of $\mathscr{G}(M, \omega)$. Applying Proposition 3.1.1 with $\mathscr{G} = \mathscr{G}(M, \omega)$, in view of the equivalence of categories $\langle M\rangle^{\otimes} \xrightarrow{\omega} \mathbf{Rep}(\mathscr{G}(M, \omega))$, for every semisimple object X of $\langle M\rangle^{\otimes}$ we have a linear isomorphism

$$Ext^1_{\langle M\rangle^{\otimes}}(\mathbb{1}, X) \cong Hom(\mathfrak{u}^{ab}(M), X), \tag{21}$$

which is functorial in X. (The notation $Ext^1_{\langle M\rangle^{\otimes}}$ means the Ext^1 group in the tannakian subcategory $\langle M\rangle^{\otimes}$ generated by M.)

We now check that this map is intrinsic to the category $\mathbf{T}$:

Lemma 3.2.1. *The isomorphism* (21) *is independent of the choice of fiber functor* ω.

Proof. Let ω and ω' be two fiber functors for $\mathbf{T}$. Set $\mathscr{G} := \mathscr{G}(M, \omega)$, $\mathscr{G}' := \mathscr{G}(M, \omega')$, $\mathscr{U} := \mathscr{U}(M, \omega)$ and $\mathscr{U}' := \mathscr{U}(M, \omega')$. We will also simply write $\mathfrak{u}$ and $\mathfrak{u}^{ab}$ for $\mathfrak{u}(M)$ and $\mathfrak{u}^{ab}(M)$. Fix an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. By adding the bar symbol to the notation for any object we mean the object obtained by extending scalars from $\mathbb{K}$ to $\overline{\mathbb{K}}$. Since the formation of unipotent radicals commutes with base field extensions (in characteristic 0), $\overline{\mathscr{U}}$ and $\overline{\mathscr{U}'}$ are the unipotent radicals of $\overline{\mathscr{G}}$ and $\overline{\mathscr{G}'}$.

By Deligne's work [16] (see §1.12 and §1.13 therein), there is an isomorphism α of fiber functors $\overline{\omega} \to \overline{\omega'}$ over $\overline{\mathbb{K}}$. This induces an isomorphism $\overline{\mathscr{G}} \to \overline{\mathscr{G}'}$. We have the following diagram:

$$
\begin{array}{ccccc}
 & Ext^1_{\mathscr{G}}(\mathbb{1},\omega X) & \xrightarrow{\simeq} & Hom_{\mathscr{G}}(\omega\mathfrak{u}^{ab},\omega X) & \\
 {\scriptstyle \omega,\simeq}\nearrow & \big\downarrow & & \big\downarrow & \nwarrow{\scriptstyle \omega,\simeq} \\
 & Ext^1_{\overline{\mathscr{G}}}(\mathbb{1},\overline{\omega X}) & \xrightarrow{\simeq} & Hom_{\overline{\mathscr{G}}}(\overline{\omega\mathfrak{u}^{ab}},\overline{\omega X}) & \\
Ext^1_{\langle M\rangle^{\otimes}}(\mathbb{1},X) & {\scriptstyle \alpha}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \alpha} & Hom(\mathfrak{u}^{ab},X) \\
 & Ext^1_{\overline{\mathscr{G}'}}(\mathbb{1},\overline{\omega' X}) & \xrightarrow{\simeq} & Hom_{\overline{\mathscr{G}'}}(\overline{\omega'\mathfrak{u}^{ab}},\overline{\omega' X}) & \\
 {\scriptstyle \omega',\simeq}\searrow & \big\uparrow & & \big\uparrow & \swarrow{\scriptstyle \omega',\simeq} \\
 & Ext^1_{\mathscr{G}'}(\mathbb{1},\omega' X) & \xrightarrow{\simeq} & Hom_{\mathscr{G}'}(\omega'\mathfrak{u}^{ab},\omega' X) &
\end{array}
\tag{22}
$$

The horizontal isomorphisms in the diagram are the ones coming from Proposition 3.1.1. We have used the fact that base change commutes with taking Lie algebras and abelianization. The vertical maps not marked as α are given by extension of scalars. It follows from the construction of the isomorphism of Proposition 3.1.1 that the top square and similarly the bottom square are commutative. The isomorphisms marked by α are induced by α in the natural ways: The isomorphism on the right is given simply by compositions with the isomorphisms $\overline{\omega'\mathfrak{u}^{ab}} \to \overline{\omega\mathfrak{u}^{ab}}$ and $\overline{\omega X} \to \overline{\omega' X}$ given by α. The fact that α is an isomorphism of functors implies that the right triangle in (22) is commutative. The isomorphism marked as α from $Ext^1_{\overline{\mathscr{G}}}(\mathbb{1},\overline{\omega X})$ to $Ext^1_{\overline{\mathscr{G}'}}(\mathbb{1},\overline{\omega' X})$ is obtained on the level of extensions by replacing the $\overline{\mathscr{G}}$-representation $\overline{\omega X}$ by the $\overline{\mathscr{G}'}$-representation $\overline{\omega' X}$ via α, keeping the middle vector space unchanged, and transforming the $\overline{\mathscr{G}}$-action on it to a $\overline{\mathscr{G}'}$-action via $\overline{\mathscr{G}} \xrightarrow{\alpha} \overline{\mathscr{G}'}$. This gives a well-defined map on the level of Ext^1 groups. If E is an extension of $\mathbb{1}$ by $\overline{\omega X}$ in $\mathbf{Rep}(\overline{\mathscr{G}})$ and E' is an extension of $\mathbb{1}$ by $\overline{\omega' X}$ in $\mathbf{Rep}(\overline{\mathscr{G}'})$, then the map

$$
Ext^1_{\overline{\mathscr{G}}}(\mathbb{1},\overline{\omega X}) \to Ext^1_{\overline{\mathscr{G}'}}(\mathbb{1},\overline{\omega' X}) \tag{23}
$$

in (22) maps the class of E to the class of E' if and only if E and E' fit in a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\omega X} & \longrightarrow & E & \longrightarrow & \mathbb{1} & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & \overline{\omega' X} & \longrightarrow & E' & \longrightarrow & \mathbb{1} & \longrightarrow & 0
\end{array}
\tag{24}
$$

such that the vertical maps are equivariant under the isomorphism $\overline{\mathscr{G}} \xrightarrow{\alpha} \overline{\mathscr{G}'}$. The commutativity of the left triangle in (22) is thus also guaranteed by the fact that α is an isomorphism of functors.

We now turn our attention to the commutativity of the middle square in (22). Suppose we have a diagram as in (24) as described earlier (so that (23) takes E to E'). Choose a linear section of $E \twoheadrightarrow \mathbb{1}$. This section gives, via the isomorphism $E \to E'$, a linear section of $E' \twoheadrightarrow \mathbb{1}$. Use these sections to compute the morphisms

$$
\overline{\mathscr{U}} \xrightarrow{(\rho_{\overline{\mathscr{U}}})_{12}} \overline{\omega X} \quad \text{and} \quad \overline{\mathscr{U}'} \xrightarrow{(\rho_{\overline{\mathscr{U}'}})_{12}} \overline{\omega' X}
$$

corresponding to the extensions E and E' (see the proof of Proposition 3.1.1). The isomorphism $\overline{\mathcal{G}} \xrightarrow{\alpha} \overline{\mathcal{G}'}$ maps $\overline{\mathcal{U}}$ isomorphically to $\overline{\mathcal{U}'}$. Then $(\rho_{\overline{\mathcal{U}}})_{12}$ and $(\rho_{\overline{\mathcal{U}'}})_{12}$ are related to one another by this isomorphism between $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}'}$ on the one hand, and the isomorphism $\overline{\omega X} \xrightarrow{\alpha} \overline{\omega' X}$ on the other. The commutativity of the middle square of (22) now follows on noting that the map $\overline{\omega\mathfrak{u}} \to \overline{\omega'\mathfrak{u}}$ obtained by applying the Lie algebra functor to $\overline{\mathcal{U}} \xrightarrow{\alpha} \overline{\mathcal{U}'}$ coincides with the map arising from the fact that α is an isomorphism of functors $\omega \to \omega'$.

The injectivity of the vertical arrows in (22) now gives the desired conclusion. □

3.3. For convenience, we summarize the conclusions of §3.1 and §3.2. For any semisimple object X of $\langle M\rangle^{\otimes}$ (where M satisfies (C1)) we have a canonical isomorphism

$$Ext^1_{\langle M\rangle^{\otimes}}(\mathbb{1}, X) \xrightarrow{\simeq} Hom(\mathfrak{u}^{ab}(M), X), \tag{25}$$

which is linear and functorial with respect to morphisms $X \to X'$. In particular, (25) is an isomorphism of $End(X)$-modules. To calculate the image of an extension E of $\mathbb{1}$ by X in $\langle M\rangle^{\otimes}$ under (25), we choose a fiber functor ω and follow the construction of the proof of Proposition 3.1.1: choose a linear section of $\omega E \twoheadrightarrow \omega\mathbb{1}$ to decompose $\omega E \cong \omega X \oplus \omega \mathbb{1}$ as vector spaces, express the action of $\mathcal{U}(M,\omega)$ on ωE with respect to this decomposition to obtain a morphism of algebraic groups $\mathcal{U}(M,\omega) \to \omega X$ over $\mathbb{K}$, and then take the logarithm of this morphism. The induced map $\omega\mathfrak{u}^{ab}(M) \to \omega X$ is the image under ω of the morphism $\mathfrak{u}^{ab}(M) \to X$ corresponding to E under the isomorphism (25).

3.4. Let X be an object of $\langle M\rangle^{\otimes}$. The map $\mathfrak{u}(M) \twoheadrightarrow \mathfrak{u}^{ab}(M)$ induces an injection

$$Hom(\mathfrak{u}^{ab}(M), X) \hookrightarrow Hom(\mathfrak{u}(M), X)$$

which is functorial in X. If X is semisimple, composing this with the isomorphism (25) we obtain an injection

$$Ext^1_{\langle M\rangle^{\otimes}}(\mathbb{1}, X) \hookrightarrow Hom(\mathfrak{u}(M), X) \tag{26}$$

that is also functorial in X. The image of an extension E under this can be calculated just like the image under (25) of E, except we skip the final step of passing to the map induced on the abelianization of $\omega\mathfrak{u}(M)$.

3.5. We will be interested in taking X of the previous subsections to be a subobject of $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$. Let us denote the injection (26) for the choices $X = \underline{Hom}(A_2, A_1)$ and $X = \underline{Hom}(A_3, A_2)$ respectively by Ψ_{12} and Ψ_{23}, and the injection for the choice $X = \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$ by Ψ. By the functoriality property of these maps, we have a commutative diagram

$$\begin{array}{ccc} Ext^1_{\langle M\rangle^{\otimes}}\left(\mathbb{1}, \bigoplus_j \underline{Hom}(A_j, A_{j-1})\right) & \xhookrightarrow{\Psi} & Hom\left(\mathfrak{u}(M), \bigoplus_j \underline{Hom}(A_j, A_{j-1})\right) \\ \| & & \| \\ \bigoplus_j Ext^1_{\langle M\rangle^{\otimes}}\left(\mathbb{1}, \underline{Hom}(A_j, A_{j-1})\right) & \xhookrightarrow{\Psi_{12}\oplus\Psi_{23}} & \bigoplus_j Hom\left(\mathfrak{u}(M), \underline{Hom}(A_j, A_{j-1})\right) \end{array} \tag{27}$$

where the vertical identifications are via the canonical isomorphisms.

3.6. Recall from §2.1 that $\mathscr{L}$ and $\mathscr{N}$ respectively denote the elements of $Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1))$ and $Ext^1(\mathbb{1}, \underline{Hom}(A_3, A_2)))$ corresponding to L and N. Both $\mathscr{L}$ and $\mathscr{N}$ belong to the subgroups $Ext^1_{\langle M\rangle^{\otimes}}$ of the corresponding Ext^1 groups. Recall that we wrote the coordinates of the canonical map π of §2.6 as π_{12} and π_{23}. Denote the composition

$$\mathfrak{u}(M) \hookrightarrow W_{-1}\underline{End}(M) \xrightarrow{\pi = (\pi_{12}, \pi_{23})} \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$$

also by $\pi = (\pi_{12}, \pi_{23})$.

Proposition 3.6.1. *We have*

$$\Psi_{12}(\mathscr{L}) = \pi_{12}$$
$$\Psi_{23}(\mathscr{N}) = \pi_{23}.$$

Proof. The two formulas are verified similarly, so we will only give the proof of the second one. We will follow the procedure summarized in §3.3 to calculate the map $\Psi_{23}(\mathscr{N})$ (see also §3.4). The extension class $\mathscr{N}$ is represented by the extension

$$0 \longrightarrow \underline{Hom}(A_3, A_2) \longrightarrow \underline{Hom}(A_3, N)^{\dagger} \longrightarrow \mathbb{1} \longrightarrow 0$$

(also denoted by $\mathscr{N}$), which after applying ω becomes the extension

$$0 \longrightarrow Hom_{\mathbb{K}}(\omega A_3, \omega A_2) \longrightarrow Hom_{\mathbb{K}}(\omega A_3, \omega N)^{\dagger} \longrightarrow \mathbb{K} \longrightarrow 0.$$

Here, $\underline{Hom}(A_3, N)^{\dagger}$ is the subobject of $\underline{Hom}(A_3, N)$ whose image

$$Hom_{\mathbb{K}}(\omega A_3, \omega N)^{\dagger} \subset Hom_{\mathbb{K}}(\omega A_3, \omega N)$$

under ω consists of all linear maps $g : \omega A_3 \to \omega N$ such that the composition $\omega A_3 \xrightarrow{g} \omega N \twoheadrightarrow \omega A_3$ (the latter map being the structure map) is a scalar multiple, denoted by $\lambda(g) \in \mathbb{K}$, of the identity map on ωA_3. The surjective arrow in $\omega\mathscr{N}$ is the map $\lambda : g \mapsto \lambda(g)$, and the injective arrow in $\omega\mathscr{N}$ is given by the inclusion $\omega A_2 \hookrightarrow \omega N$. (See [19, §3.2] for more details.)

Choose a splitting of ωM (in the sense of §2.9). This splitting induces a section for the exact sequence $\omega\mathscr{N}$ above. We identify $Hom_{\mathbb{K}}(\omega A_3, \omega N)^{\dagger}$ with $Hom_{\mathbb{K}}(\omega A_3, \omega A_2) \oplus \mathbb{K}$ via this section.

Let $f = (f_{ij}) \in \omega\mathfrak{u}(M) \subset End_{\mathbb{K}}(\omega M)$, where we write endomorphisms of ωM as 3×3 matrices using our splitting of ωM. Set $\sigma := \exp(f) = 1 + f + f^2/2$. Note that σ has the same super-diagonal entries as f (i.e. f_{12} and f_{23}). The element $\sigma \in \mathcal{U}(M, \omega)$ acts on elements of $\omega\underline{Hom}(A_3, N)$ by sending $g : \omega A_3 \to \omega N$ to $\sigma_N g \sigma_{A_3}^{-1}$, where σ_N and σ_{A_3} are the actions of σ on ωN and ωA_3, respectively. Writing elements of $GL(\omega N)$ as 2×2 matrices via the isomorphism $\omega N \cong \omega A_2 \oplus \omega A_3$ given by our splitting, given any $g = (g_{23}, \lambda) \in Hom_{\mathbb{K}}(\omega A_3, \omega N)^{\dagger}$, on recalling that $\sigma_{A_3} = 1$ and σ_N is the bottom right 2×2 submatrix of σ, we have

$$\sigma \cdot g = \sigma_N g \sigma_{A_3}^{-1} = \begin{pmatrix} 1 & f_{23} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{23} \\ \lambda \end{pmatrix}.$$

Thus the image of $\Psi_{23}(\mathscr{N}) : \mathfrak{u}(M) \to \underline{Hom}(A_3, A_2)$ under ω is the logarithm of the morphism

$$\mathcal{U}(M, \omega) \to Hom_{\mathbb{K}}(\omega A_3, \omega A_2) \qquad \sigma = \exp(f) \mapsto f_{23}$$

of algebraic groups. Thus $\Psi_{23}(\mathscr{N})$ after applying ω is simply the projection $f \mapsto f_{23}$, as desired. □

3.7. We are ready to give our characterization of $\mathfrak{u}_{-1}(M)$. To shorten the notation, let us set

$$V := \underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2).$$

We shall consider the two vertical identifications in (27) as equality. In particular, we have an element

$$(\mathscr{L}, \mathscr{N}) \in Ext^1_{\langle M \rangle^{\otimes}}(\mathbb{1}, V) \subset Ext^1(\mathbb{1}, V).$$

Theorem 3.7.1. *The subobject $\mathfrak{u}_{-1}(M)$ of V is the intersection of the kernels of all the endomorphisms ϕ of V such that $\phi_*(\mathscr{L}, \mathscr{N}) = 0$:*

$$\mathfrak{u}_{-1}(M) = \bigcap_{\substack{\phi \in End(V) \\ \phi_*(\mathscr{L}, \mathscr{N}) = 0}} \ker(\phi).$$

Proof. Thanks to (C1), we can write the subobject $\mathfrak{u}_{-1}(M)$ of V as the intersection of the kernels of all the endomorphisms ϕ of V such that $\phi \circ \pi$ is zero on $\mathfrak{u}(M)$ (i.e. such that ϕ vanishes on $\mathfrak{u}_{-1}(M)$). By the commutativity of (27), we have

$$\Psi(\mathscr{L}, \mathscr{N}) = (\Psi_{12}(\mathscr{L}), \Psi_{23}(\mathscr{N})) \overset{\text{Prop. 3.6.1}}{=} \pi.$$

The assertion follows by the fact that Ψ is $End(V)$-linear and injective. □

3.8. Combining Theorem 3.7.1 and Proposition 2.17.1 we immediately obtain the following criterion for maximality of $\mathfrak{u}(M)$.

Corollary 3.8.1. *The following statements are equivalent.*

(i) The annihilator of the element $(\mathscr{L}, \mathscr{N})$ of the $End(V)$-module $Ext^1(\mathbb{1}, V)$ is trivial. That is, the only endomorphism ϕ of V with $\phi_*(\mathscr{L}, \mathscr{N}) = 0$ is zero.

(ii) $\mathfrak{u}_{-1}(M) = V$

Moreover, if $A_2 \neq 0$, then these are also equivalent to the following statement.

(iii) $\mathfrak{u}(M) = W_{-1}\underline{End}(M)$

The equivalent conditions of Corollary 3.8.1 imply that $\mathscr{L}$ and $\mathscr{N}$ must be totally nonsplit (see [20] or a footnote in the Introduction to recall the definition of a totally nonsplit extension). Indeed, condition (ii) above combined with the fact that $\mathfrak{u}_{-1}(M) \subset \mathfrak{u}(L) \oplus \mathfrak{u}(N)$ implies that $\mathfrak{u}(L) = \underline{Hom}(A_2, A_1)$ and $\mathfrak{u}(N) = \underline{Hom}(A_3, A_2)$. These conditions are respectively equivalent to total nonsplitting of L and N (see §2.4).

The above corollary implies and gives a more conceptual reason for Theorem 4.3.2 of [20] in the case of three graded components (as well as Corollary 6.7.1 of [19]).

3.9. We will give a refinement of Corollary 3.8.1 that can be more useful when $\mathscr{L}$ or $\mathscr{N}$ is not necessarily totally nonsplit. Set

$$V' := \mathfrak{u}(L) \oplus \mathfrak{u}(N) \ \subset V.$$

We always have $\mathfrak{u}_{-1}(M) \subset V'$. We will give an equivalent condition to $\mathfrak{u}_{-1}(M) = V'$ analogous to condition (i) of Corollary 3.8.1. For this, we will first give a slight variant of Theorem 3.7.1.

The extension $\mathscr{L}$ of $\mathbb{1}$ by $\underline{Hom}(A_2, A_1)$ is the pushforward of an extension of $\mathbb{1}$ by $\mathfrak{u}(L)$. Moreover, $\mathfrak{u}(L)$ is the smallest subobject of $\underline{Hom}(A_2, A_1)$ with this property (see §2.4). Thus any extension of $\mathbb{1}$ by $\mathfrak{u}(L)$ that pushes forward to $\mathscr{L}$ is totally nonsplit. Condition (C1) forces the pushforward map

$$Ext^1(\mathbb{1}, \mathfrak{u}(L)) \to Ext^1(\mathbb{1}, \underline{Hom}(A_2, A_1))$$

to be injective (as $\underline{Hom}(A_2, A_1)$ is semisimple). With abuse of notation, we denote the unique element of $Ext^1(\mathbb{1}, \mathfrak{u}(L))$ pushing forward to $\mathscr{L}$ also by $\mathscr{L}$. Similarly, we use the same notation for $\mathscr{N} \in Ext^1(\mathbb{1}, \underline{Hom}(A_3, A_2))$ and the unique element of $Ext^1(\mathbb{1}, \mathfrak{u}(N))$ that pushes forward to it. Both $\mathscr{L}$ and $\mathscr{N}$, when considered as extensions of $\mathbb{1}$ by $\mathfrak{u}(L)$ and $\mathfrak{u}(N)$, are totally nonsplit.

Let Ψ' be the canonical injection (26) for $X = V'$. In view of (C1) and functoriality of (26) applied to the inclusion $V' \hookrightarrow V$, Proposition 3.6.1 implies

$$\Psi'(\mathscr{L}, \mathscr{N}) = \pi,$$

where $(\mathscr{L}, \mathscr{N})$ is considered as an element of $Ext^1(\mathbb{1}, V')$ and the map π (with abuse of notation) means the restriction of its namesake to a map $\mathfrak{u}(M) \to V'$. The following variant of Theorem 3.7.1 now follows by the same argument.

Theorem 3.9.1. *Consider $(\mathscr{L}, \mathscr{N})$ as an element of $Ext^1(\mathbb{1}, V')$. Then $\mathfrak{u}_{-1}(M)$ is the intersection of the kernels of all the endomorphisms ϕ of V' such that $\phi_*(\mathscr{L}, \mathscr{N}) = 0$:*

$$\mathfrak{u}_{-1}(M) = \bigcap_{\substack{\phi \in End(V') \\ \phi_*(\mathscr{L}, \mathscr{N}) = 0}} \ker(\phi).$$

We thus obtain the following criterion for maximality of $\mathfrak{u}_{-1}(M)$ (in a refined sense):

Corollary 3.9.2. *We have $\mathfrak{u}_{-1}(M) = V'$ if and only if the annihilator (in $End(V')$) of the element $(\mathscr{L}, \mathscr{N})$ of the $End(V')$-module $Ext^1(\mathbb{1}, V')$ is trivial.*

Note that if $\mathscr{L}$ and $\mathscr{N}$ are totally nonsplit in $Ext^1(A_2, A_1)$ and $Ext^1(A_3, A_2)$, then $V' = V$ and the last result becomes the equivalence of (i) and (ii) in Corollary 3.8.1.

One also obtains the following corollary, which generalizes the "graded-independent" situation of [19] and [20] for three graded components.

Corollary 3.9.3. *Suppose that $Hom(\mathfrak{u}(L), \mathfrak{u}(N)) = 0$. Then $\mathfrak{u}_{-1}(M) = V'$.*

Proof. We show that the annihilator of $(\mathscr{L}, \mathscr{N})$ in $End(V')$ is trivial. Suppose $\phi \in End(V')$ annihilates $(\mathscr{L}, \mathscr{N})$. There are no nonzero morphisms between $\mathfrak{u}(L)$ and $\mathfrak{u}(N)$ (recall that A_1, A_2 and A_3 are semisimple). Thus we have $\phi = \phi_1 \oplus \phi_2$, where ϕ_1 (resp. ϕ_2) is an endomorphism of $\mathfrak{u}(L)$ (resp. $\mathfrak{u}(N)$). The condition $\phi_*(\mathscr{L}, \mathscr{N}) = 0$ implies $(\phi_1)_*\mathscr{L}$ and $(\phi_2)_*\mathscr{N}$ vanish. As extensions of $\mathbb{1}$ by $\mathfrak{u}(L)$ and $\mathfrak{u}(N)$ respectively, $\mathscr{L}$ and $\mathscr{N}$ are totally nonsplit. Thus ϕ_1 and ϕ_2 are both zero. □

4. The characterizations of $\mathfrak{u}_{-2}(M)$ and its subquotients

Our first goal in this section is to give a characterization of $\mathfrak{u}_{-2}(M)$ in the spirit of the characterization of the tannakian group of an extension given in [18, Theorem 3.3.1]. This will be the subject of §4.1 - §4.3 below, and is in the full generality of the data of §2.1, without necessarily satisfying any of the conditions of §2.16. After that, we will turn our focus to the derived Lie algebra $[\mathfrak{u}(M), \mathfrak{u}(M)] \subset \mathfrak{u}_{-2}(M)$ and the quotient $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M), \mathfrak{u}(M)]$.

4.1. Recall from §2.2 that for every objects X of $\mathbf{T}$ and Y of $\langle X \rangle^{\otimes}$ the notation $\mathfrak{g}(X, Y)$ means the canonical subobject of $\underline{End}(X)$ (or $\mathfrak{g}(X)$) whose image under any fiber functor ω is the Lie algebra of the kernel of the natural surjection $\mathcal{G}(X, \omega) \to \mathcal{G}(Y, \omega)$. We denoted this kernel by $\mathcal{G}(X, Y, \omega)$.

Lemma 4.1.1. *We have*

$$\mathfrak{u}_{-2}(M) = \mathfrak{g}(M, L \oplus N).$$

Proof. Let ω be a fiber functor. Since $A_1\oplus A_2\oplus A_3$ belongs to $\langle L\oplus N\rangle^{\otimes}$, we have

$$\mathscr{G}(M,L\oplus N,\omega)\subset\mathscr{G}(M,A_1\oplus A_2\oplus A_3,\omega).$$

In particular, $\mathscr{G}(M,L\oplus N,\omega)$ is unipotent and $\omega\mathfrak{g}(M,L\oplus N)\subset\omega\mathfrak{u}(M)$.

We show that $\omega\mathfrak{u}_{-2}(M)$ and $\omega\mathfrak{g}(M,L\oplus N)$ coincide in $W_{-1}End_{\mathbb{K}}(\omega M)$. Choose a splitting of ωM (see §2.9) to express elements of $GL(\omega M)$ and $End_{\mathbb{K}}(\omega M)$ as 3×3 matrices. Let $f=(f_{ij})\in\omega\mathfrak{u}_{-2}(M)$ (so f_{13} is the corresponding map $\omega A_3\to\omega A_1$ and the rest of the f_{ij} are all zero). Then

$$exp(f)=\begin{pmatrix}1&0&f_{13}\\0&1&0\\0&0&1\end{pmatrix}\in\mathscr{G}(M,\omega) \tag{28}$$

acts trivially on both L and N. Thus $\omega\mathfrak{u}_{-2}(M)\subset\omega\mathfrak{g}(M,L\oplus N)$.

On the other hand, if $f\in\omega\mathfrak{g}(M,L\oplus N)$, then $\exp(f)\in\mathscr{G}(M,L\oplus N,\omega)$ acts trivially on ωL and ωN and hence is of the form (28) for some f_{13}. That is, f belongs to the subspace $Hom_{\mathbb{K}}(\omega A_3,\omega A_1)\hookrightarrow W_{-1}End_{\mathbb{K}}(\omega M)$. □

4.2. Let X be an object of $\mathbf{T}$ and $\mathscr{E}\in Ext^1(\mathbb{1},X)$. In [19] with Murty we defined what it means for $\mathscr{E}$ to originate from a full tannakian subcategory $\mathbf{S}$ of $\mathbf{T}$ that is closed under taking subobjects (or subquotients): we say $\mathscr{E}$ originates from $\mathbf{S}$ if there exist a subobject X' of X in $\mathbf{S}$ and an extension $\mathscr{E}'\in Ext^1_{\mathbf{S}}(\mathbb{1},X')$ that pushes forward to $\mathscr{E}$.

In what follows, we shall only deal with instances of the above definition that X is in $\mathbf{S}$. In that case, the definition simplifies: $\mathscr{E}$ originates from $\mathbf{S}$ if and only if $\mathscr{E}$ is in the subgroup

$$Ext^1_{\mathbf{S}}(\mathbb{1},X)\subset Ext^1(\mathbb{1},X)$$

if and only if the middle object of $\mathscr{E}$ belongs to $\mathbf{S}$.

4.3. Let

$$\mathscr{M}^h\in Ext^1(\mathbb{1},\underline{Hom}(N,A_1))$$

be the element corresponding to the horizontal extension on the middle row of (1) under the canonical isomorphism

$$Ext^1(N,A_1)\cong Ext^1(\mathbb{1},\underline{Hom}(N,A_1)).$$

We are ready to give our characterization of $\mathfrak{u}_{-2}(M)$.

Theorem 4.3.1. *Let $\mathfrak{v}$ be a subobject of $\underline{Hom}(A_3,A_1)$. Then the extension*

$$\mathscr{M}^h/\mathfrak{v}\ \in\ Ext^1(\mathbb{1},\underline{Hom}(N,A_1)/\mathfrak{v})$$

originates from $\langle L\oplus N\rangle^{\otimes}$ if and only if $\mathfrak{u}_{-2}(M)\subset\mathfrak{v}$. That is, $\mathfrak{u}_{-2}(M)$ is the smallest subobject of $\underline{Hom}(A_3,A_1)$ such that $\mathscr{M}^h/\mathfrak{u}_{-2}(M)$ originates from $\langle L\oplus N\rangle^{\otimes}$.

Proof. We will prove the apparently[5] stronger statement that for every subobject $\mathfrak{v}$ of $\underline{Hom}(N,A_1)$ the extension $\mathscr{M}^h/\mathfrak{v}$ originates from $\langle L\oplus N\rangle^{\otimes}$ if and only if $\mathfrak{u}_{-2}(M)\subset\mathfrak{v}$. We may assume that $N\neq 0$. Fix a fiber functor ω. The extension $\mathscr{M}^h$ is given by

$$0\longrightarrow\underline{Hom}(N,A_1)\longrightarrow\underline{Hom}(N,M)^{\dagger}\longrightarrow\mathbb{1}\longrightarrow 0,$$

where the notation is in line with our notation in the proof of Proposition 3.6.1. That is, the image of $\underline{Hom}(N,M)^{\dagger}$ under ω consists of linear maps $g:\omega N\to\omega M$ whose composition with the structure map $\omega M\twoheadrightarrow\omega N$ is a scalar multiple $\lambda(g)$ of the identity map on ωN. After applying ω, the map $\underline{Hom}(N,M)^{\dagger}\to\mathbb{1}$ sends g to $\lambda(g)$.

[5] And indeed only in appearance, as the two statements are equivalent.

By [19, Lemma 3.4.2], $\mathscr{M}^h/\mathfrak{v}$ originates from $\langle L\oplus N\rangle^{\otimes}$ if and only if $\omega(\mathscr{M}^h/\mathfrak{v})$ splits in the category of representations of $\mathscr{G}(M, L\oplus N, \omega)$. Since $\underline{Hom}(N, A_1)/\mathfrak{v}$ belongs to $\langle L\oplus N\rangle^{\otimes}$, the action of $\mathscr{G}(M, L\oplus N, \omega)$ on $\omega(\underline{Hom}(N, A_1)/\mathfrak{v})$ is trivial. It follows that $\mathscr{M}^h/\mathfrak{v}$ originates from $\langle L\oplus N\rangle^{\otimes}$ if and only if, identifying $\underline{Hom}(N, M)^{\dagger} \subset W_{-1}\underline{End}(M)$ and choosing a splitting of ωM to identify $\omega M = \omega A_1 \oplus \omega A_2 \oplus \omega A_3$ as vector spaces and hence writing maps $\omega M \to \omega M$ as matrices, the element

$$\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \omega\mathfrak{v} \;\in\; Hom_{\mathbb{K}}(\omega N, \omega M)^{\dagger}/\omega\mathfrak{v} \;\subset\; W_{-1}End_{\mathbb{K}}(\omega M)/\omega\mathfrak{v} \tag{29}$$

is fixed by $\mathscr{G}(M, L \oplus N, \omega)$. By Lemma 4.1.1 and the fact that $\mathscr{G}(M, L \oplus N, \omega)$ is unipotent,

$$\mathscr{G}(M, L \oplus N, \omega) = \exp(\omega\mathfrak{u}_{-2}(M)).$$

For every $f \in Hom_{\mathbb{K}}(\omega A_3, \omega A_1) \subset W_{-1}End_{\mathbb{K}}(\omega M)$, a direct computation gives

$$\exp(f)\begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}\exp(f)^{-1} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} = f.$$

It follows that $\mathscr{G}(M, L \oplus N, \omega)$ fixes (29) if and only if $\omega\mathfrak{u}_{-2}(M) \subset \omega\mathfrak{v}$. ☐

Thus roughly speaking, $\mathfrak{u}_{-2}(M)$ captures the obstruction to the extension $\mathscr{M}^h$ being originated from $\langle L \oplus N\rangle^{\otimes}$, or rather, the obstruction to M being an object of $\langle L \oplus N\rangle^{\otimes}$. In particular, one has the following corollary:

Corollary 4.3.2. *We have $\mathfrak{u}_{-2}(M) = 0$ if and only if M is contained in $\langle L \oplus N\rangle^{\otimes}$.*

Proof. By Theorem 4.3.1, the object $\mathfrak{u}_{-2}(M)$ is zero if and only if $\underline{Hom}(N, M)^{\dagger}$ (i.e. the middle object of $\mathscr{M}^h$, see the proof of Theorem 4.3.1) belongs to the subcategory $\langle L \oplus N\rangle^{\otimes}$. The claim now follows on noting that $\langle L\oplus N\rangle^{\otimes}$ contains M if and only if it contains $\underline{Hom}(N, M)^{\dagger}$. (Note for the *if* implication: For nonzero N the evaluation morphism $\underline{Hom}(N, M)^{\dagger} \otimes N \to M$ is surjective.) ☐

Remark 4.3.3. One can give an analogous characterization of $\mathfrak{u}_{-2}(M)$ in terms of the extension

$$\mathscr{M}^v \in Ext^1(\mathbb{1}, \underline{Hom}(A_3, L))$$

corresponding to the middle column of (1). By a similar argument to the one above, $\mathfrak{u}_{-2}(M)$ is the smallest subobject of $\underline{Hom}(A_3, A_1)$ such that the pushforward

$$\mathscr{M}^v/\mathfrak{u}_{-2}(M) \in Ext^1(\mathbb{1}, \underline{Hom}(A_3, L)/\mathfrak{u}_{-2}(M))$$

originates from $\langle L \oplus N\rangle^{\otimes}$.

4.4. Our subject of study in the next two subsections is the derived algebra $[\mathfrak{u}(M), \mathfrak{u}(M)]$. In this subsection we shall see that this subobject of $\underline{Hom}(A_3, A_1)$ is completely determined by $\mathfrak{u}_{-1}(M)$; in fact, we will see that one can very easily compute $[\mathfrak{u}(M), \mathfrak{u}(M)]$ from $\mathfrak{u}_{-1}(M)$. In particular, when condition (C1) of §2.16 holds, combining with Theorem 3.7.1 we will have an explicit description of the derived algebra. We do not need to assume any conditions for the discussion of this subsection.

As before, let V be $\underline{Hom}(A_2, A_1) \oplus \underline{Hom}(A_3, A_2)$. Let

$$V \otimes V \to \underline{Hom}(A_3, A_1) \tag{30}$$

be the morphism that after applying a fiber functor ω is given by

$$(f_{12}, f_{23}) \otimes (g_{12}, g_{23}) \mapsto f_{12} \circ g_{23} - g_{12} \circ f_{23}.$$

That this is a morphism is easily seen by verifying compatibility with the actions of $\mathcal{G}(A_1 \oplus A_2 \oplus A_3, \omega)$. We use the notation $\{\,,\}$ for the map (30) and the associated pairing.

One has a commutative diagram

$$\begin{array}{ccc} \mathfrak{u}(M) \otimes \mathfrak{u}(M) & \xrightarrow{\pi\otimes\pi} & \mathfrak{u}_{-1}(M) \otimes \mathfrak{u}_{-1}(M) \\ & {\scriptstyle [\,,]} \searrow \quad \swarrow {\scriptstyle \{\,,\}} & \\ & \underline{Hom}(A_3, A_1). & \end{array}$$

Indeed, choose a fiber functor ω and a splitting of ωM to identify ωM as $\omega A_1 \oplus \omega A_2 \oplus \omega A_3$. Recall that $\omega\pi$ simply sends an element $f = (f_{ij})$ of $\mathfrak{u}(M) \subset W_{-1}End_{\mathbb{K}}(\omega M)$ to (f_{12}, f_{23}). The commutativity of the diagram above after applying ω is checked by a direct computation.

We thus obtain the following:

Proposition 4.4.1. *The derived algebra* $[\mathfrak{u}(M), \mathfrak{u}(M)]$ *is the image of* $\mathfrak{u}_{-1}(M) \otimes \mathfrak{u}_{-1}(M)$ *under the map* (30)*.*

Combining with §2.11 we obtain the following corollary:

Corollary 4.4.2. *The subobject* $[\mathfrak{u}(M), \mathfrak{u}(M)]$ *of* $\underline{Hom}(A_3, A_1)$ *only depends on the extensions* L *and* N *(and not on the particular blended extension* M*). In particular, whether or not the Lie algebra* $\mathfrak{u}(M)$ *of a blended extension* M *of* N *by* L *is abelian only depends on the extensions* N *and* L*.*

4.5. The previous subsection allows us to explicitly calculate the derived algebra $[\mathfrak{u}(M), \mathfrak{u}(M)]$ from $\mathfrak{u}_{-1}(M)$. Here we record a more conceptual (although possibly less practical) characterization of $[\mathfrak{u}(M), \mathfrak{u}(M)]$ under condition (C1). The characterization is rather basic and surely well known; it is included here mainly for the purpose of completeness of the discussion.

Proposition 4.5.1. *Assume condition (C1) of §2.16. For every subobject* $\mathfrak{v}$ *of* $\mathfrak{u}_{-2}(M)$ *the following statements are equivalent:*

(i) The pushforward of the extension

$$0 \longrightarrow \mathfrak{u}_{-2}(M) \longrightarrow \mathfrak{u}(M) \longrightarrow \mathfrak{u}_{-1}(M) \longrightarrow 0 \tag{31}$$

along the quotient map $\mathfrak{u}_{-2}(M) \twoheadrightarrow \mathfrak{u}_{-2}(M)/\mathfrak{v}$ *splits.*

(ii) The quotient $\mathfrak{u}(M)/\mathfrak{v}$ *is a semisimple object of* $\mathbf{T}$*.*

(iii) We have $[\mathfrak{u}(M), \mathfrak{u}(M)] \subset \mathfrak{v}$*.*

In particular, $[\mathfrak{u}(M), \mathfrak{u}(M)]$ *is zero if and only if* $\mathfrak{u}(M)$ *is a semisimple object of* $\mathbf{T}$*.*

Proof. The equivalence of (i) and (ii) is clear upon recalling that thanks to condition (C1), $\mathfrak{u}_{-1}(M)$ and $\mathfrak{u}_{-2}(M)$ are semisimple. We will argue that (ii) and (iii) are equivalent. The object $\mathfrak{u}(M)/\mathfrak{v}$ is semisimple if and only if the adjoint action of $\mathcal{G}(M, \omega)$ on it (after applying a fiber functor ω) factors through an action of $\mathcal{G}(A_1 \oplus A_2 \oplus A_3, \omega)$, or equivalently, if and only if the action of $\mathcal{G}(M, A_1 \oplus A_2 \oplus A_3, \omega)$ on it is trivial. The latter statement is equivalent to the triviality of the action of $\omega\mathfrak{u}(M)$ on $\omega\mathfrak{u}(M)/\omega\mathfrak{v}$ induced by the Lie bracket, which is in turn equivalent to the statement that $\omega\mathfrak{v}$ contains $[\omega\mathfrak{u}(M), \omega\mathfrak{u}(M)]$.

The final assertion follows from specializing to the case $\mathfrak{v} = 0$. □

Thus $[\mathfrak{u}(M), \mathfrak{u}(M)]$ is the smallest subobject of $\mathfrak{u}_{-2}(M)$ such that $\mathfrak{u}(M)/[\mathfrak{u}(M), \mathfrak{u}(M)]$ is a semisimple object of $\mathbf{T}$, or equivalently, such that the pushforward of the extension of (31) along $\mathfrak{u}_{-2}(M) \twoheadrightarrow \mathfrak{u}_{-2}(M)/[\mathfrak{u}(M), \mathfrak{u}(M)]$ splits. Since every subobject of $\mathfrak{u}(M)$ is a Lie subobject (in fact, a Lie ideal subobject), we may replace the term *the smallest subobject* in the previous sentence by *the smallest Lie subobject.*

4.6. We now turn our attention to the quotient of $\mathfrak{u}_{-2}(M)$ by the derived algebra. The following result can be useful for characterizing this quotient when M satisfies conditions (C1) and (C2) of §2.16.

Proposition 4.6.1. *Assume (C1) and (C2). There is a canonical isomorphism*

$$Ext^1_{\langle M\rangle^{\otimes}}(A_3, A_1) \;\cong\; Hom(\frac{\mathfrak{u}_{-2}(M)}{[\mathfrak{u}(M), \mathfrak{u}(M)]}, \underline{Hom}(A_3, A_1)). \tag{32}$$

Proof. Thanks to (C1), by §3.1 - §3.3 (taking $X = \underline{Hom}(A_3, A_1)$ in (25)) we have a canonical isomorphism

$$Ext^1_{\langle M\rangle^{\otimes}}(A_3, A_1) \cong\; Ext^1_{\langle M\rangle^{\otimes}}(\mathbb{1}, \underline{Hom}(A_3, A_1)) \cong Hom(\mathfrak{u}^{ab}(M), \underline{Hom}(A_3, A_1)).$$

Since $\mathfrak{u}^{ab}(M)$ is a semisimple object of $\mathbf{T}$, the restriction map

$$Hom(\mathfrak{u}^{ab}(M), \underline{Hom}(A_3, A_1)) \to Hom(\frac{\mathfrak{u}_{-2}(M)}{[\mathfrak{u}(M), \mathfrak{u}(M)]}, \underline{Hom}(A_3, A_1))$$

induced by the inclusion of $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M), \mathfrak{u}(M)]$ in $\mathfrak{u}^{ab}(M)$ is surjective. Condition (C2) guarantees that this map is an isomorphism, as there are no nonzero morphisms from $\mathfrak{u}_{-1}(M)$ to $\underline{Hom}(A_3, A_1)$. Composing this isomorphism with the previous isomorphism we obtain (32). □

The object $\mathfrak{u}_{-2}(M)/[\mathfrak{u}(M), \mathfrak{u}(M)]$ is a subquotient of $\underline{Hom}(A_3, A_1)$, and hence assuming (C1), also a subobject of it. Thus the last proposition has the following immediate consequence:

Corollary 4.6.2. *Assume (C1) and (C2). Then* $[\mathfrak{u}(M), \mathfrak{u}(M)] = \mathfrak{u}_{-2}(M)$ *if and only if*

$$Ext^1_{\langle M\rangle^{\otimes}}(A_3, A_1) = 0.$$

In particular, one has $\mathfrak{u}_{-2}(M) = 0$ *if and only if* $\mathfrak{u}(M)$ *is an abelian Lie algebra and the group* $Ext^1_{\langle M\rangle^{\otimes}}(A_3, A_1)$ *vanishes.*

4.7. Building on the last corollary, we end this section with another consequence of Proposition 4.6.1, which is an observation on blended extensions with vanishing $\mathfrak{u}_{-2}$.

Proposition 4.7.1. *Assume conditions (C1) and (C2). Fixing L and N, there exists at most one blended extension M in $Extpan(N, L)$ such that $\mathfrak{u}_{-2}(M)$ vanishes.*

Proof. Suppose that there exist blended extensions M_1 and M_2 of N by L both with vanishing $\mathfrak{u}_{-2}$. Then in particular, $\mathfrak{u}(M_1)$ and $\mathfrak{u}(M_2)$ are abelian and hence as objects of $\mathbf{T}$, they are semisimple (see Proposition 4.5.1). There is a natural injection

$$\mathfrak{u}(M_1 \oplus M_2) \hookrightarrow \mathfrak{u}(M_1) \oplus \mathfrak{u}(M_2),$$

induced by the natural embedding of the tannakian group of $M_1 \oplus M_2$ in the product of the tannakian groups of M_1 and M_2. Here, $\mathfrak{u}(M_1 \oplus M_2)$ is the object $\mathfrak{u}$ of the direct sum blended extension $M_1 \oplus M_2$ (a blended extension of N^2 by L^2), defined according to §2.3; thanks to the semisimplicity of the A_j, it is also the Lie algebra of the unipotent radical of the tannakian group of the object $M_1 \oplus M_2$ of $\mathbf{T}$. Since $\mathfrak{u}(M_1)$ and $\mathfrak{u}(M_2)$ are semisimple objects of $\mathbf{T}$, the restriction map

$$\bigoplus_{j=1}^{2} Hom(\mathfrak{u}(M_j), \underline{Hom}(A_3, A_1)) \;\to\; Hom(\mathfrak{u}(M_1 \oplus M_2), \underline{Hom}(A_3, A_1))$$

is surjective. Moreover, since $\mathfrak{u}(M_1)$ and $\mathfrak{u}(M_2)$ are abelian, so is $\mathfrak{u}(M_1 \oplus M_2)$. In view of Proposition 3.1.1, we thus obtain a surjection

$$\bigoplus_{j=1}^{2} Ext^1_{\langle M_j\rangle^{\otimes}}(A_3, A_1) \twoheadrightarrow Ext^1_{\langle M_1\oplus M_2\rangle^{\otimes}}(A_3, A_1).$$

Since $\mathfrak{u}_{-2}(M_1)$ and $\mathfrak{u}_{-2}(M_2)$ are zero, by Corollary 4.6.2 the two Ext groups on the left vanish. We thus get

$$Ext^1_{\langle M_1\oplus M_2\rangle^{\otimes}}(A_3, A_1) = 0.$$

This forces M_1 and M_2 to be the same in $Extpan(N, L)$. Indeed, the isomorphism classes of M_1 and M_2 (as blended extensions) belong to the subset $Extpan_{\langle M_1\oplus M_2\rangle^{\otimes}}(N, L)$ of $Extpan(N, L)$ consisting of the isomorphism classes of blended extensions of N by L in the category $\langle M_1 \oplus M_2\rangle^{\otimes}$. The set $Extpan_{\langle M_1\oplus M_2\rangle^{\otimes}}(N, L)$ is a torsor for $Ext^1_{\langle M_1\oplus M_2\rangle^{\otimes}}(A_3, A_1)$. Since this Ext group vanishes, $Extpan_{\langle M_1\oplus M_2\rangle^{\otimes}}(N, L)$ is a singleton. □

By Proposition 4.4.1, an obvious necessary condition for existence of a blended extension of N by L with vanishing $\mathfrak{u}_{-2}$ is that $\{\mathfrak{u}_{-1}, \mathfrak{u}_{-1}\}$ must vanish (where $\{,\}$ is the pairing of (30)). When $\mathbf{T}$ is the category of differential modules over a differential field of characteristic zero with an algebraically closed constant field, a result of Hardouin [23, Théorème 1.1(i)] asserts that the condition $\{\mathfrak{u}_{-1}, \mathfrak{u}_{-1}\} = 0$ is also sufficient for existence of a blended extension with vanishing $\mathfrak{u}_{-2}$ (assuming $Extpan(N, L)$ is nonempty). The same should be true in general, but we have not tried to prove it. For arbitrary $\mathbf{T}$, Bertrand has proved in [10] that in a rather special setting where he defines a notion of self-duality for blended extensions, assuming $Extpan(N, L)$ is nonempty, there always exists a blended extension M of N by L such that $\mathfrak{u}_{-2}(M) = [\mathfrak{u}(M), \mathfrak{u}(M)]$. (See *loc. cit.*, Theorems 1 and 2, also the proof of the latter.) The same might also be true in general.

5. Application to the Hodge-Nori conjecture for 1-motives

Throughout this section $\mathbb{F}$ is an algebraically closed subfield of $\mathbb{C}$.

5.1. Let $\mathbf{NMM}(\mathbb{F})$ be Nori's tannakian category of mixed motives over $\mathbb{F}$ ([28], see also the more easily available [26]). We refer to the objects of $\mathbf{NMM}(\mathbb{F})$ as Nori motives. Let **MHS** be the category of rational mixed Hodge structures. In [2] André proves the following theorem:

Theorem 5.1.1 (André). *Let M be a 1-motive[6] over $\mathbb{F}$ (in the sense of Deligne [13]). Then the motivic Galois group of the Nori motive associated with M coincides with its Mumford-Tate group.*

Here, by the motivic Galois group of a Nori motive one means the tannakian group of the motive with respect to the "Betti" fiber functor, which is the composition of the Hodge realization functor $\mathbf{NMM}(\mathbb{F}) \to \mathbf{MHS}$ and the forgetful functor from **MHS** to the category of finite-dimensional rational vector spaces. The Mumford-Tate group of the motive is the tannakian group of the Hodge realization of M with respect to the forgetful fiber functor. The latter group is always canonically a subgroup of the former, and the Hodge-Nori conjecture (which is a variant of the Hodge conjecture) predicts that the two groups are the same for every object of $\mathbf{NMM}(\mathbb{F})$ (as long as $\mathbb{F}$ is algebraically closed, which is what we assume throughout).

[6]The reader should beware of the slight change in notation in this section; we will be using M for a Deligne 1-motive, not the motive associated to it in a tannakian category of motives.

André's proof of the Hodge-Nori conjecture for 1-motives uses a deformation argument to reduce the problem to the case of semisimple 1-motives. This semisimple case was proved earlier by André himself in [1] in the setting of motives via motivated correspondences. Arapura [3] has proved that André's category of motives constructed via motivated correspondences is canonically equivalent to the full subcategory of $\mathbf{NMM}(\mathbb{F})$ consisting of semisimple objects, so that André's result in [1] also resolves the case of semisimple 1-motives for the Nori setting.

In [2] André wonders if it is possible to give a proof of the reduction to the semisimple case that is not based on a deformation argument. Here, as an application of the earlier results of the paper, we propose such an argument. The argument is rather formal, and it works in any tannakian category of motives as long as every extension of $\mathbb{1}$ by $\mathbb{Q}(1)$ in it comes from a 1-motive (as it is predicted to be the case by Deligne, see the next subsection).

5.2. Let $\mathbf{DM}_1(\mathbb{F})$ be Deligne's abelian category of 1-motives over $\mathbb{F}$ up to isogeny. Suppose that $\mathbf{MM}(\mathbb{F})$ is a tannakian category over $\mathbb{Q}$ of mixed motives over $\mathbb{F}$, with a $\mathbb{Q}$-linear (covariant) functor

$$h : \mathbf{DM}_1(\mathbb{F}) \to \mathbf{MM}(\mathbb{F})$$

the composition of which with the (exact, faithful, linear, tensor) Hodge realization functor $\mathfrak{R}^H : \mathbf{MM}(\mathbb{F}) \to \mathbf{MHS}$ is the Hodge realization functor on 1-motives constructed by Deligne in [13] (thus in particular, h is faithful and exact). Suppose moreover that h behaves well with respect to duals, that is, for every 1-motive M we have a canonical isomorphism $h(M^*) \cong h(M)^\vee(1)$, where M^* is the Cartier dual of M and $-^\vee$ is the dualizing functor in $\mathbf{MM}(\mathbb{F})$ in the tannakian sense. These requirements are basic, and any of the known constructions of a tannakian category of mixed motives (in particular, Nori's $\mathbf{NMM}(\mathbb{F})$) satisfies these conditions.

Given a 1-motive M over $\mathbb{F}$, thanks to its weight filtration, M fits into a blended extension as in (1) in the abelian category $\mathbf{DM}_1(\mathbb{F})$. The top left object, the top right object, and the bottom right object (in places of A_1, A_2, and A_3) respectively are the torus part, the abelian part, and the lattice part of M. Applying h and then further $\mathfrak{R}^H$ to this blended extension we obtain blended extensions $h(M)$ and $\mathfrak{R}^H h(M)$ in the tannakian categories $\mathbf{MM}(\mathbb{F})$ and $\mathbf{MHS}$. We thus may speak of $\mathfrak{u}(h(M))$, $\mathfrak{u}_{-1}(h(M))$ and $\mathfrak{u}_{-2}(h(M))$, as well as the corresponding objects $\mathfrak{u}(\mathfrak{R}^H h(M))$, $\mathfrak{u}_{-1}(\mathfrak{R}^H h(M))$ and $\mathfrak{u}_{-2}(\mathfrak{R}^H h(M))$ for the Hodge realization of $h(M)$. The latter three objects, respectively, are canonically contained in the Hodge realizations of the former three objects. In fact, the top extension in (8) for $\mathfrak{R}^H h(M)$ is contained in the Hodge realization of the analogous extension for $h(M)$.

Since $\mathbb{F}$ is algebraically closed and the composition

$$\mathbf{DM}_1(\mathbb{F}) \xrightarrow{h} \mathbf{MM}(\mathbb{F}) \xrightarrow{\mathfrak{R}^H} \mathbf{MHS}$$

coincides with the usual Hodge realization of 1-motives, the composition is a full functor (see [2, Proposition 2.1]). Since h and $\mathfrak{R}^H$ are both faithful, it follows that the functor h is also full. We thus have an injection

$$Ext^1_{\mathbf{DM}_1(\mathbb{F})}(M, M') \hookrightarrow Ext^1_{\mathbf{MM}(\mathbb{F})}(h(M), h(M'))$$

for every 1-motives M and M' over $\mathbb{F}$. Deligne conjectures that in a good tannakian category of mixed motives the essential image of $\mathbf{DM}_1(\mathbb{F})$ should be closed under extensions ([15], §2.4). Thus the above map should be an isomorphism. In particular, taking $M = \mathbb{Z}$ and $M' = \mathbb{G}_m$, we should have that

$$Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1}, \mathbb{Q}(1)) \cong Ext^1_{\mathbf{DM}_1(\mathbb{F})}(\mathbb{Z}, \mathbb{G}_m). \tag{33}$$

By the fullness of $\mathfrak{R}^H h$, the composition

$$Ext^1_{\mathbf{DM}_1(\mathbb{F})}(\mathbb{Z},\mathbb{G}_m) \xrightarrow{h} Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1},\mathbb{Q}(1)) \xrightarrow{\mathfrak{R}^H} Ext^1_{\mathbf{MHS}}(\mathbb{1},\mathbb{Q}(1))$$

is injective. (In fact, identifying $Ext^1_{\mathbf{DM}_1(\mathbb{F})}(\mathbb{Z},\mathbb{G}_m)$ with $\mathbb{F}^\times \otimes \mathbb{Q}$ and $Ext^1_{\mathbf{MHS}}(\mathbb{1},\mathbb{Q}(1))$ with $\mathbb{C}/2\pi i\mathbb{Q}$, the composition above is just given by the logarithm function.) Thus the special case (33) of Deligne's conjecture would imply that the Hodge realization map on $Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1},\mathbb{Q}(1))$ is injective.

We shall prove the following:

Theorem 5.2.1. *Assume that* $\mathbf{MM}(\mathbb{F})$ *is a tannakian category of mixed motives*[7] *over* $\mathbb{F}$ *with a* $\mathbb{Q}$*-linear functor* $h : \mathbf{DM}_1(\mathbb{F}) \to \mathbf{MM}(\mathbb{F})$ *which behaves well with respect to duals (in the sense mentioned above), and whose composition with the Hodge realization* $\mathfrak{R}^H : \mathbf{MM}(\mathbb{F}) \to \mathbf{MHS}$ *is the usual Hodge realization of 1-motives constructed by Deligne. Let* M *be a 1-motive over* $\mathbb{F}$*. Suppose that the Hodge realization map*

$$Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1},\mathbb{Q}(1)) \to Ext^1_{\mathbf{MHS}}(\mathbb{1},\mathbb{Q}(1))$$

is injective on the subgroup $Ext^1_{\langle h(M)\rangle^\otimes}(\mathbb{1},\mathbb{Q}(1))$ *of* $Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1},\mathbb{Q}(1))$*. Then*

$$\mathfrak{u}(\mathfrak{R}^H h(M)) = \mathfrak{R}^H \mathfrak{u}(h(M)). \tag{34}$$

Proof. Let $M = [\mathbb{Z}^n \xrightarrow{v} G]$, where G is a semiabelian variety over $\mathbb{F}$, an extension of an abelian variety A by a torus $\mathbb{G}_m^s$. Consider the blended extension in $\mathbf{DM}_1(\mathbb{F})$ given by the weight filtration on M. Its top row is the extension G of A by $\mathbb{G}_m^s$, and its right column is the extension $M/\mathbb{G}_m^s = [\mathbb{Z}^n \to A]$ (the map being the projection of v) of $\mathbb{Z}^n$ by A. The blended extensions to which the earlier results of the paper will be applied are $h(M)$ and $\mathfrak{R}^H h(M)$. Both of these blended extensions do indeed satisfy conditions (C1) and (C2) of §2.16. The equality (34) holds if and only if we have equalities of $\mathfrak{u}_{-1}$'s and $\mathfrak{u}_{-2}$'s.

Step one: We will show

$$\mathfrak{u}_{-1}(\mathfrak{R}^H h(M)) = \mathfrak{R}^H \mathfrak{u}_{-1}(h(M))$$

by comparing the descriptions of $\mathfrak{u}_{-1}(h(M))$ and $\mathfrak{u}_{-1}(\mathfrak{R}^H h(M))$ given by Theorem 3.7.1. Set $N = h(M/\mathbb{G}_m^s)$ and $L = h(G)$, the former an extension of $\mathbb{1}^n$ by $h(A)$ and the latter an extension of $h(A)$ by $\mathbb{Q}(1)^s$. Consistent with the notation used in §3, the corresponding extensions of $\mathbb{1}$ by

$$\underline{Hom}(\mathbb{1}^n, h(A)) \cong h(A^n) \quad \text{and} \quad \underline{Hom}(h(A),\mathbb{Q}(1)^s) \overset{(\dagger)}{\cong} h(A^{*s})$$

will be respectively denoted by $\mathscr{N}$ and $\mathscr{L}$. The identification (†) here uses the canonical isomorphism $h(A^*) \cong h(A)^\vee(1)$. Using the functoriality of the isomorphisms $h(-^*) \cong h(-)^\vee(1)$, it is not difficult to see that up to a sign, the extension $\mathscr{L}$ in $Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1}, h(A^{*s}))$ coincides with the element associated with $h(G^*)$.

We have an element

$$(\mathscr{L},\mathscr{N}) \in Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1}, h(A^{*s}) \oplus h(A^n)) \cong Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1}, h(A^{*s} \times A^n)).$$

[7]To be clear, in addition to the hypotheses explicitly mentioned in the statement, all that here is needed from $\mathbf{MM}(\mathbb{F})$ is the following: $\mathbf{MM}(\mathbb{F})$ is a filtered tannakian category over $\mathbb{Q}$ in the sense of [19], there is an exact faithful tensor $\mathbb{Q}$-linear functor $\mathfrak{R}^H : \mathbf{MM}(\mathbb{F}) \to \mathbf{MHS}$ preserving the weight filtration, and the images of abelian varieties under h are semisimple. Working in this generality, the object $\mathbb{Q}(1)$ of $\mathbf{MM}(\mathbb{F})$ is defined to be $h(\mathbb{G}_m)$.

By Theorem 3.7.1, the subobject $\mathfrak{u}_{-1}(h(M))$ of $h(A^{*s}\times A^n)$ is the intersection of the kernels of all the endomorphisms of $h(A^{*s}\times A^n)$ which annihilate $(\mathscr{L},\mathscr{N})$. Applying $\mathfrak{R}^H$ to this, we get

$$\mathfrak{R}^H\mathfrak{u}_{-1}(h(M)) = \bigcap_{\substack{\phi\in End(h(A^{*s}\times A^n))\\ \phi_*(\mathscr{L},\mathscr{N})=0}} \ker(\mathfrak{R}^H\phi). \tag{35}$$

On the other hand, applying Theorem 3.7.1 to $\mathfrak{R}^H h(M)$ we have

$$\mathfrak{u}_{-1}(\mathfrak{R}^H h(M)) = \bigcap_{\substack{\psi\in End(\mathfrak{R}^H h(A^{*s}\times A^n))\\ \psi_*(\mathfrak{R}^H\mathscr{L},\mathfrak{R}^H\mathscr{N})=0}} \ker(\psi). \tag{36}$$

It thus suffices to show that every endomorphism ψ of $\mathfrak{R}^H h(A^{*s}\times A^n)$ which annihilates $(\mathfrak{R}^H\mathscr{L},\mathfrak{R}^H\mathscr{N})$ is of the form $\mathfrak{R}^H\phi$ for some (unique) endomorphism ϕ of $h(A^{*s}\times A^n)$ that annihilates $(\mathscr{L},\mathscr{N})$.

By fullness of $\mathfrak{R}^H h$, we have isomorphisms

$$End_{\mathbf{DM}_1(\mathbb{F})}(A^{*s}\times A^n) \xrightarrow{h} End(h(A^{*s}\times A^n)) \xrightarrow{\mathfrak{R}^H} End(\mathfrak{R}^H h(A^{*s}\times A^n)). \tag{37}$$

Thus it suffices to argue that for every $\alpha\in End_{\mathbf{DM}_1(\mathbb{F})}(A^{*s}\times A^n)$, we have an implication

$$(\mathfrak{R}^H h(\alpha))_*(\mathfrak{R}^H\mathscr{L},\mathfrak{R}^H\mathscr{N})=0 \quad\Rightarrow\quad (h(\alpha))_*(\mathscr{L},\mathscr{N})=0. \tag{38}$$

There is a commutative diagram

$$\begin{array}{ccc} Ext^1_{\mathbf{DM}_1(\mathbb{F})}(\mathbb{Z},A^{*s}\times A^n) & \overset{h}{\hookrightarrow} & Ext^1_{\mathbf{MM}(\mathbb{F})}(\mathbb{1},h(A^{*s}\times A^n)) \\ & \searrow^{\mathfrak{R}^H h} & \downarrow{\scriptstyle\mathfrak{R}^H} \\ & & Ext^1_{\mathbf{MHS}}(\mathbb{1},\mathfrak{R}^H h(A^{*s}\times A^n)). \end{array} \tag{39}$$

Each endomorphism algebra in (37) acts by pushforwards on the corresponding Ext^1 group above. Moreover, the maps in (39) commute with the actions of these endomorphism algebras (as h and $\mathfrak{R}^H$ are exact). The implication (38) now follows, since the extension $(\mathscr{L},\mathscr{N})$ is in the image of $Ext^1_{\mathbf{DM}_1(\mathbb{F})}(\mathbb{Z},A^{*s}\times A^n)$.

Step two: We now show that

$$\mathfrak{u}_{-2}(\mathfrak{R}^H h(M)) = \mathfrak{R}^H\mathfrak{u}_{-2}(h(M)). \tag{40}$$

By the previous step and Proposition 4.4.1 (and the fact that $\mathfrak{R}^H$ takes the bracket $\{,\}$ of $h(M)$ to the counterpart for $\mathfrak{R}^H h(M)$), we have

$$[\mathfrak{u}(\mathfrak{R}^H h(M)),\mathfrak{u}(\mathfrak{R}^H h(M))] = \mathfrak{R}^H[\mathfrak{u}(h(M)),\mathfrak{u}(h(M))].$$

Thus we have an injection

$$\frac{\mathfrak{u}_{-2}(\mathfrak{R}^H h(M))}{[\mathfrak{u}(\mathfrak{R}^H h(M)),\mathfrak{u}(\mathfrak{R}^H h(M))]} \overset{i}{\hookrightarrow} \frac{\mathfrak{R}^H\mathfrak{u}_{-2}(h(M))}{\mathfrak{R}^H[\mathfrak{u}(h(M)),\mathfrak{u}(h(M))]}.$$

The isomorphisms of Proposition 4.6.1 (or rather, its proof with $\mathbb{Q}(1)$ replacing $\underline{Hom}(A_3, A_1)$) for $h(M)$ and $\mathfrak{R}^H h(M)$ give rise to a commutative diagram

$$\begin{array}{ccc}
Ext^1_{\langle h(M)\rangle^{\otimes}}(\mathbb{1},\mathbb{Q}(1)) & \xrightarrow{\ \simeq\ } & Hom\left(\dfrac{\mathfrak{u}_{-2}(h(M))}{[\mathfrak{u}(h(M)),\mathfrak{u}(h(M))]},\mathbb{Q}(1)\right) \\
 & & \Big\downarrow{\scriptstyle \mathfrak{R}^H}\ {\scriptstyle\simeq} \\
\Big\downarrow{\scriptstyle \mathfrak{R}^H} & & Hom\left(\dfrac{\mathfrak{R}^H\mathfrak{u}_{-2}(h(M))}{\mathfrak{R}^H[\mathfrak{u}(h(M)),\mathfrak{u}(h(M))]},\mathbb{Q}(1)\right) \\
 & & \Big\downarrow{\scriptstyle i^*} \\
Ext^1_{\langle \mathfrak{R}^H h(M)\rangle^{\otimes}}(\mathbb{1},\mathbb{Q}(1)) & \xrightarrow{\ \simeq\ } & Hom\left(\dfrac{\mathfrak{u}_{-2}(\mathfrak{R}^H h(M))}{[\mathfrak{u}(\mathfrak{R}^H h(M)),\mathfrak{u}(\mathfrak{R}^H h(M))]},\mathbb{Q}(1)\right)
\end{array}$$

The Hodge realization map on the right is an isomorphism since $\mathfrak{u}_{-2}(h(M))/[\mathfrak{u}(h(M)),\mathfrak{u}(h(M))]$ is a subquotient of $\underline{Hom}(\mathbb{1}^n,\mathbb{Q}(1)^s)$ and hence is a direct sum of copies of $\mathbb{Q}(1)$. The pullback map i^* is surjective because i admits a retraction (by semisimplicity). The Hodge realization map on the left is crucially assumed to be injective in the statement of the theorem. It follows that i^* is injective, so that

$$Hom\left(\frac{\mathfrak{R}^H\mathfrak{u}_{-2}(h(M))}{\mathfrak{u}_{-2}(\mathfrak{R}^H h(M))},\mathbb{Q}(1)\right)=0.$$

But $\dfrac{\mathfrak{R}^H\mathfrak{u}_{-2}(h(M))}{\mathfrak{u}_{-2}(\mathfrak{R}^H h(M))}$ is a direct sum of copies of $\mathbb{Q}(1)$, so that we get (40). □

Remark 5.2.2. We should mention that prior to [2], Jossen had proved that for 1-motives over $\mathbb{C}$, the unipotent radicals of the motivic Galois and Mumford-Tate groups coincide (see [27, Theorem 6.2]). (Jossen has a concrete geometric construction for the unipotent radical of the motivic Galois group of a 1-motive. But the object he constructs does indeed agree with the more abstract one coming from tannakian formalism. See the appendix of [27].)

Remark 5.2.3. The argument in the first step of the proof of Theorem 5.2.1 proves, in fact, the following more precise statement: Let $\mathbf{T}$ be any tannakian category over $\mathbb{Q}$ with an exact faithful $\mathbb{Q}$-linear functor $h : \mathbf{DM}_1(\mathbb{F}) \to \mathbf{T}$ such that $\xi := h(\mathbb{G}_m)$ is invertible and there is an isomorphism of functors $h(-^*) \cong h(-)^\vee \otimes \xi$. Let $M = [\mathbb{Z}^n \to G]$ be a 1-motive over $\mathbb{F}$, with G an extension of an abelian variety A by $\mathbb{G}_m^s$. Suppose the points $\underline{P} = (P_1,\dots,P_n) \in A^n(\mathbb{F})$ and $\underline{Q} = (Q_1,\dots,Q_s) \in A^{*s}(\mathbb{F})$ describe the extensions $M/\mathbb{G}_m^s$ of $\mathbb{Z}^n$ by A and G of A by $\mathbb{G}_m^s$. Let B be the smallest abelian subvariety of $A^{*s} \times A^n$ over $\mathbb{F}$ that contains a nonzero multiple of the point $(\underline{Q},\underline{P}) \in A^{*s} \times A^n$. If $h(A)$ is semisimple and the map $End_{\mathbf{DM}_1(F)}(A \times A^*) \xrightarrow{h} End(h(A \times A^*))$ is an isomorphism, then

$$\mathfrak{u}_{-1}(h(M)) = h(B).$$

Note that here no assumption on extensions of $\mathbb{1}$ by $h(\mathbb{G}_m)$ in $\mathbf{T}$ were made.

This fact may shed some further light on the weight -1 graded part of the geometric constructions of the unipotent radical of the motivic Galois group of a 1-motive due to Bertolin [6] and Jossen [27].

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