Special values of the Riemann zeta function - a journey from concrete to abstract

Payman Eskandari
University of Winnipeg, Nov 2023

Chapter 1: The concrete - I

L. Euler (1707-1783)

- $$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \begin{cases} \text{converges if } \alpha > 1 \\ \text{diverges if } \alpha = 1 \end{cases}$$

- First analytic proof of the infinitude of primes (Euler):

$$\prod_{p=prime} \frac{1}{1 - 1/p} = \prod_{p=prime} (1 + 1/p + 1/p^2 + \cdots) = \underbrace{\sum_{n=1}^{\infty} 1/n}_{<\infty \text{ if } \#\text{ primes} < \infty}$$

- Euler (around 1740):

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Euler's proof:

- $$\frac{\sin x}{x} = 1 - x^2/3! + x^4/5! - \cdots$$
- $$\frac{\sin(x)}{x} = (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi})\cdots$$
$$= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2})\cdots$$
- Compare coefficients of $1/x^2$:

$$-1/3! = -(1/\pi^2)(1 + 1/4 + 1/9 + \cdots)$$

Definition:

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad (\alpha > 1)$$

- Euler:

$$\zeta(2) = \pi^2/6 \qquad \zeta(4) = \pi^4/90 \qquad \zeta(6) = \pi^6/945$$

$$\vdots$$

$$\boxed{\zeta(2k) = \pi^{2k} \cdot (\text{a rational number})}$$

- Lambert (1760): π is irrational.

- Lindemann (1882): π is transcendental (i.e. there is no nonzero $f \in \mathbb{Q}[x]$ such that $f(\pi) = 0$).

Corollary: $\zeta(2k)$ is transcendental.

A different picture : Odd zeta values

- Almost nothing known about transcendence/irrationality.

- Likely expectation in modern times: $\{\pi, \zeta(3), \zeta(5), \zeta(7), \ldots\}$ should be algebraically independent (i.e. $\nexists$ nonzero $f \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$ such that $f(\pi, \zeta(3), \zeta(5), \ldots) = 0$).

- Apéry (1978): $\zeta(3)$ is irrational.

- We dont even Know irrationality of $\zeta(5)$. Nothing is known about transcendence of any odd zeta value.

Chapter 2:
The concrete - II

B. Riemann (1826-1866)

- Distribution of primes: $\pi(x) = \#\{p \leq x : \ p = \text{prime}\}$

$\boxed{\text{Prime number theorem (Hadamard, de la Valée Poussin, 1896): } \lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1}$

- Riemann's visionary work (1859)

Consider ζ as a function of a complex variable:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (Re(s) > 1)$$

Analytic on the half plane $\{s \in \mathbb{C} : Re(s) > 1\}$

Riemann's visionary work (cont.)
Analytic continuation and functional equation

Riemann (1859): $\zeta(s)$ extends to an analytic function on $\mathbb{C}\backslash\{1\}$ with a single pole of order 1 at $s = 1$, and satisfies a functional equation relating $\zeta(s)$ and $\zeta(1-s)$:

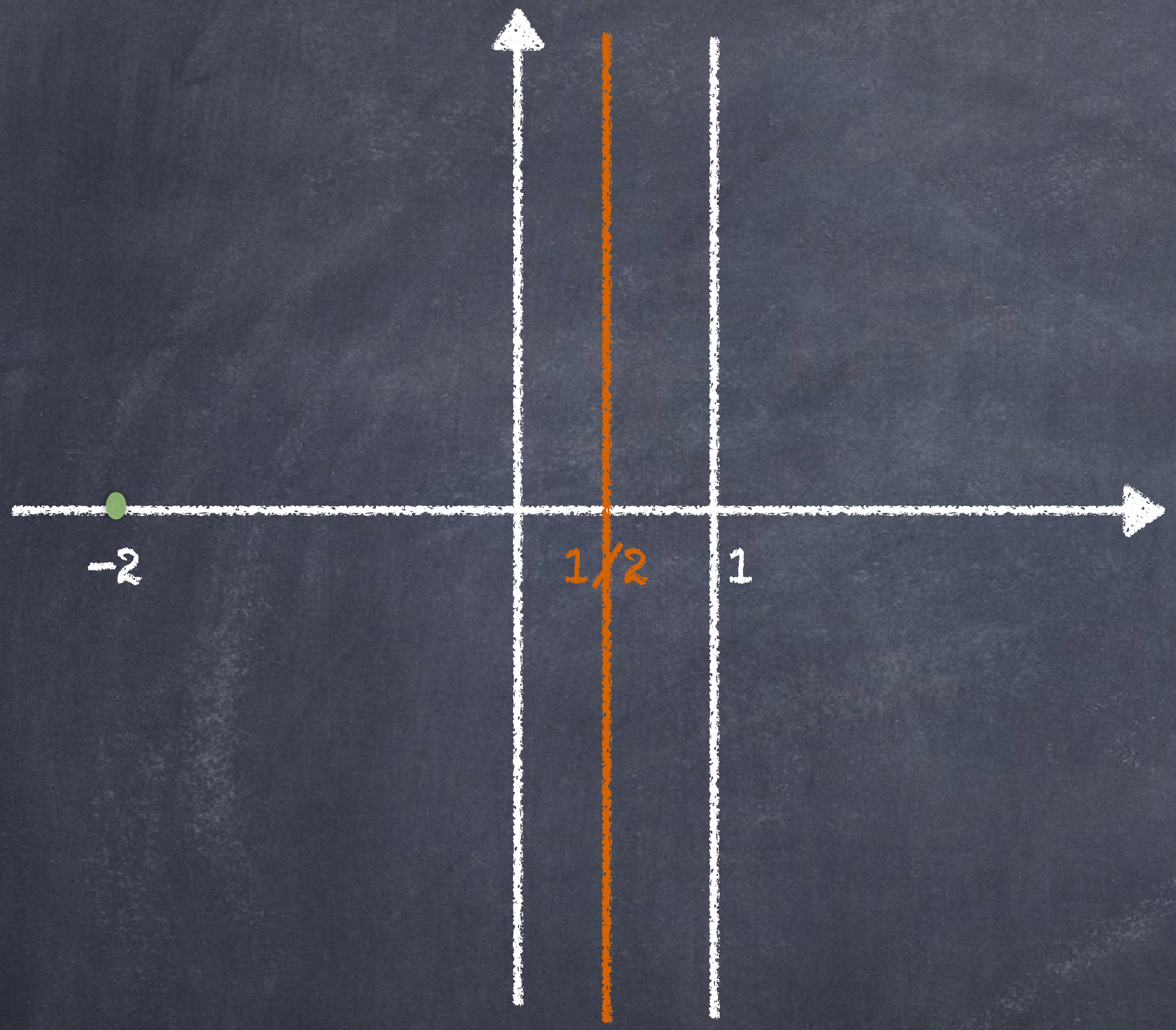
- 1) $\zeta(s) = \dfrac{s}{s-1} - s\displaystyle\int_1^\infty \frac{\{x\}}{x^{s+1}}dx \Rightarrow \zeta$ extends to an analytic function on $\{s : Re(s) > 0\}\backslash\{1\}$ with a single pole at 1.

- 2) Set $\Lambda(s) := \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ where $\Gamma(s) = \displaystyle\int_0^\infty x^{s-1}e^{-x}dx$ for Re(s)>0. Then:

$$\boxed{\Lambda(s) = \Lambda(1-s)}$$

(Λ is called the completed zeta function.)

$$\boxed{\Lambda(s) = \Lambda(1-s)}$$



Corollary: Λ extends to an analytic function on $\mathbb{C}\setminus\{0,1\}$ with single poles at 0,1. It has no zeros outside the critical strip $\{s : 0 \leq Re(s) \leq 1\}$.

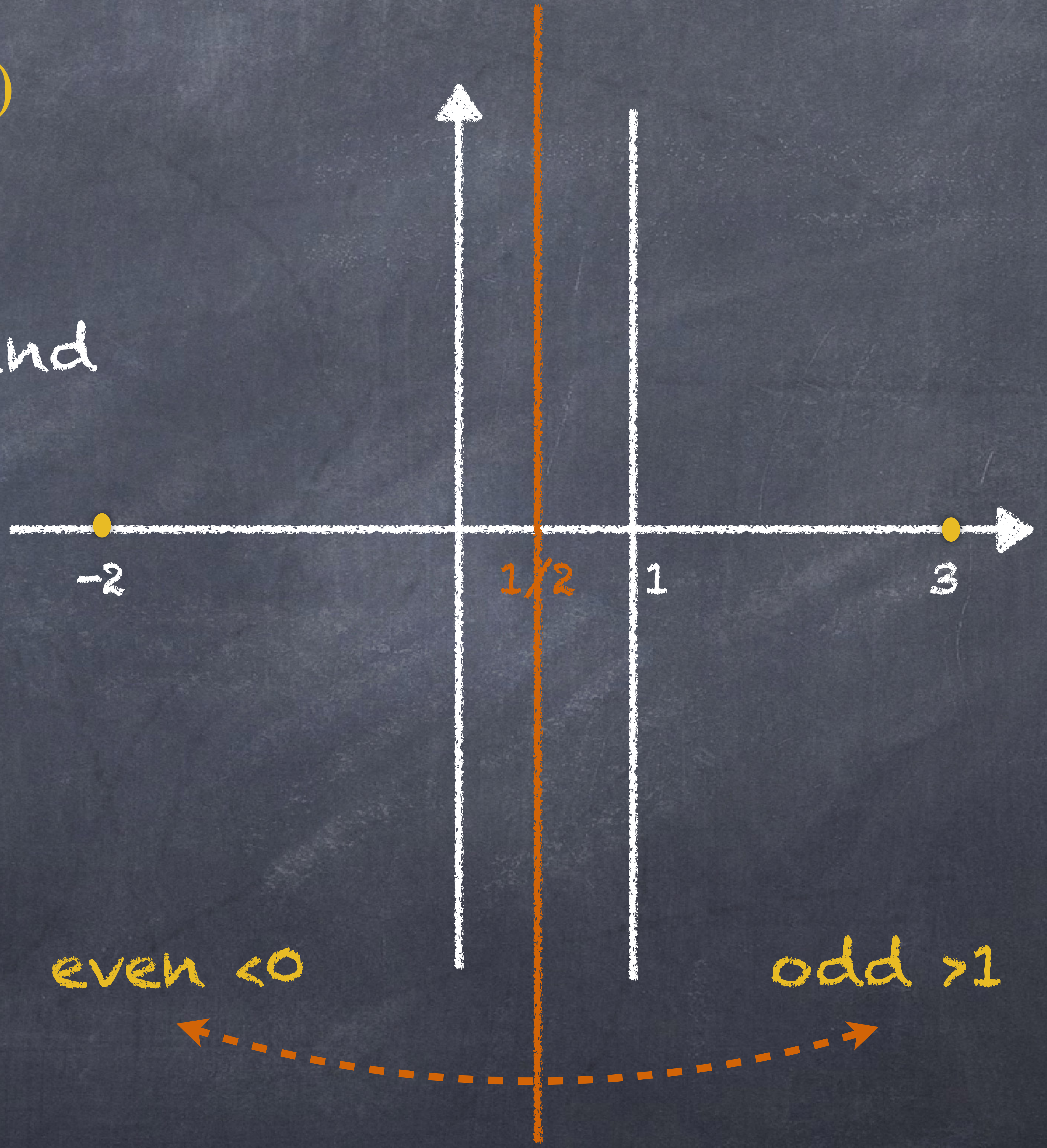
ζ extends to an analytic function on $\mathbb{C}\setminus\{1\}$ with a single pole at 1. It has zeros of order 1 at negative even integers. These are the only zeros of ζ outside the critical strip $\{s : 0 \leq Re(s) \leq 1\}$.

(Riemann Hypothesis: All other zeros are on the line Re(s)=1/2. (Riemann proved the PNT assuming this.))

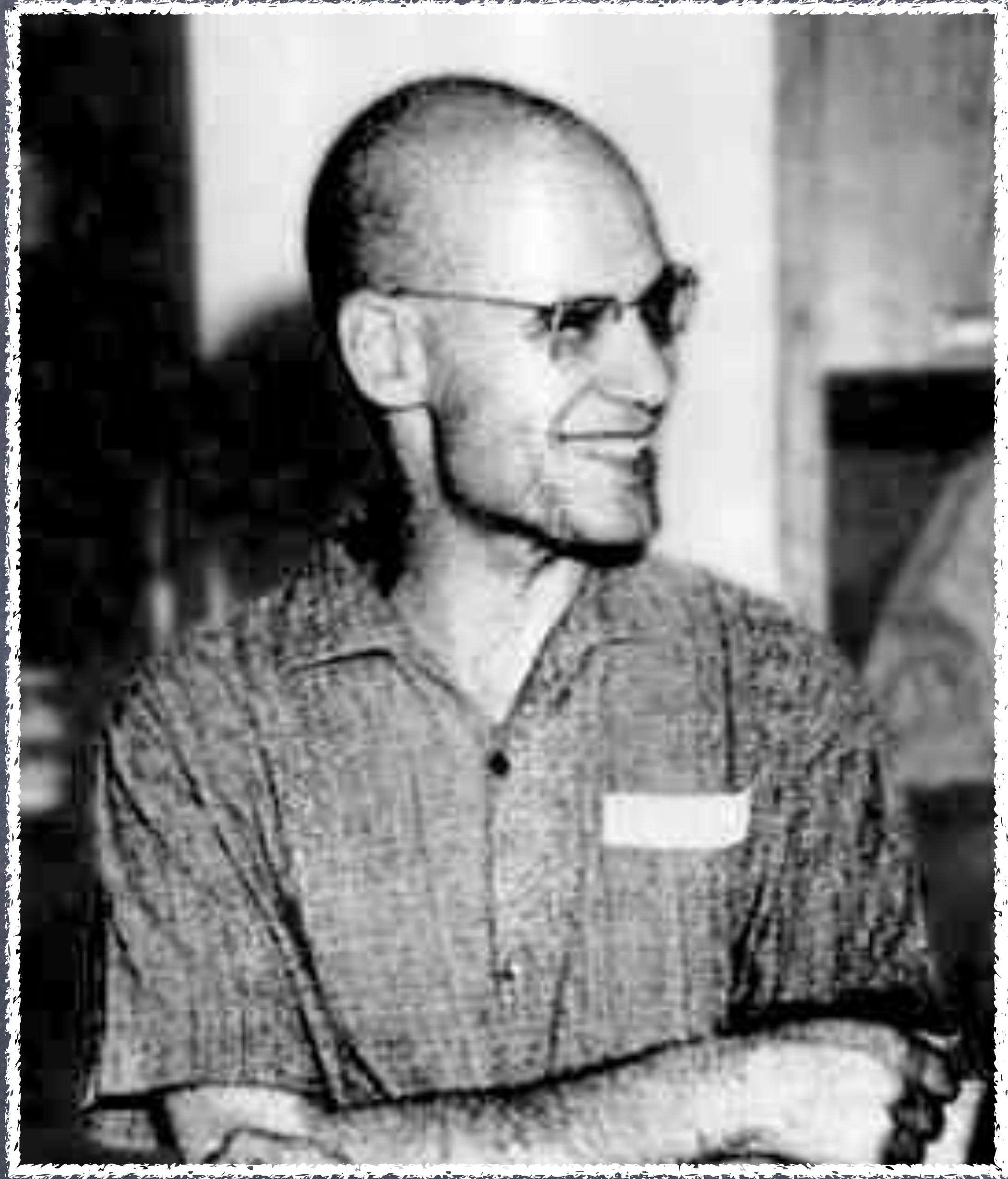
Summary: Two mysterious phenomena:

A. Contrast in the situations for $\zeta(\text{odd} > 1)$ and $\zeta(\text{even} > 1)$.

B. $\zeta(\text{even} < 0) = 0$ with a zero of order 1, and $\zeta(\text{odd} < 0) \neq 0$.



Chapter 3: The abstract



A. Grothendieck
(1928-2014)

P. Deligne
(born in 1944)

- An affine variety over $\mathbb{Q}$ is the zero set of a collection of polynomials with rational coefficients:

$$\{z \in \mathbb{C}^n : f_1(z) = \cdots = f_k(z) = 0\} \qquad (f_1, \ldots, f_k \in \mathbb{Q}[z_1, \ldots, z_n])$$

- A variety over $\mathbb{Q}$ is obtained by "patching together" affine varieties.

- Examples: (1) $\mathbb{C}^n$ (2) $\mathbb{C}^\times = \mathbb{C} - \{0\} \cong \{(x, y) \in \mathbb{C}^2 : xy = 1\}$

(3) the projective line: $\mathbb{P}^1 = \mathbb{C}^2 - \{0\}/(z \sim z'$ iff $z' = cz$ for some $c \in \mathbb{C})$

(4) elliptic curves $\{(x, y) : y^2 = x^3 + ax + b\}$ $(a, b \in \mathbb{Q})$

(5) Fermat curves $\{(x, y) : x^n + y^n = 1\}$

X

(Var./$\mathbb{Q}$)

Betti (singular) cohomology $H^n_B(X)$ — V. S./ $\mathbb{Q}$

de Rham cohomology $H^n_{dR}(X)$ — V. S./ $\mathbb{Q}$

l-adic cohomology $H^n_l(X)$ — V. S./ $\mathbb{Q}_l$

$\vdots$

Vector spaces with additional structure

Example 1) For all connected X, $H^0_B(X) \cong H^0_{dR}(X) \cong \mathbb{Q}$

Example 2) $\mathbb{C}^\times = \mathbb{C} \backslash \{0\}$

dim $H^1_B(\mathbb{C}^\times)$ = dim $H^1_{dR}(\mathbb{C}^\times)$ = 1

$H_1^B(\mathbb{C}^\times)$ = spanned by 

$H^1_{dR}(\mathbb{C}^\times)$ spanned by dz/z

Paired by integration: $\underbrace{\int_{|z|=1} dz/z = 2\pi i}$

A period of $\mathbb{C}^\times$

Philosophy of motives (Grothendieck's dream)

Alg. Geometry

Various non-alg. geometric linearizations

Var./$\mathbb{Q}$

X

Betti (singular) cohomology $H^n_B(X)$

de Rham cohomology $H^n_{dR}(X)$

l-adic cohomology $H^n_l(X)$

$\vdots$

Periods

Philosophy of motives (Grothendieck's dream)

Alg. Geometry

Various non-alg. geometric linearizations

$$\begin{array}{ccc} \text{Var.}/\mathbb{Q} & \xrightarrow{\text{linearization}} & \text{Motives} \\ X & & H^n(X) \end{array}$$

(realizations)

- Betti (singular) cohomology $H^n_B(X)$
- de Rham cohomology $H^n_{dR}(X)$
- l-adic cohomology $H^n_l(X)$
- $\vdots$

$\left.\begin{array}{l} H^n_B(X) \\ H^n_{dR}(X) \end{array}\right\} \rightsquigarrow$ Periods

The category of motives should be abelian (there should be kernels and cokernels).

Nowadays, we finally have non-conjectural geometric constructions of the category of motives (Ayoub '14, Nori '00s).

Extensions

- $\mathscr{A}$ = an abelian category (e.g. category of abelian groups)
- $B \subset E \quad \rightsquigarrow \quad B \overset{i}{\hookrightarrow} E \overset{q}{\twoheadrightarrow} E/B$ with $\mathrm{Im}(i) = \mathrm{Ker}(q)$
- Given A and B, an **extension of A by B** is diagram $B \overset{i}{\hookrightarrow} E \overset{q}{\twoheadrightarrow} A$ such that $\mathrm{Im}(i) = \mathrm{Ker}(q)$ (and i is injective And q is surjective).
- Equivalence of extensions: Two extensions E and E′ of A by B are called equivalent if there is a map E -> E′ fitting into a commutative diagram:

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\ B \arrow[r, hook] & E' \arrow[r, two heads] & A \end{tikzcd}$$

Extensions (cont.)

- Ext(A,B) := the set of equivalence classes of extensions of A by B

- Theorem (Baer '34, Cartan-Eilenberg '42, Yoneda '60): There is a natural binary operation under which Ext(A,B) is an abelian group.

  The identity element of Ext(A,B) is the extension class of

$$B \hookrightarrow B \oplus A \twoheadrightarrow A$$

- In our case of interest, Ext groups will be vector spaces.

Back to motives

- 1-dimensional motives: for each integer n, there is a 1-dimensional motive $\mathbb{Q}(n)$.

$\mathbb{Q}(-1) := H^1(\mathbb{C}^\times)$; period $2\pi i$

$\mathbb{Q}(-n) := \mathbb{Q}(-1)^{\otimes n} = H^{2n}(\mathbb{P}^n)$; period $(2\pi i)^n$

$\mathbb{Q}(0) = \mathbf{1}$ = the unit object = $H^0(\mathbb{C})$

- Theorem (Borel '72, Soulé '78, Voevodsky/Levine): Let n>0. Then

$$\dim \mathrm{Ext}(\mathbf{1}, \mathbb{Q}(n)) = \begin{cases} 1 & n=3,5,7,\ldots \\ 0 & n=2,4,6,\ldots \end{cases}$$

Chapter 4: From the abstract to the concrete

A. Beilinson
(born 1957)

S. Bloch
(born 1944)

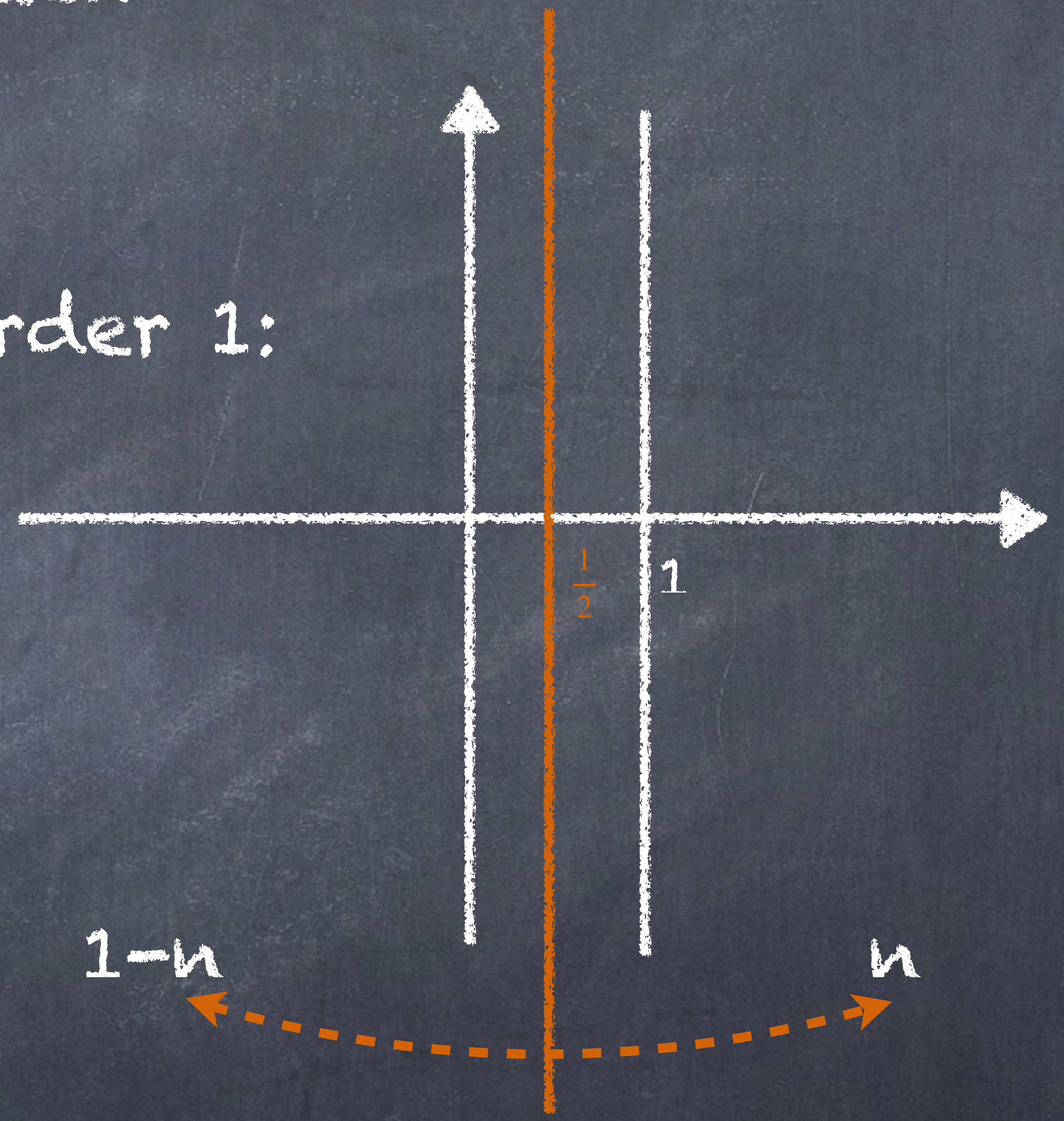
We had 2 phenomena:

A) $\zeta(2k) \underset{\mathbb{Q}^\times}{\sim} \pi^{2k}$ for k>0, while $\zeta(2k+1)$ (k>0) seemed much more mysterious.

(Up to a rational factor)

B) $\zeta(\text{odd} < 0) \neq 0$, while $\zeta(\text{even} < 0)=0$ with a zero of order 1:

$$\mathrm{ord}_{s=1-n}\,\zeta(s) = \begin{cases} 1 & n=3,5,7,\ldots \\ 0 & n=2,4,6,\ldots \end{cases}$$



Borel's theorem restated:

$$\mathrm{ord}_{s=1-n}(\zeta(s)) = \dim \mathrm{Ext}(\mathbf{1}, \mathbb{Q}(n))$$

Beilinson's conjecture (around '82) - p. I:

Let M=$H^i(X)$ for a smooth projective variety X over $\mathbb{Q}$.

Then for i-2n<-2,

$\mathrm{ord}_{s=i+1-n} L(M,s) = \dim \mathrm{Ext}(\mathbf{1}, M(n))$.

→ In the background of B

→ Modified statements for i-2n $\in \{-2,-1\}$

Borel's theorem verifies this for the trivial motive $\mathbf{1}$.

Beilinson's conjecture: With M, n as above, the "higher regulator map"

$$\mathrm{Ext}(\mathbf{1}, M(n)) \otimes \mathbb{R} \to \mathrm{Ext}_{\mathscr{M}\mathscr{H}^+_{\mathbb{R}}}(\mathbf{1}, M(n))$$

is an isomorphism and its determinant with respect to natural rational structures is up to a rational factor equal to the leading coefficient of the Taylor expansion of L(M,s) as s=i+1-n.

In background of $\zeta(2k) \sim_{\mathbb{Q}^\times} \pi^{2k}$: $\mathrm{Ext}(\mathbf{1}, \mathbb{Q}(-2k)) = 0 \Rightarrow \det(r) = 1 \Rightarrow \zeta(1-2k) \in \mathbb{Q} \underset{\text{F. Eq.}}{\Rightarrow} \zeta(2k) \sim_{\mathbb{Q}^\times} \pi^{2k}$

BC known for very few cases: X = a single point (<-> Dedekind zeta functions) by work of Borel in '70s.

Grothendieck Period conjecture

- We havent yet explained half of Phenomenon A.
- Motivic Galois Groups: The category of motives is actually tannakian (has tensor products and dual objects).

In any tannakian category: Each object $\rightsquigarrow$ An alg. group

The group attached to a motive M via tannakian formalism is called the motivic Galois group of M.

Grothendieck period conjecture (mid-late '60s): For every motive M over $\mathbb{Q}$,

$$\dim \mathrm{Gal}^{mot}(M) = \text{transcendence degree of field generated by periods of } M.$$

- Recall: $\dim \mathrm{Ext}(\mathbf{1},\mathbb{Q}(n)) = 1$ for n=3,5,7,...
- Theorem (Deligne '89): For odd n >1, the nontrivial extension of $\mathbf{1}$ by $\mathbb{Q}(n)$ has a period matrix

$$\begin{bmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n) \\ 0 & 1 \end{bmatrix}.$$

- Corollary: Let n>1 be odd.

GPC for the nontrivial extension of $\mathbf{1}$ by $\mathbb{Q}(n)$ $\Longrightarrow$ $\{\pi, \zeta(n)\}$ is algebraically independent.

Follows by an easy argument using the machinery of tannakian categories and unipotent radicals of Galois groups.

Pf: Since the extension is nontrivial, dim $\mathrm{Gal}^{\mathrm{mot}}$ = 2. Combining with Deligne's theorem we get the implication.

Some recent work

- Ext groups in categories of motives $\longleftrightarrow$ unipotent radicals of motivic Galois groups.
- Unipotent radicals of motivic Galois groups have been studied previously by Deligne, Bertrand, Hardouin, Bertolin, Jossen, etc.
- Recent work (PJM '23, ANT '23 with K. Murty and a recent preprint on arXiv) further studies unipotent radicals of motivic Galois groups.
- Theorem (E.): Fix integers a,b,c>1 with b,c distinct and odd. There is a family of motives with period matrices of the form

$$\begin{bmatrix} (2\pi i)^{-b-c-1} & (2\pi i)^{-b-c-1}\zeta(c) & *_{1,3} & *_{1,4} \\ 0 & (2\pi i)^{-b-1} & (2\pi i)^{-b-1}\zeta(b) & *_{2,4} \\ 0 & 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log(a) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and 7-dimensional motivic Galois groups. In particular, if GPC holds, then $\{\pi, \zeta(b), \zeta(c), \log(a), *_{1,3}, *_{2,4}, *_{1,4}\}$ is algebraically independent.

Historical note: In 1785 Euler speculated that perhaps there are $\alpha, \beta \in \mathbb{Q}$ such that

$$\zeta(3) = \alpha(\log 2)^3 + \beta\pi^2 \log 2.$$

GPC would imply that this speculation is false.

Thank you! :)