Algebraic Cycles, Fundamental Group of a Punctured Curve, and Applications in Arithmetic

by

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Abstract

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The results of this thesis can be divided into two parts, geometric and arithmetic. Let X be a smooth projective curve over $\mathbb{C}$, and $e, \infty \in X(\mathbb{C})$ be distinct points. Let L_n be the mixed Hodge structure of functions on $\pi_1(X - \{\infty\}, e)$ given by iterated integrals of length $\leq n$ (as defined by Hain). In the geometric part, inspired by a work of Darmon, Rotger, and Sols [6], we express the mixed Hodge extension $\mathbb{E}^{\infty}_{n,e}$ given by the weight filtration on $\dfrac{L_n}{L_{n-2}}$ in terms of certain null-homologous algebraic cycles on X^{2n-1}. These cycles are constructed using the diagonal embeddings of X^{n-1} into X^n.

The arithmetic part of the thesis gives some number-theoretic applications of the geometric part. We assume that $X = X_0 \otimes_K \mathbb{C}$ and $e, \infty \in X_0(K)$, where K is a subfield of $\mathbb{C}$ and X_0 is a projective curve over K. Let Jac be the Jacobian of X_0. We use the extension $\mathbb{E}^{\infty}_{n,e}$ to associate to each $Z \in CH_{n-1}(X_0^{2n-2})$ a point $P_Z \in \mathrm{Jac}(K)$, which can be described analytically in terms of iterated integrals. The proof of K-rationality of P_Z uses that the algebraic cycles constructed in the first part are defined over K. Assuming a certain plausible hypothesis on the Hodge filtration on L_n holds, we show that an algebraic cycle $Z \in CH_{n-1}(X_0^{2n-2})$ for which P_Z is torsion, gives rise to relations between periods of L_2. Interestingly, these relations are non-trivial even when one takes Z to be the diagonal in X_0^2. In the elliptic curve case, we show unconditionally that a certain relation between periods of L_2 (which is induced by the diagonal in X_0^2) exists if and only if $e - \infty$ is torsion.

The geometric result of the thesis in $n = 2$ case, and the fact that one can asso-

ciate to $\mathbb{E}_{2,e}^{\infty}$ a family of points in $\mathrm{Jac}(K)$, are due to Darmon, Rotger, and Sols [6]. Our contribution is in generalizing the picture to higher weights.

To my parents, my brother Payam, and my grandmother

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Contents

Chapter 1

Introduction

The theory of the fundamental group in algebraic topology is strikingly beautiful and simple. Things become even more beautiful once one leaves the category of topological spaces and goes into categories whose objects carry more structure. Similar to the cohomology situation, the theory of the fundamental group becomes extremely rich once one considers varieties. In fact, now there are various *compatible* versions of the fundamental group, each of which is a shadow, or a *realization*, of a single more abstract object. This object, or rather the collection of its realizations together with their compatibility data, is a wonderful source of arithmetic and geometric information. The theory now is no longer simple as it was in the primitive topological setting, but it is very powerful with deep connections and applications to various parts of mathematics.

In this thesis, we will only work with the so called Hodge realization of the fundamental group. In very rough terms, to a smooth[†] complex variety U and a point $e \in U(\mathbb{C})$, one can associate a (or rather a direct system of) mixed Hodge structure(s). More precisely, thanks to the works of Chen, Hain, Deligne, Morgan and others one has, for each n, a mixed Hodge structure $L_n(U, e)$ with integral lattice

$$\left(\frac{\mathbb{Z}[\pi_1(U, e)]}{I^{n+1}}\right)^{\vee},$$

where $I \subset \mathbb{Z}[\pi_1(U, e)]$ is the augmentation ideal. The filtrations (Hodge and weight)

[†]The smoothness assumption is not necessary for some of the statements.

are defined using the characterization of

$$\left(\frac{\mathbb{C}[\pi_1(U,e)]}{I^{n+1}}\right)^{\vee}, \tag{1.1}$$

where I is again the augmentation ideal, as the space of closed (i.e. homotopy invariant) iterated integrals of length $\leq n$ on U. One has

$$L_1(U,e) \simeq \mathbb{Z}(0) \oplus H^1(U),$$

but the $L_n(U,e)$ are more complicated for $n > 1$. In particular, they may not be pure even if U is projective.

There are two aspects of the Hodge realization of the fundamental group that are of particular interest to us:

1. Connections to null-homologous algebraic cycles: Over the past few decades, a number of connections have been found between the Hodge theory of the fundamental group and null-homologous algebraic cycles. See for instance [21], [25], [5], and the expository paper [19]. More recently, Darmon, Rotger, and Sols in [6] considered the extension

$$0 \to \frac{L_1}{L_0}(U,e) \to \frac{L_2}{L_0}(U,e) \to \frac{L_2}{L_1}(U,e) \to 0,$$

where U is obtained from a smooth projective curve X over a subfield $K \subset \mathbb{C}$ by removing a K-rational point, and $e \in U(K)$. They related this extension to the modified diagonal cycle of Gross, Kudla, and Schoen in X^3. Using this relation they were able to define a family of rational points on the Jacobian of X parametrized by algebraic cycles in X^2. One of the primary goals of this thesis is to generalize this picture to higher weights. We will discuss this in more detail shortly.

2. Periods: Similar to the cohomology case, if U and e are defined over a subfield $K \subset \mathbb{C}$, $L_n(U,e)$ is endowed with a *de Rham lattice*, which is a K-lattice inside (1.1). One then has a K-vector space of *periods* of $L_n(U,e)$, which contains the periods of U if $n \geq 1$. The new phenomenon here is that because of a formal property of iterated integrals, namely the so called *shuffle product*, periods of $\cup L_n(U,e)$ that correspond to the same path in $\pi_1(U,e)$, are closed under multiplication, and form a K-subalgebra of $\mathbb{C}$. One may refer to the periods of $\cup L_n(U,e)$ also as the periods of $\pi_1(U,e)$. The celebrated multiple zeta values arise as periods of π_1 of $\mathbb{P}^1 - \{0,1,\infty\}$.

We proceed to give a review of the results of the thesis. The work can be divided into two parts, geometric and arithmetic. Before we discuss the contents of each part, let us fix some notation. We use $\mathrm{CH}_i(-)$ for Chow groups. (As usual, the subscript is the dimension.) By $\mathrm{CH}_i^{\hom}(-)$ we mean the subgroup of $\mathrm{CH}_i(-)$ consisting of homologically trivial cycles. We denote by $\underline{\mathrm{Hom}}$ the internal Hom in the category of mixed Hodge structures, and for a pure Hodge structure A of odd weight $2k-1$, by JA we refer to the "middle" Carlson Jacobian

$$JA := \frac{A_{\mathbb{C}}}{F^k A_{\mathbb{C}} + A_{\mathbb{Z}}},$$

where $F^{\cdot}$ denotes the Hodge filtration. For instance, if $A = H^{2k-1}(U)$ for a smooth projective complex variety U, JA is nothing but the Griffiths' intermediate Jacobian.

From now on, X is a smooth (connected) projective curve over $\mathbb{C}$. Let $e, \infty \in X(\mathbb{C})$ be distinct. We write H^1 for $H^1(X)$, the mixed Hodge structure associated to the degree one cohomology of X.

1. Geometric part: (Chapter 3) Darmon, Rotger and Sols in [6] relate the extension $\mathbb{E}_{2,e}^{\infty}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{L_1}{L_0}(X-\{\infty\},e) & \longrightarrow & \frac{L_2}{L_0}(X-\{\infty\},e) & \longrightarrow & \frac{L_2}{L_1}(X-\{\infty\},e) & \longrightarrow & 0, \\ & & \wr\| & & & & \wr\| & & \\ & & H^1 & & & & (H^1)^{\otimes 2} & & \end{array}$$

to the modified diagonal cycle of Kudla, Gross and Schoen[†]

$$\begin{aligned} \Delta_{2,e} &:= \{(x,x,x) : x \in X\} - \{(e,x,x) : x \in X\} - \{(x,e,x) : x \in X\} - \{(x,x,e) : x \in X\} \\ &+ \{(e,e,x) : x \in X\} + \{(e,x,e) : x \in X\} + \{(x,e,e) : x \in X\} \in \mathrm{CH}_1^{\hom}(X^3) \end{aligned}$$

and the cycle

$$Z_{2,e}^{\infty} := \{(x,x,\infty) : x \in X\} - \{(x,x,e) : x \in X\} \in \mathrm{CH}_1^{\hom}(X^3).$$

[†]The reason for this non-standard choice of notation will be clear shortly.

Let h_2 be the composition

$$\mathrm{CH}_1^{\mathrm{hom}}(X^3) \xrightarrow{\text{Abel-Jacobi}} \mathrm{J}\underline{\mathrm{Hom}}(H^3(X^3), \mathbb{Z}(0)) \xrightarrow{\text{Kunneth}} \mathrm{J}\underline{\mathrm{Hom}}((H^1)^{\otimes 3}, \mathbb{Z}(0)), \tag{1.2}$$

and identify

$$\mathrm{Ext}((H^1)^{\otimes 2}, H^1) \cong \mathrm{J}\underline{\mathrm{Hom}}((H^1)^{\otimes 2}, H^1) \cong \mathrm{J}\underline{\mathrm{Hom}}((H^1)^{\otimes 2} \otimes H^1, \mathbb{Z}(0)),$$

where the first isomorphism is that of Carlson [1], and the second is given by Poincare duality. Theorem 2.5 of [6] asserts[†] that

$$\mathbb{E}_{2,e}^{\infty} = h_2(-\Delta_{2,e} + Z_{2,e}^{\infty}). \tag{1.3}$$

In Chapter 3, our goal is to generalize this result to higher weights. For each $n \geq 2$, we consider the extension $\mathbb{E}_{n,e}^{\infty}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{L_{n-1}}{L_{n-2}}(X-\{\infty\}, e) & \longrightarrow & \dfrac{L_n}{L_{n-2}}(X-\{\infty\}, e) & \longrightarrow & \dfrac{L_n}{L_{n-1}}(X-\{\infty\}, e) & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & (H^1)^{\otimes n-1} & & & & (H^1)^{\otimes n} & & \end{array}$$

of mixed Hodge structures as an element of $\mathrm{Ext}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right)$. One can show that the weight filtration on

$$\frac{L_n}{L_{n-2}}(X-\{\infty\}, e)$$

is given by

$$W_{n-2} = 0, \quad W_{n-1} = \frac{L_{n-1}}{L_{n-2}}(X-\{\infty\}, e), \quad \text{and} \quad W_n = \frac{L_n}{L_{n-2}}(X-\{\infty\}, e),$$

so that it gives rise to only one interesting extension, namely $\mathbb{E}_{n,e}^{\infty}$.

Let h_n be the composition

$$\mathrm{CH}_{n-1}^{\mathrm{hom}}(X^{2n-1}) \xrightarrow{\text{Abel-Jacobi}} \mathrm{J}\underline{\mathrm{Hom}}(H^{2n-1}(X^{2n-1}), \mathbb{Z}(0)) \xrightarrow{\text{Kunneth}} \mathrm{J}\underline{\mathrm{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0)),$$

[†]The result in [6] is slightly weaker, but a small modification of its proof implies (1.3). See Section 3.6.

and identify

$$\mathrm{Ext}((H^1)^{\otimes n},(H^1)^{\otimes n-1}) \overset{\text{Carlson}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n},(H^1)^{\otimes n-1}\right) \overset{\text{Poincare duality}}{\cong} J\underline{\mathrm{Hom}}((H^1)^{\otimes 2n-1},\mathbb{Z}(0)).$$

For each n, we define algebraic cycles

$$\Delta_{n,e}\ ,\ Z^{\infty}_{n,e} \in CH^{\mathrm{hom}}_{n-1}(X^{2n-1})$$

such that (1.3) generalizes to the following result.

Theorem 1.0.1.

$$\mathbb{E}^{\infty}_{n,e} = (-1)^{\frac{n(n-1)}{2}} h_n\left(\Delta_{n,e} - Z^{\infty}_{n,e}\right)$$

The cycle $\Delta_{n,e}$ is constructed by first taking an alternating sum

$$\sum_i (-1)^{i-1}\,{}^t\Gamma_{\delta_i}$$

of the transposes of the graphs of the diagonal embeddings $\delta_i : X^{n-1} \longrightarrow X^n$ defined by

$$(x_1,\ldots,x_{n-1}) \mapsto (x_1,\ldots,x_i,x_i,\ldots,x_{n-1}), \tag{1.4}$$

and then using the method of Gross and Schoen [17] to produce a null-homologous cycle. The cycle $Z^{\infty}_{n,e}$ is defined as

$$\sum_{i=1}^{n-1}(-1)^{i-1}\left((\pi_{n+i,\infty})_* - (\pi_{n+i,e})_*\right)({}^t\Gamma_{\delta_i}),$$

where for $x \in X$, $\pi_{i,x}$ is the map $X^{2n-1} \to X^{2n-1}$ that replaces the i^{th} coordinate by x, and leaves the other coordinated unchanged[†].

Note that the fact that the diagonal embeddings $\delta_i : X^{n-1} \to X^n$ appear in the constructions is not surprising. Wojtkowiak used these maps in [29] to form a cosimplicial scheme that gives rise to the de Rham fundamental group, and Deligne and

[†]This map is only mentioned one time in the thesis after this point, namely when we define $Z^{\infty}_{n,e}$ in Section 3.4. The map is not used in any of the arguments, and hence hopefully the use of the letter π for it will not lead to any confusion.

Goncharov used these maps in [12] to construct their motivic fundamental group.

We should mention that one motivation for considering extensions of the form

$$0 \longrightarrow \frac{L_{n-1}}{L_{n-2}} \longrightarrow \frac{L_n}{L_{n-2}} \longrightarrow \frac{L_n}{L_{n-1}} \longrightarrow 0,$$

rather than

$$0 \longrightarrow L_{n-1} \longrightarrow L_n \longrightarrow \frac{L_n}{L_{n-1}} \longrightarrow 0,$$

is that the quotients $\{\frac{L_n}{L_{n-1}}\}$ are independent of the base point, so that we can think of extensions coming from different base points as elements of the same Ext group. The reason for looking at extensions coming from π_1 of the punctured curve, rather than the curve X itself, is that the successive quotients $\frac{L_n}{L_{n-1}}(X, e)$ for $n > 2$ are much more complicated than their counterparts for $X - \{\infty\}$. (See [26].)

2. Arithmetic part: Here we find some number theoretic applications for Theorem 1.0.1. Suppose $K \subset \mathbb{C}$ is a subfield, $X = X_0 \otimes_K \mathbb{C}$, where X_0 is a (smooth) projective curve over K, and $e, \infty \in X_0(K)$. Let g be the genus. Denote the Jacobian of X_0 by Jac.

2A. <u>Application to rational points on the Jacobian</u>: (Chapter 4) Following the ideas of [6], we associate to the extension $\mathbb{E}^{\infty}_{n,e}$ a family of points in $\mathrm{Jac}(K)$ parametrized by algebraic cycles of the appropriate dimension in a certain power of X_0. Our approach is in line with Darmon's general philosophy of constructing rational points on Jacobians of curves using algebraic cycles on higher dimensional varieties.
Throughout, we identify

$$\mathrm{Jac}(\mathbb{C}) \cong J\underline{\mathrm{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0)).$$

For a Hodge class

$$\xi \in (H^1)^{\otimes 2n-2},$$

let ξ^{-1} be the map

$$J\underline{\mathrm{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0)) \to J\underline{\mathrm{Hom}}(H^1, \mathbb{Z}(0)) \cong \mathrm{Jac}(\mathbb{C})$$

defined by

$$\big(\text{ class of } f : (H^1_{\mathbb{C}})^{\otimes 2n-1} \to \mathbb{C}\ \big) \quad \mapsto \quad \big(\text{ class of } f(\xi \otimes -)\ \big)\ .$$

For $Z \in CH_{n-1}(X_0^{2n-2})$, let ξ_Z be the $(H^1)^{\otimes 2n-2}$ Kunneth component of the class of Z. We now state the main result of Chapter 4.

Theorem 1.0.2. Let $Z \in CH_{n-1}(X_0^{2n-2})$. Then $\xi_Z^{-1}(\mathbb{E}^{\infty}_{n,e}) \in \mathrm{Jac}(K)$.

Note that this is not a priori obvious, as to define $\mathbb{E}^{\infty}_{n,e}$ one first goes to analytic topology. The result is a consequence of Theorem 1.0.1 in view of the following two facts:

(i) The map ξ_Z^{-1} is given by a correspondence. More precisely, it is induced by an element of

$$CH_n(X_0^{2n}) = CH_n(X_0^{2n-1} \times X_0)$$

whose class is the $(H^1)^{\otimes 2n}$ component of

$$Z \times \Delta(X_0),$$

where $\Delta(X_0)$ is the diagonal of X_0. Denoting the composition

$$CH^{\mathrm{hom}}_{n-1}(X_0^{2n-1}) \overset{\text{natural map}}{\to} CH^{\mathrm{hom}}_{n-1}(X^{2n-1}) \overset{h_n}{\to} J\underline{\mathrm{Hom}}((H^1)^{\otimes 2n-1}, \mathbb{Z}(0))$$

also by h_n, this gives us a commutative diagram

$$\begin{tikzcd} CH^{\mathrm{hom}}_{n-1}(X_0^{2n-1}) \arrow[r, "h_n"] \arrow[d] & J((H^1)^{\otimes 2n-1})^{\vee} \arrow[d, "\xi_Z^{-1}"] \\ CH^{\mathrm{hom}}_0(X_0) \arrow[r, "\text{Abel-Jacobi}"] & J(H^1)^{\vee}. \end{tikzcd} \tag{1.5}$$

(ii) The algebraic cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ are defined over K.

Theorem 1.0.2 is due to Darmon, Rotger, and Sols [6] in the case $n = 2$. For each n, it associates to $\mathbb{E}^{\infty}_{n,e}$, or equivalently to $\{W_{n-2}, W_{n-1}, W_n\}$ part of the weight filtration on $L_n(X - \{\infty\}, e)$, a family of rational points on Jac parametrized by $CH_{n-1}(X_0^{2n-2})$.

To simplify the notation, we will write P_ξ for $\xi^{-1}(\mathbb{E}^{\infty}_{n,e})$ and P_Z for P_{ξ_Z}. The point P_ξ (and in particular P_Z) can be described analytically using iterated integrals. Ideally, we would like to have a description in terms of algebraic 1-forms on X_0. Let $\Omega^1_{\mathrm{hol}}(X)$ be the space of holomorphic 1-forms on X. Identify

$$\mathrm{Jac}(\mathbb{C}) \cong \frac{\Omega^1_{\mathrm{hol}}(X)^\vee}{H_1(X,\mathbb{Z})}.$$

Let $\alpha_1, \dots, \alpha_{2g}$ be regular algebraic 1-forms on $X_0 - \{\infty\}$ whose classes form a basis $H^1_{dR}(X_0)$. Moreover, suppose $\alpha_1, \dots, \alpha_g$ are holomorphic on X. Let $d_1, \dots, d_{2g}$ form a basis of $H^1_{\mathbb{Z}}$ such that

$$\int_X d_i \wedge d_j = 1 \qquad \text{if } i < j.$$

Let ω_i be the representative of d_i in $\sum_j \mathbb{C}\alpha_j$. Write

$$\alpha_i = \sum_j p_{ij}\omega_j.$$

For each i, let $\beta_i \in \pi_1(X - \{\infty\}, e)$ be such that

$$\int_{\beta_i} - = \int_X d_i \wedge -$$

on H^1. Then, assuming the α_i satisfy a certain hypothesis, which we refer to as Hypothesis $\star(n)$ (see Paragraph 4.2.1), the point

$$P_\xi \in \frac{\Omega^1_{\mathrm{hol}}(X)^\vee}{H_1(X,\mathbb{Z})}$$

is represented by

$$f_\xi : \alpha_l \mapsto \sum_{i,j,k \le 2g} \mu'_{i,j,k}(\xi;\alpha_l) \int_{\beta_k} \omega_i\omega_j.$$

Here the coefficients

$$\mu'_{i,j,k}(\xi;\alpha_l) \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) := \sum_r p_{lr}\mathbb{Q}$$

are explicit linear combinations (in fact, with integer coefficients) of the p_{lr}.

Hypothesis $\star(n)$ is related to whether the Hodge filtration on $L_n(X - \{\infty\}, e)$ can be described "nicely" in terms of iterated integrals formed by the α_i. (See the remark in Paragraph 4.2.1.) It is possible that this hypothesis always holds. We were not able to prove this, but we show that in the case $g = 1$, Hypothesis $\star(n)$ is satisfied if α_2 has a pole of order 2 at ∞. For instance, if X_0 is given by the affine equation

$$y^2 = 4x^3 - g_2 x - g_3 \tag{1.6}$$

and ∞ is the point at infinity, Hypothesis $\star(n)$ holds (for all n) if $\alpha_2 = \frac{xdx}{y}$.

2B. Application to periods: (Chapter 5) Assume for the moment that the Mordell-Weil group $\mathrm{Jac}(K)$ has rank ≥ 1. A natural question one can ask is whether the families

$$\{P_Z : Z \in CH_{n-1}(X_0^{2n-2}) \subset \mathrm{Jac}(K)$$

contain non-torsion points.† This led us to ask whether P_Z being torsion will have any interesting consequences.

It is well-known that Hodge classes in tensor powers of H^1 induce polynomial relations (with integer coefficients) between the periods of X_0. In Chapter 5, we observe that, assuming the α_i can be chosen to satisfy Hypothesis $\star(n)$, a Hodge class $\xi \in (H^1)^{\otimes 2n-2}$ for which P_ξ is torsion induces relations between periods of $L_2(X - \{\infty\}, e)$. This is an easy consequence of the analytic description of P_ξ. Indeed, setting

$$\mu_{i,j,k}(\xi;\alpha_l) = \mu'_{i,j,k}(\xi;\alpha_l) - \mu'_{j,i,k}(\xi;\alpha_l) \quad (i,j,k \leq 2g, i < j),$$

it is easy to see that if the α_i satisfy Hypothesis $\star(n)$ and P_ξ is torsion, then

$$\sum_{\substack{i,j,k\leq 2g\\ i<j}} \mu_{i,j,k}(\xi;\alpha_l) \int_{\beta_k} \omega_i\omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) \quad (l \leq g). \tag{1.7}$$

The reason for writing these only in terms of the triples (i,j,k) satisfying $i < j$ is that thanks to the shuffle product property of iterated integrals,

$$\int_{\beta_k} \omega_i\omega_j + \int_{\beta_k} \omega_j\omega_i = \int_{\beta_k} \omega_i \int_{\beta_k} \omega_i.$$

Let $\mathbb{Q}(X_0)$ be the field generated over $\mathbb{Q}$ by all the numbers p_{ij} $(i,j \leq 2g)$. The relations

†One should keep in mind that for different n these families arise from different parts of the weight filtration on the mixed Hodge structure on $\pi_1(X - \{\infty\}, e)$.

(1.7) can be considered as linear relations in

$$1, \int_{\beta_k} \omega_i \omega_j \quad (i, j, k \leq 2g, i < j) \tag{1.8}$$

with coefficients in $\mathrm{Per}_{\mathbb{Q}}(X_0)$. By multi-linearity of iterated integrals, they can also be rewritten as linear relations between

$$1, \int_{\beta_k} \alpha_i \alpha_j \quad (i, j, k \leq 2g, i < j)$$

with coefficients in $\mathbb{Q}(X_0)$.

We then proceed to specialize to the Hodge classes coming from the diagonal of X_0 and X_0^2. Even these simplest cases lead to interesting statements.

Proposition 1.0.1. Suppose the α_i are chosen so that they satisfy Hypothesis $\star(n)$ for $n = 2, 3$.
(a) Suppose $P_{\Delta(X_0)}$ is torsion. Then the g relations (1.7), which in this case take the form

$$\sum_{\substack{i,j,k \leq 2g \\ i<j}} (-1)^{i+j} p_{lk} \int_{\beta_k} \omega_i \omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) \qquad (l \leq g),$$

are independent (as linear relations among (1.8) with coefficients in $\mathbb{Q}(X_0)$).
(b) For $1 \leq i, j \leq 2g$ define the numbers λ_{ij} by $\lambda_{ij} = (-1)^{i+j}$ if $i < j$ and $\lambda_{ij} = -\lambda_{ji}$. Suppose $P_{\Delta(X_0^2)}$ is torsion. Then the relations (1.7), which in this case are

$$\sum_{\substack{i,j,k \\ i<j}} \left(\lambda_{jk} p_{li} - \lambda_{ik} p_{lj} - 2(-1)^{i+j} p_{lk} \right) \int_{\beta_k} \omega_i \omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) \qquad (l \leq g),$$

are independent.
(c) Let $g = 2$. Suppose $P_{\Delta(X_0)}$ and $P_{\Delta(X_0^2)}$ are torsion. Then at least three of the relations given in (a) and (b) are independent.

Part (c) of the proposition is particularly interesting, as it shows that by digging deeper into the weight filtration the method might indeed give new information about the periods. Also note that thanks to Theorem 1.0.2, $P_{\Delta(X_0)}$ and $P_{\Delta(X_0^2)}$ are K-rational, so that they are guaranteed to be torsion if it happens that $\mathrm{Jac}(K)$ is finite. This happens for instance when $K = \mathbb{Q}$ and X_0 is a Fermat curve of degree an odd prime ≤ 7 [13].

In the elliptic curve case, one can say more:

Theorem 1.0.3. Let $g = 1$. Suppose that α_2 has a pole of order 2 at ∞. Then

$$p_{11} \int_{\beta_1} \omega_1\omega_2 + p_{12} \int_{\beta_2} \omega_1\omega_2 \in \mathrm{Per}_{\mathbb{Q}}(\alpha_1) \tag{1.9}$$

if and only if $\infty - e$ is torsion in $\mathrm{CH}_0^{\mathrm{hom}}(X_0)$ (or equivalently, in $X_0(K)$).

The condition on the order of the pole at ∞ is included only to guarantee that Hypothesis $\star$ is satisfied. The relation (1.9) is indeed equivalent to $P_{\Delta(X_0)}$ being torsion. To prove Theorem 1.0.3 one uses the fact that $h_2(\Delta_{2,e})$ is torsion when $g = 1$. I personally find the "only if" statement particularly satisfying, as it confirms that this relation is not merely a consequence of the formal properties of iterated integrals, and that it cannot be proved using methods that are insensitive to whether or not $\infty - e$ is torsion. Also, (1.9) can be written equivalently as

$$\frac{1}{p_{11}p_{22} - p_{12}p_{21}} \left(p_{11} \int_{\beta_1} \alpha_1\alpha_2 + p_{12} \int_{\beta_2} \alpha_1\alpha_2 - \frac{1}{2} p_{11}p_{12}(p_{21} + p_{22}) \right) \in \mathrm{Per}_{\mathbb{Q}}(\alpha_1). \tag{1.10}$$

Let X_0 be given by the affine equation (1.6) and ∞ be the point at infinity. Take $\alpha_1 = \frac{dx}{y}$ and $\alpha_2 = \frac{xdx}{y}$. Then the classical Legendre relation says $p_{11}p_{22} - p_{12}p_{21} = 2\pi i$, and (1.10) can be equivalently rewritten as

$$\int_{\beta_1} \alpha_1 \int_{\beta_2} (\alpha_1\alpha_2 - \alpha_2\alpha_1) \; - \; \int_{\beta_2} \alpha_1 \int_{\beta_1} (\alpha_1\alpha_2 - \alpha_2\alpha_1) \; \in 2\pi i \mathrm{Per}_{\mathbb{Q}}(\alpha_1).$$

By Theorem 1.0.3 this holds if and only if $e \in X_0(K)$ is torsion.

We close this introduction with a word on the structure of the thesis. We recall some background material in the next chapter. Chapter 3 contains the geometric component of the thesis. The goal of the chapter is to state and prove Theorem 1.0.1. The last two chapters contain the arithmetic part of the thesis. In Chapter 4, we prove Theorem 1.0.2 and give an analytic description for the point P_ξ. Chapter 5 applies the earlier results of the thesis to periods. Section 5.2 explains the methodology in detail, namely how Hodge classes may induce relations between periods of $L_2(X - \{\infty\}, e)$.

Section 5.3 discusses Proposition 1.0.1 and Theorem 1.0.3 above.

Chapter 2

Some Background

In this chapter we briefly go over some of the background material we shall be needing throughout the thesis. Of course, we could not recall every relevant fact or definition. Choices had to be made, and mostly we have only included what was rather essential towards understanding of the thesis. We highlight that nothing in this chapter is original. References will be given in each section.

2.1 Recollections from Hodge theory

The comparison isomorphism with de Rham cohomology induces a certain structure on the singular cohomology of a smooth projective complex variety. Historically, the notion of an (abstract) Hodge structure was defined as an abstraction of this structure. The notion of a mixed Hodge structure was defined as a generalization of that of a Hodge structure, in order to generalize the theory to arbitrary complex varieties. In this section, first we recall the definitions of a Hodge and mixed Hodge structure, and review some basic facts about them. Then we go over the motivating example of the cohomology of a complex variety.

2.1.1

Throughout this paragraph, unless otherwise stated, $\mathbb{K}$ can be any of $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Recall that a Hodge structure of weight n over $\mathbb{K}$ consists of the following data:

(i) a finitely generated $\mathbb{K}$-module $H_{\mathbb{K}}$

(ii) a decomposition of $H_{\mathbb{C}} := H_{\mathbb{K}} \otimes \mathbb{C}$ as

$$H_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}$$

such that $H^{p,q} = 0$ if $p + q \neq n$, and moreover $\overline{H^{p,q}} = H^{q,p}$ for all p, q.

One can equivalently replace (ii) by

(ii)′ a decreasing filtration $F^{\cdot}$ on $H_{\mathbb{C}}$ satisfying $H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{n-p+1} H_{\mathbb{C}}}$.

The passage between (ii) and (ii)′ is given by

$$F^p H_{\mathbb{C}} = \bigoplus_{\substack{p',q' \\ p' \geq p}} H^{p',q'}$$

and

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^{n-p} H_{\mathbb{C}}}\,.$$

If the weight n is even, a Hodge class is an element of $H_{\mathbb{K}}$ whose image in $H_{\mathbb{C}}$ belongs to $H^{\frac{n}{2},\frac{n}{2}}$, or equivalently, to $F^{\frac{n}{2}} H_{\mathbb{C}}$ (as $F^{\frac{n}{2}} H_{\mathbb{C}} \cap H_{\mathbb{K}} = H^{\frac{n}{2},\frac{n}{2}} \cap H_{\mathbb{K}}$).

The filtration (resp. decomposition) in (ii)′ (resp. (ii)) is referred to as the Hodge filtration (resp. decomposition). The advantage of the characterization in terms of the filtration (as opposed to the decomposition) is that it is more suitable for generalizations.

Definition. (1) For $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$, a *mixed* Hodge structure over $\mathbb{K}$ consists of the following data:

(i) a finite dimensional $\mathbb{K}$-vector space $H_{\mathbb{K}}$

(ii) a decreasing filtration $F^{\cdot}$ on $H_{\mathbb{C}} := H_{\mathbb{K}} \otimes \mathbb{C}$ called the Hodge filtration

(iii) an increasing filtration $W_{\cdot}$ on $H_{\mathbb{K}}$ called the weight filtration

such that for each n,

$$\mathrm{Gr}_n^W := \frac{W_n H_{\mathbb{K}}}{W_{n-1} H_{\mathbb{K}}}$$

together with the filtration induced by F on its complexification, i.e. given by

$$F^p \mathrm{Gr}_n^W \otimes \mathbb{C} := \frac{F^p W_n H_{\mathbb{C}} + W_{n-1} H_{\mathbb{C}}}{W_{n-1} H_{\mathbb{C}}}$$

is a Hodge structure of weight n over $\mathbb{K}$.
(2) A mixed Hodge structure over $\mathbb{Z}$ is a finitely generated abelian group $H_{\mathbb{Z}}$ together with a mixed Hodge structure over $\mathbb{Q}$ with the underlying rational vector space $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$.

We will often denote a Hodge or mixed Hodge structure by a capital English letter, and then decorate it with the subscript $\mathbb{K}$ to refer to its corresponding $\mathbb{K}$-module. For example, if H is a mixed Hodge structure over $\mathbb{Z}$, by $H_{\mathbb{Z}}$, $H_{\mathbb{Q}}$, and $H_{\mathbb{C}}$ we refer to the corresponding $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ modules.

A morphism of mixed Hodge structures over $\mathbb{K}$ is a map between the underlying $\mathbb{K}$-modules which (after extending the scalars) respects the filtrations. One can show that morphisms of mixed Hodge structures are *strict* with respect to the filtrations, i.e. given a morphism $f : A \to B$,

$$(F^p B_{\mathbb{C}}) \cap f(A_{\mathbb{C}}) = f(F^p A_{\mathbb{C}})$$

and

$$(W_n B_{\mathbb{C}}) \cap f(A_{\mathbb{C}}) = f(W_n A_{\mathbb{C}}).$$

The category $\mathbf{MHS}_{\mathbb{K}}$ of mixed Hodge structures over $\mathbb{K}$ is abelian. If H is a Hodge structure of weight n, we can think of it as a mixed Hodge structure in the obvious way: The weight filtration on it is given by $W_{n-1}H_{\mathbb{Q}} = 0$ and $W_n H_{\mathbb{Q}} = H_{\mathbb{Q}}$. By a Hodge structure we mean a direct sum (in $\mathbf{MHS}_{\mathbb{K}}$) of Hodge structures of possibly different weights. The category $\mathbf{HS}_{\mathbb{K}}$ of Hodge structures over $\mathbb{K}$ is the full subcategory of $\mathbf{MHS}_{\mathbb{K}}$ whose objects are Hodge structures. One sometimes refers to an object of $\mathbf{MHS}_{\mathbb{K}}$ which is a Hodge structure of weight n as a *pure* Hodge structure.

For each $n \in \mathbb{Z}$, one has a unique Hodge structure $\mathbb{K}(-n)$ over $\mathbb{K}$ of weight $2n$ with underlying module $\mathbb{K}$, with Hodge filtration given by $F^n\mathbb{C} = \mathbb{C}$ and $F^{n+1}\mathbb{C} = 0$. For a Hodge structure H over $\mathbb{K}$ of weight $2n$, a Hodge class can be equivalently thought of as a morphism

$$\mathbb{K}(-n) \to H.$$

For $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, each W_n is a functor $\mathbf{MHS}_{\mathbb{K}} \to \mathbf{MHS}_{\mathbb{K}}$. To have this over $\mathbb{Z}$ as well, one wants to have the weight filtration defined on the level of the underlying abelian group of a mixed Hodge structure over $\mathbb{Z}$. This is done by setting $W_n H_{\mathbb{Z}}$ to be the pre-image of $W_n H_{\mathbb{Q}}$ under the natural map

$$H_{\mathbb{Z}} \to H_{\mathbb{Q}}.$$

With this convention, now the W_n are functors $\mathbf{MHS}_{\mathbb{Z}} \to \mathbf{MHS}_{\mathbb{Z}}$ as well. The highest (resp. lowest) weight of a mixed Hodge structure H is defined to be the smallest n for which $W_n H = H$ (resp. $W_n H \neq 0$).

Tensor product and internal Homs: Given mixed Hodge structures A and B over $\mathbb{K}$, one has an object $A \otimes B$ in $\mathbf{MHS}_{\mathbb{K}}$ defined in the obvious way. For each n, the *twist* $A(n) := A \otimes \mathbb{K}(n)$ is obtained from A by shifting the filtrations†. One clearly has $A(0) = A$. The category $\mathbf{MHS}_{\mathbb{K}}$ is a tensor abelian category with $\mathbb{K}(0)$ as the identity of the tensor product.

Given objects A and B of $\mathbf{MHS}_{\mathbb{K}}$, their *internal hom* $\underline{\mathrm{Hom}}(A, B)$ is a mixed Hodge structure over $\mathbb{K}$ defined as follows: For $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, the underlying $\mathbb{K}$-vector space is $\mathrm{Hom}_{\mathbb{K}}(A_{\mathbb{K}}, B_{\mathbb{K}})$. The filtrations are given by

$$W_n \mathrm{Hom}_{\mathbb{K}}(A_{\mathbb{K}}, B_{\mathbb{K}}) = \{f : A_{\mathbb{K}} \to B_{\mathbb{K}} \mid f(W_l A_{\mathbb{K}}) \subset W_{n+l} B_{\mathbb{K}} \text{ for all } l\}$$

and

$$F^p \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}) = \{f : A_{\mathbb{C}} \to B_{\mathbb{C}} \mid f(F^l A_{\mathbb{C}}) \subset F^{p+l} B_{\mathbb{C}} \text{ for all } l\}.$$

For $\mathbb{K} = \mathbb{Z}$, $\underline{\mathrm{Hom}}(A, B)$ has $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}})$ as its underlying abelian group, and the mixed Hodge structure on

$$\underline{\mathrm{Hom}}(A, B)_{\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}}) \otimes \mathbb{Q} = \mathrm{Hom}_{\mathbb{Q}}(A_{\mathbb{Q}}, B_{\mathbb{Q}})$$

is the one just described.

If A and B are pure of weights a and b, $\underline{\mathrm{Hom}}(A, B)$ is pure of weight $b - a$. The *dual* to a mixed Hodge structure A is defined to be $A^\vee := \underline{\mathrm{Hom}}(A, \mathbb{K}(0))$. It is easy to see that if $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, one has a natural isomorphism $A^{\vee\vee} \cong A$, and the category $\mathbf{MHS}_{\mathbb{K}}$ is in fact neutral Tannakian (with fiber functor $H \mapsto H_{\mathbb{K}}$). In other words, in this case, $\mathbf{MHS}_{\mathbb{K}}$ is equivalent to the category of representations of an affine group scheme over $\mathbb{K}$†. Over $\mathbb{Z}$ however, $A^{\vee\vee} \not\cong A$ if $A_{\mathbb{Z}}$ has torsion.

†For any $\mathbb{K}$-module M, we identify $M \otimes \mathbb{K} = M$.

†For an introduction to the theory of Tannakian categories, the reader can see [10]. We do not say

We adopt the convention $A^{\otimes n} := (A^{\otimes -n})^\vee$ for n negative. One clearly has $\mathbb{K}(n) = \mathbb{K}(1)^{\otimes n}$ for all n.

In most of this thesis, we work with mixed Hodge structures over $\mathbb{Z}$. For simplicity, unless otherwise stated, the term "Hodge structure" means "Hodge structure over $\mathbb{Z}$" from now on. We simply write **MHS** for $\mathbf{MHS}_{\mathbb{Z}}$. For references on the material so far, see [7], [27], and [10].

Carlson Jacobians: Motivated by Griffiths' intermediate Jacobians of a variety, given a mixed Hodge structure A, Carlson [1] defined its n^{th} *Jacobian*[‡] by

$$J^n(A) := \frac{A_{\mathbb{C}}}{F^n A_{\mathbb{C}} + A_{\mathbb{Z}}},$$

where by $A_{\mathbb{Z}}$ we obviously mean its image in $A_{\mathbb{C}}$. It is easy to see that for n bigger than half the highest weight of A, the natural map

$$A_{\mathbb{R}} := A_{\mathbb{Z}} \otimes \mathbb{R} \to \frac{A_{\mathbb{C}}}{F^n A_{\mathbb{C}}} \tag{2.1}$$

(given by the inclusion $A_{\mathbb{R}} \subset A_{\mathbb{C}}$) is injective, whence $J^n(A)$ is the quotient of a complex vector space by a discrete subgroup. It is easy to see that in general J^n is a functor from **MHS** to the category of abelian groups that respects direct sums.

Of special interest to us is the case of the "middle Jacobian" $JA := J^n A$ of a pure Hodge structure A of weight $2n - 1$ (possibly negative). It is easy to see that in this case, the map (2.1) is an isomorphism, and hence induces an isomorphism of real tori

$$\frac{A_{\mathbb{R}}}{A_{\mathbb{Z}}} \cong JA. \tag{2.2}$$

We close this paragraph by recording, for future reference, a few easy statements in the following lemma.

more about Tannakian categories here as they are not explicitly used in the thesis.

[‡]One should not be misled by the use of the word Jacobian here: Carlson Jacobians of a mixed Hodge structure are often not algebraic.

Lemma 2.1.1. Let A, B and C be mixed Hodge structures.

(a) If $B_{\mathbb{Z}}$ is free, the canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}} \otimes C_{\mathbb{Z}}) \cong \mathrm{Hom}_{\mathbb{Z}}\left(A_{\mathbb{Z}} \otimes B_{\mathbb{Z}}^{\vee}, C_{\mathbb{Z}}\right)$ induces an isomorphism $\underline{\mathrm{Hom}}(A, B \otimes C) \cong \underline{\mathrm{Hom}}\left(A \otimes B^{\vee}, C\right)$.

(b) The canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}}) \otimes C_{\mathbb{Z}} \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}} \otimes C_{\mathbb{Z}})$ induces an isomorphism $\underline{\mathrm{Hom}}(A, B) \otimes C \cong \underline{\mathrm{Hom}}(A, B \otimes C)$.

(c) $J^n A(-p) = J^{n-p} A$

(d) If A is pure of odd weight, $JA(-p) = JA$.

(e) $J^n \underline{\mathrm{Hom}}(A(-p), B) = J^{n+p} \underline{\mathrm{Hom}}(A, B)$.

(f) If A and B are pure of opposite parity weights, then $J\underline{\mathrm{Hom}}(A(-p), B) = J\underline{\mathrm{Hom}}(A, B)$.

The proofs are all straightforward. For (a) (resp. (b)) one notes that the canonical isomorphisms $\mathrm{Hom}_{\mathbb{K}}(A_{\mathbb{K}}, B_{\mathbb{K}} \otimes C_{\mathbb{K}}) \cong \mathrm{Hom}_{\mathbb{K}}\left(A_{\mathbb{K}} \otimes B_{\mathbb{K}}^{\vee}, C_{\mathbb{K}}\right)$ (resp. $\mathrm{Hom}_{\mathbb{K}}(A_{\mathbb{K}}, B_{\mathbb{K}}) \otimes C_{\mathbb{K}} \cong \mathrm{Hom}_{\mathbb{K}}(A_{\mathbb{K}}, B_{\mathbb{K}} \otimes C_{\mathbb{K}})$) for $\mathbb{K} = \mathbb{Q}, \mathbb{C}$ come from their $\mathbb{K} = \mathbb{Z}$ counterpart by extending the scalars, and then checks that the isomorphisms respect the filtrations W and F. Parts (c) and (e) are immediate from that $F^n A(-p)_{\mathbb{C}} = F^{n-p} A_{\mathbb{C}}$. Part (d) (resp. (f)) is a special case of (c) (resp. (e)).

2.1.2 Classical example: Cohomology of a complex variety

Let U be a complex variety. If U is smooth and projective, its degree n cohomology is a pure Hodge structure of weight n: The underlying abelian group is the Betti (singular) cohomology group $H^n(U, \mathbb{Z})$ (U with analytic topology). Identifying

$$H^n(U, \mathbb{Z}) \otimes \mathbb{C} \cong H^n(U, \mathbb{C}) \overset{\text{de Rham iso.}}{\cong} H^n_{dR}(U), \tag{2.3}$$

where $H^n_{dR}(U)$ here is the n^{th} cohomology of the complex $E^{\cdot}_{\mathbb{C}}(U)$ of complex-valued smooth differential forms on U, the Hodge decomposition is given by the classical result

$$H^n_{dR}(U) = \bigoplus_{p+q=n} H^{p,q},$$

where the elements of $H^{p,q}$ are the classes that are represented by forms of type (p, q).

More generally, thanks to a theorem of Deligne, the degree n cohomology of U (which is no longer assumed to be projective or smooth), naturally carries a mixed

Hodge structure, which we denote by $H^n(U)$. If U is smooth, $H^n(U)$ can be described as follows: The underlying abelian group is again Betti cohomology with integral coefficients. Via the identifications (2.3), we define the weight and Hodge filtrations on $H^n_{dR}(U)$. Realize U as $Y \backslash D$, where Y is smooth projective and D is a normal crossing divisor. Then the complex $E^{\cdot}(Y \log D)$ of smooth differential forms on U with at most logarithmic singularity along D calculates the cohomology of U, i.e. the inclusion

$$E^{\cdot}(Y \log D) \hookrightarrow E^{\cdot}_{\mathbb{C}}(U)$$

is a quasi-isomorphism. Recall that $E^{\cdot}(Y \log D)$ consists of elements ω of $E^{\cdot}_{\mathbb{C}}(U)$ which satisfy the following local condition: Near each point $x \in D$, if the z_i are local holomorphic coordinates such that D is given by $\prod_{i=1}^{m} z_i = 0$, ω is of the form

$$\sum \omega_I \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_t}}{z_{i_t}},$$

where $i_l \in \{1 \cdots, m\}$ and ω_I is smooth (near x) along D as well. One defines a filtration $F^{\cdot}$ (resp. $W_{\cdot}$) on the complex $E^{\cdot}(Y \log D)$ by holomorphic degree (resp. order of poles along D). More precisely, $F^p E^{\cdot}(Y \log D)$ consists of elements with at least p forms of type $(1,0)$ in their local expansions, and $W_n E^{\cdot}(Y \log D)$ consists of those elements which (with the notation as above) near $x \in D$ can be expressed in the form

$$\sum \omega_I \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_t}}{z_{i_t}} \quad (i_l \in \{1, \cdots, m\}),$$

where $t \leq n$ and ω_I is smooth along D as well. Then the Hodge and Weight filtration on $H^n_{dR}(U)$ are given by

$$F^p H^n_{dR}(U) = \mathrm{Im}\left(H^n F^p E^{\cdot}(Y \log D) \longrightarrow H^n_{dR}(U) \right)$$

and

$$W_l H^n_{dR}(U) = \mathrm{Im}\left(H^n W_{l-n} E^{\cdot}(Y \log D) \longrightarrow H^n_{dR}(U) \right).$$

One can show that $W_{\cdot}$ is defined over $\mathbb{Q}$, i.e. is induced by a filtration on

$$H^n(U, \mathbb{Q}) \subset H^n(U, \mathbb{C}) \cong H^n_{dR}(U),$$

and that the structure just defined only depends on U (and not the compactification used in the process).

As for references for the mixed Hodge structure on the cohomology of a variety, the original articles are Deligne's [7] (for the smooth case) and [8] (for the general case). The reader can also consult [23] and [27]. For more details on the complex $E^{\cdot}(Y \log D)$ see [23]. For the classical smooth projective case, one can refer to [28] or again [27].

Throughout the thesis, our varieties will all be smooth and for such a variety U we continue to identify $H^n(U, \mathbb{C})$ and $H^n_{dR}(U)$ via the isomorphism of de Rham.

2.2 Extensions in MHS

2.2.1 Extensions in an abelian category

Let A and B be objects in an abelian category **C**. Given short exact sequences

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 \tag{2.4}$$

and

$$0 \longrightarrow B \longrightarrow E' \longrightarrow A \longrightarrow 0\,, \tag{2.5}$$

say (2.4) ~ (2.5) if there is a morphism $E \to E'$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"'] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commute. It is clear that ~ is an equivalence relation on the collection of all short exact sequences

$$0 \longrightarrow B \longrightarrow \cdot \longrightarrow A \longrightarrow 0\,.$$

An equivalence class is called an *extension* of A by B. The set of all such extensions is naturally equipped with a binary operation, making it into an abelian group denoted by

$$\mathrm{Ext}_{\mathbf{C}}(A, B).$$

With abuse of terminology, we might simply say the extension (2.4) to refer to the one represented by it.

In what follows, Ext always means $\mathrm{Ext}_{\mathbf{MHS}}$.

2.2.2 Carlson's theorem on classifying extensions in MHS

Let A and B be mixed Hodge structures such that the highest weight of B is less than the lowest weight of A. Suppose moreover that $A_{\mathbb{Z}}$ is torsion-free. Carlson [1] gave a functorial isomorphism

$$\mathrm{Ext}(A, B) \cong J^0\underline{\mathrm{Hom}}(A, B).$$

Given an extension $\mathbb{E}$ given by a short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0\,,$$

the corresponding element in the Jacobian is obtained in one of the following ways:

(i) Choose a Hodge (resp. integral) section σ_F (resp. $\sigma_{\mathbb{Z}}$) of $E_{\mathbb{C}} \to A_{\mathbb{C}}$. Then $\mathbb{E}$ corresponds to the class of $\sigma_F - \sigma_{\mathbb{Z}}$. (By a Hodge section we mean a section that is compatible with the Hodge filtrations, and by integral we mean a map that is induced by a map between the underlying $\mathbb{Z}$-modules.)

(ii) Choose a Hodge section σ_F as above, and an integral retraction $\rho_{\mathbb{Z}} : E_{\mathbb{C}} \to B_{\mathbb{C}}$. The extension $\mathbb{E}$ corresponds to the class of $\rho_{\mathbb{Z}} \circ \sigma_F$.

In the interest of simplifying notation, we shall identify $\mathrm{Ext}(A, B)$ and $J^0\underline{\mathrm{Hom}}(A, B)$ via the isomorphism of Carlson.

2.3 Review of the reduced Bar construction

In this section, we briefly review certain aspects of the reduced bar construction on a differential graded algebra. The construction is due to K.T. Chen, and the reader can refer to [3] and [18] for references. We only discuss a special case that is of interest to us. Throughout this section $\mathbb{K}$ is a field of characteristic 0.

Let us start by recalling the definition of a differential graded algebra. A differential graded algebra over $\mathbb{K}$ is a graded $\mathbb{K}$-algebra $A^{\cdot} = \bigoplus_{n\geq 0} A^n$, equipped with a differential d of degree 1 (so that one has a complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

of $\mathbb{K}$-vector spaces) such that the graded Leibniz rule holds, i.e.

$$d(ab) = (da)b + (-1)^{\deg(a)} a(db)$$

for homogeneous elements $a, b \in A^{\cdot}$, where deg is the degree. Moreover, we say $A^{\cdot}$ is commutative if

$$ab = (-1)^{\deg(a)\deg(b)} ba$$

for all homogeneous a, b.

Note that $\mathbb{K}$ itself can be thought of as a differential graded algebra over $\mathbb{K}$ in an obvious way. Suppose $A^{\cdot} = \bigoplus_{n\geq 0} A^n$ is a differential graded algebra over $\mathbb{K}$, with the differential denoted by d. Denote the positive degree part by A^+. Let $\epsilon : A^{\cdot} \to \mathbb{K}$ be an augmentation (i.e. a morphism of differential graded algebras into $\mathbb{K}$). For any integers r, s ($r \geq 0$), let $T^{-r,s}(A^{\cdot})$ be the degree s part of $(A^+)^{\otimes r}$, i.e. the $\mathbb{K}$-span of all terms of the form

$$a_1 \otimes \ldots \otimes a_r, \tag{2.6}$$

where $a_i \in A^+$ and $\sum \deg a_i = s$. (By convention, $(A^+)^{\otimes 0} = \mathbb{K}$.) It is customary to use the notation

$$[a_1|\ldots|a_r]$$

for the element (2.6). The $T^{-r,s}(A^{\cdot})$ form a second quadrant bicomplex $T^{\cdot,\cdot}(A^{\cdot})$, with $T^{-r,s}(A^{\cdot})$ being the $(-r, s)$ bidegree component, and anti-commuting differentials both of degree 1 defined below. Here $Ja = (-1)^{\deg a} a$ for any homogeneous element $a \in A^{\cdot}$.

- The horizontal differential d_h:

$$d_h([a_1|\ldots|a_r]) = \sum_{i=1}^{r-1} (-1)^{i+1} [Ja_1|\ldots|Ja_{i-1}|(Ja_i)a_{i+1}|a_{i+2}|\ldots|a_r]$$

- The vertical differential d_v:

$$d_v([a_1|\dots|a_r]) = \sum_{i=1}^{r} (-1)^i [Ja_1|\dots|Ja_{i-1}|da_i|a_{i+1}|\dots|a_r].$$

The formulas for the differentials are particularly important for us when all the a_i are of degree 1. In this case the formulas simplify to

$$d_h[a_1\dots|a_r] = -\sum_i [a_1|\dots|a_i a_{i+1}|\dots|a_r] \tag{2.7}$$

and

$$d_v[a_1\dots|a_r] = -\sum_i [a_1|\dots|da_i|\dots|a_r]. \tag{2.8}$$

The associated total complex $\mathrm{Tot}\,(T^{\cdot,\cdot}(A^{\cdot}))$ is concentrated in non-negative degrees, and its degree zero part is $\bigoplus_{s\geq 0} T^{-s,s}(A^{\cdot}) = \bigoplus_{s\geq 0}(A^1)^{\otimes s}$. The reduced bar construction $\overline{B}(A^{\cdot},\epsilon) = \bigoplus_{n\geq 0} \overline{B}^n(A^{\cdot},\epsilon)$ of $A^{\cdot}$ relative to ϵ is by definition a certain quotient of $\mathrm{Tot}\,(T^{\cdot,\cdot}(A^{\cdot}))$, where the subcomplex by which one quotients depends on ϵ. The image of $[a_1|\dots|a_r]$ is denoted by $(a_1|\dots|a_r)$. If $A^0 = \mathbb{K}$, then $\overline{B}(A^{\cdot},\epsilon)$ is simply $\mathrm{Tot}\,(T^{\cdot,\cdot}(A^{\cdot}))$. From now on we drop the augmentation ϵ from our notation for $\overline{B}$ if it will not lead to any confusion.

The reduced bar construction is naturally filtered by tensor length: Let

$$\mathcal{T}_n = \bigoplus_{r\leq n} \left(T^{-r,s}(A^{\cdot})\right).$$

The filtration $\{\mathcal{T}_n\}$ of the double complex $(T^{\cdot,\cdot}(A^{\cdot}))$ induces a filtration $\{\mathcal{B}_n\}$ on the reduced bar construction. We denote the filtration induced on the cohomology of $\overline{B}(A^{\cdot})$ also by $\{\mathcal{B}_n\}$.

The reduced bar construction is functorial. In particular, if $A^{\cdot}$ and $\tilde{A}^{\cdot}$ are differential graded $\mathbb{K}$-algebras, and $\epsilon : A^{\cdot} \to \mathbb{K}$ and $\tilde{\epsilon} : \tilde{A}^{\cdot} \to \mathbb{K}$ are augmentations, a morphism $f : A^{\cdot} \to \tilde{A}^{\cdot}$ of differential graded algebras satisfying $\tilde{\epsilon}\circ f = \epsilon$ induces a morphisms of complexes $\overline{B}(A^{\cdot}) \to \overline{B}(\tilde{A}^{\cdot})$ compatible with the filtrations $\{\mathcal{B}_n\}$. Moreover, if f is a quasi-isomorphism and $H^0(A^{\cdot}) = \mathbb{K}$, then the induced maps between the reduced bar constructions or the $\mathcal{B}_n$ are also quasi-isomorphisms.

The map

$$(a_1|\ldots|a_r) \mapsto \sum_i (a_1|\ldots|a_i) \otimes (a_{i+1}|\ldots|a_r)$$

makes $\overline{B}(A^\cdot)$ into a differential graded coalgebra. If $A^\cdot$ is commutative, then $\overline{B}(A^\cdot)$ is in fact a differential graded Hopf algebra, with multiplication given by the so called *shuffle product*, which for degree zero elements is given by the formula†

$$(a_1|\ldots|a_r)\cdot(a_{r+1}|\ldots|a_{r+s}) = \sum_{\text{(r,s) shuffles } \sigma} (a_{\sigma(1)}|\ldots|a_{\sigma(r+s)}).$$

The general formula is an alternating sum of the $(a_{\sigma(1)}|\ldots|a_{\sigma(r+s)})$, where the coefficients take into account the signs of the σ and the degrees of the a_i. In particular, when $A^\cdot$ is commutative, $H^0\overline{B}(A^\cdot)$ is a Hopf algebra, which is commutative as it can be seen from the definition of the shuffle product given above. If $f : A^\cdot \to B^\cdot$ is a quasi-isomorphism of (commutative) differential graded algebras, then the induced map between the reduced bar constructions respects the comultiplications (and multiplications).

2.4 Hodge theory of the fundamental group- the general case

2.4.1

Let G be a finitely generated group and $\mathbb{K}$ a field of characteristic zero. The Malcev or (pro)-unipotent completion of G over $\mathbb{K}$ is a pro-unipotent algebraic group G^{un} over $\mathbb{K}$, together with a homomorphism $G \to G^{un}(\mathbb{K})$, such that for any pro-unipotent group U over $\mathbb{K}$ and any homomorphism $G \to U(\mathbb{K})$, there is a unique morphism $G^{un} \to U$ of group schemes over $\mathbb{K}$ making the obvious diagram commute. It follows immediately that the image of G is dense in G^{un}. The group G^{un} can be defined explicitly as $\mathrm{Spec}(\mathcal{O}_{G^{un}_{\mathbb{K}}})$, where

$$\mathcal{O}_{G^{un}_{\mathbb{K}}} = \varinjlim \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee},$$

†Recall that $\sigma \in S_{r+s}$ is an (r,s) shuffle if

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(r) \quad \text{and} \quad \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$$

.

and I is the augmentation ideal. One can think of

$$\left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee}$$

as the space of $\mathbb{K}$-valued functions on G which (after being extended linearly to $\mathbb{K}[G]$) vanish on I^{m+1}. For the very last sentence, $\mathbb{K}$ can be a ring.

2.4.2 Chen's theory of iterated integrals and the description of $\mathcal{O}(\pi_1^{un})$

We review some results of K.T Chen in this paragraph. For details and proofs, see [2], [3] and [4].

Let U be a path-connected manifold, $e \in U$, $G = \pi_1(U, e)$, and Ω_e be the (smooth) loop space at e. Throughout this paragraph $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Using iterated integrals, Chen [2] defined the differential graded algebra of $\mathbb{K}$-valued differential forms on Ω_e, which following [2] we denote by $A'_{\mathbb{K}}$. The space $A'^d_{\mathbb{K}}$ of $\mathbb{K}$-valued iterated integrals of degree d, or d-forms on Ω_e, is spanned by elements of the form $\int \omega_1 \ldots \omega_r$, where the ω_i are $k-$valued differential forms of positive degree on U, and $\sum \deg \omega_i = r + d$. An element of $A'^d_{\mathbb{K}}$ that can be expressed as a linear combination of iterated integrals as above with $r \leq m$ is said to be of length $\leq m$. The complex $A'_{\mathbb{K}}$ is naturally filtered by length. Let $A'_{\mathbb{K}}(m)$ denote the subcomplex formed by iterated integrals of length $\leq m$. Since $A'_{\mathbb{K}}$ is concentrated in degree ≥ 0, one has

$$H^0(A'_{\mathbb{K}}(m)) \subset H^0(A'_{\mathbb{K}}),$$

and the $\{H^0(A'_{\mathbb{K}}(m))\}$ is a filtration on $H^0(A'_{\mathbb{K}})$.

In the case that $\omega_1, \ldots, \omega_r$ are all 1-forms on U, the zero form, i.e. function, $\int \omega_1 \ldots \omega_r$ on the loop space is defined by

$$\left(\gamma : [0,1] \to U \right) \quad \mapsto \quad \int\limits_{0 \leq t_1 \leq \ldots \leq t_r \leq 1} f_1(t_1)dt_1 \ldots f_r(t_r)dt_r,$$

where $f_i(t)dt = \gamma^*(\omega_i)$. If $r = 0$, the "empty" iterated integral is defined to be the constant function 1. The value of $\int \omega_1 \ldots \omega_r$ on γ is denoted by $\int\limits_{\gamma} \omega_1 \ldots \omega_r$. It is clear that for $r = 1$, this coincides with the usual integral.

From now on, by an iterated integral we mean one of degree zero. The following formula describes how iterated integrals behave relative to composition of paths. Here α and β are loops at e.

$$\int_{\alpha\beta} \omega_1 \dots \omega_r = \sum_{i=0}^{r} \int_{\alpha} \omega_1 \dots \omega_i \int_{\beta} \omega_{i+1} \dots \omega_r \tag{2.9}$$

One can show that iterated integrals also satisfy the following relations (as functions on Ω_e). Here f is a (smooth) function on U.

$$\begin{aligned} \int (df)\omega_2 \dots \omega_r &= \int (f\omega_2) \dots \omega_r - f(e) \int \omega_2 \dots \omega_r \\ \int \omega_1 \dots \omega_{i-1}(df)\omega_{i+1} \dots \omega_r &= \int \omega_1 \dots \omega_{i-1}(f\omega_{i+1}) \dots \omega_r - \int \omega_1 \dots (f\omega_{i-1})\omega_{i+1} \dots \omega_r \\ \int \omega_1 \dots \omega_{r-1}(df) &= f(e) \int \omega_1 \dots \omega_{r-1} - \int \omega_1 \dots (f\omega_{r-1}) \end{aligned} \tag{2.10}$$

Also, one has the *shuffle product* property

$$\int_{\gamma} \omega_1 \dots \omega_r \int_{\gamma} \omega_{r+1} \dots \omega_{r+s} = \sum_{\text{(r,s) shuffles } \sigma} \int_{\gamma} \omega_{\sigma(1)} \dots \omega_{\sigma(r+s)}. \tag{2.11}$$

It follows from Chen's construction of the complex $A'_{\mathbb{K}}$ that an iterated integral induces a function on G if and only if it is closed. Moreover, it follows from (2.9) that a closed iterated integral of length $\leq m$ vanishes on $I^{m+1} \subset \mathbb{K}[G]$, so that one has a natural inclusion

$$H^0(A'_{\mathbb{K}}(m)) \subset \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee}.$$

The main theorem of [2] (Theorem 5.3) asserts that indeed

$$H^0(A'_{\mathbb{K}}(m)) = \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee}.$$

The algebraic structure of $H^0(A'_{\mathbb{K}}(m))$ can be described using the reduced bar construction on the complex $E^{\cdot}_{\mathbb{K}}(U)$ of smooth $\mathbb{K}$-valued differential forms on U, augmented by "evaluation at e". One has a natural map of differential graded algebras

$\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \to A'_{\mathbb{K}}$ given by integration

$$(\omega_1|\ldots|\omega_r) \;\mapsto\; \int \omega_1 \ldots \omega_r.$$

This map† induces an isomorphism $H^0\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \to H^0(A'_{\mathbb{K}})$ strictly compatible with the length filtrations, i.e. we have a natural isomorphism

$$\mathcal{B}_m H^0\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \xrightarrow{\int} H^0(A'_{\mathbb{K}}(m)) = \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee}.$$

Remark. (1) Let $A^{\cdot}_{\mathbb{Q}} = A^{\cdot}_{\mathbb{Q}}(U)$ be the complex of Sullivan's rational polynomial forms on U. Under the inclusion

$$\mathcal{B}_m H^0\overline{B}(A^{\cdot}_{\mathbb{Q}}) \subset \mathcal{B}_m H^0\overline{B}(A^{\cdot}_{\mathbb{Q}} \otimes \mathbb{K}) \xrightarrow{\simeq} \mathcal{B}_m H^0\overline{B}(E^{\cdot}_{\mathbb{K}}(U)) \simeq \left(\frac{\mathbb{K}[G]}{I^{m+1}}\right)^{\vee},$$

$\mathcal{B}_m H^0\overline{B}(A^{\cdot}_{\mathbb{Q}})$ is identified with $\left(\frac{\mathbb{Q}[G]}{I^{m+1}}\right)^{\vee}$.

(2) If U is (the associated complex manifold to) a smooth complex variety, and $U = Y \setminus D$ where Y is smooth projective and D is a normal crossing divisor, one can replace $E^{\cdot}_{\mathbb{C}}(U)$ by the complex $E^{\cdot}(Y \log D)$. (See Paragraph 2.1.2.)

2.4.3 Digression to mixed Hodge complexes

Roughly speaking, a rational mixed Hodge complex ($\mathbb{Q}$-MHC) **A** is the data of a filtered complex $(A^{\cdot}_{\mathbb{Q}}, W_{\cdot})$ of $\mathbb{Q}$-vectors spaces, and a bifiltered complex $(A^{\cdot}_{\mathbb{C}}, W_{\cdot}, F^{\cdot})$ of $\mathbb{C}$-vector spaces together with quasi-isomorphisms between $A^{\cdot}_{\mathbb{Q}} \otimes \mathbb{C}$ and $A^{\cdot}_{\mathbb{C}}$ strictly compatible with $W_{\cdot}$ filtrations, such that each cohomological data $H^m\mathbf{A}$ with the filtrations $F^{\cdot}$ and $W_{\cdot}$ defined by

$$\begin{aligned} F^pH^mA^{\cdot}_{\mathbb{C}} &= \mathrm{Im}\,(H^m(F^pA^{\cdot}_{\mathbb{C}}) \to H^mA^{\cdot}_{\mathbb{C}}) \\ W_nH^mA^{\cdot}_{\mathbb{Q}} &= \mathrm{Im}\left(H^m(W_{n-m}A^{\cdot}_{\mathbb{Q}}) \to H^mA^{\cdot}_{\mathbb{Q}}\right) \end{aligned}$$

is a mixed Hodge structure over $\mathbb{Q}$. One refers to the filtrations $W_{\cdot}$ and $F^{\cdot}$ as the weight and Hodge filtrations. A $\mathbb{Q}$-MHC **A** is called a (commutative) multiplicative $\mathbb{Q}$-MHC if both $A^{\cdot}_{\mathbb{Q}}$ and $A^{\cdot}_{\mathbb{C}}$ are (commutative) differential graded algebras and the

†The relations by which one mods out $\mathrm{Tot}\,(T^{\cdot,\cdot}(E^{\cdot}_{\mathbb{K}}(U))$ to get $\overline{B}(E^{\cdot}_{\mathbb{K}}(U))$ are defined exactly based on relations (2.10) satisfied by iterated integrals, so that the map just described is well-defined.

quasi-isomorphism between $A^{\cdot}_{\mathbb{Q}} \otimes \mathbb{C}$ and $A^{\cdot}_{\mathbb{C}}$ is also that of differential graded algebras. A theorem of Hain asserts that if $\mathbf{A}$ is a positively graded multiplicative $\mathbb{Q}$-MHC such that $H^0(A^{\cdot}_{\mathbb{Q}}) = \mathbb{Q}$, then each $\mathcal{B}_m\overline{B}(\mathbf{A})$ is a $\mathbb{Q}$-MHC. The Hodge filtration is just the natural extension of the Hodge filtration on $A^{\cdot}_{\mathbb{C}}$, whereas the weight filtration on $\mathcal{B}_m\overline{B}(A^{\cdot}_{\mathbb{Q}})$ (resp. $\mathcal{B}_m\overline{B}(A^{\cdot}_{\mathbb{C}})$) is the convolution $W'_{\cdot} * \mathcal{B}_{\cdot}$, where $W'_{\cdot}$ is the natural extension of the weight filtration on $A^{\cdot}_{\mathbb{Q}}$ (resp. $A^{\cdot}_{\mathbb{C}}$). We record, for future reference, the description of Hodge and weight filtrations on $\mathcal{B}_m H^0\overline{B}(\mathbf{A})$:

- $F^p\mathcal{B}_m H^0\overline{B}(A^{\cdot}_{\mathbb{C}})$ consists of all closed† elements in $\overline{B}^0(A^{\cdot}_{\mathbb{C}})$ that can be written as a linear combination of elements of the form $(a_1|\ldots|a_r)$ satisfying $r \leq m$ and $[a_1|\ldots|a_r] \in F^p(A^{\cdot}_{\mathbb{C}})^{\otimes r}$.
- $W_n\mathcal{B}_m H^0\overline{B}(A^{\cdot}_{\mathbb{C}})$ consists of all closed elements in $\overline{B}^0(A^{\cdot}_{\mathbb{C}})$ that can be written as a linear combination of elements of the form $(a_1|\ldots|a_r)$ satisfying $r \leq m$ and $[a_1|\ldots|a_r] \in W_{n-r}(A^{\cdot}_{\mathbb{C}})^{\otimes r}$. (This is defined over $\mathbb{Q}$.)

We close this paragraph by giving some references. For definitions see [18] and [8]. For the material on the reduced bar construction on multiplicative mixed Hodge complexes, see [18] and [20].

2.4.4 Mixed Hodge structure on π_1 of a smooth complex variety

Let U be a smooth variety over $\mathbb{C}$, $e \in U(\mathbb{C})$, $G = \pi_1(U, e)$, where with abuse of notation we denote a smooth complex variety and its associated complex manifold by the same symbol. Here we briefly review a result of Hain, in which he defines a mixed Hodge structure on the integral lattice

$$\left(\frac{\mathbb{Z}[G]}{I^{m+1}}\right)^{\vee},$$

which we denote by $L_m = L_m(U, e)$. For details and proofs, see [18].

Let $U = Y \setminus D$, where Y is a smooth projective variety and D is a normal crossing divisor. Then there is a commutative multiplicative $\mathbb{Q}-$MHC $\mathbf{K}$ defined as follows: $K^{\cdot}_{\mathbb{C}}$ is the complex $E^{\cdot}(Y \log D)$ of smooth $\mathbb{C}$-valued differential forms on U with at most logarithmic singularity along D, equipped with the usual filtrations $W_{\cdot}$ (by the order of poles) and $F^{\cdot}$ (by holomorphic degree). For the rational data $(K^{\cdot}_{\mathbb{Q}}, W_{\cdot})$, see Subsection 5.6 of [18]. This data is defined in such a way that there is an explicit filtered

†Note that $\mathcal{B}_m\overline{B}(A^{\cdot}_{\mathbb{C}})$ is concentrated in non-negative degrees.

quasi-isomorphism between $(K^{\cdot}_{\mathbb{Q}} \otimes \mathbb{C}, W_{\cdot} \otimes \mathbb{C})$ and $(E^{\cdot}(Y \log D), W_{\cdot})$, and an explicit quasi-isomorphism between $K^{\cdot}_{\mathbb{Q}}$ and Sullivan's complex $A^{\cdot}_{\mathbb{Q}}(U)$ of rational polynomial forms on U. The $\mathbb{Q}$-MHC $\mathbf{K}$ calculates the rational mixed Hodge structure on the cohomology of U. Evaluation at e defines an augmentation on $\mathbf{K}$. Then the $\mathcal{B}_m H^0 \overline{B}(\mathbf{K})$ form a filtered family of rational mixed Hodge structures. The underlying complex (resp. rational) vector space of $\mathcal{B}_m H^0 \overline{B}(\mathbf{K})$ is $\mathcal{B}_m H^0 \overline{B}(E^{\cdot}(Y \log D))$ (resp. $\mathcal{B}_m H^0 \overline{B}(A^{\cdot}_{\mathbb{Q}}(U))$). Via the commutative diagram

$$\begin{tikzcd} \mathcal{B}_m H^0 \overline{B}(A^{\cdot}_{\mathbb{Q}}(U)) \arrow[r, "\simeq"] \arrow[d] & \left(\frac{\mathbb{Q}[G]}{I^{m+1}}\right)^{\vee} \arrow[d] \\ \mathcal{B}_m H^0 \overline{B}(E^{\cdot}(Y \log D)) \arrow[r, "\int, \simeq"] & \left(\frac{\mathbb{C}[G]}{I^{m+1}}\right)^{\vee}, \end{tikzcd}$$

the rational mixed Hodge structure $\mathcal{B}_m H^0 \overline{B}(\mathbf{K})$ translates to a rational mixed Hodge structure $L_m = L_m(U, e)$ with underlying complex vector space $\left(\frac{\mathbb{C}[G]}{I^{m+1}}\right)^{\vee}$ and $\mathbb{Q}$ vector space $\left(\frac{\mathbb{Q}[G]}{I^{m+1}}\right)^{\vee}$. The integral lattice is defined to be $\left(\frac{\mathbb{Z}[G]}{I^{m+1}}\right)^{\vee}$. It is clear that $L_m \subset L_{m+1}$. In view of the description of the weight and Hodge filtrations on the cohomology of the reduced bar construction on a $\mathbb{Q}$-MHC given in Paragraph 2.4.3, the filtrations for L_m are described as follows:

- The weight filtration: $W_n(L_m)_{\mathbb{C}}$ is the space of those closed iterated integrals that can be expressed as a sum of (not necessarily closed) iterated integrals of the form $\int \omega_1 \ldots \omega_r$, with $r \le m$ and $\omega_i \in E^1(Y \log D)$, such that at most $n - r$ of the ω_i are not smooth along D. This filtration is defined over $\mathbb{Q}$. It is easy to see that $W_n(L_m) \subset L_n$.

- The Hodge filtration: $F^p(L_m)_{\mathbb{C}}$ is the space of those closed iterated integrals that can be expressed as a sum of (not necessarily closed) iterated integrals of the form $\int \omega_1 \ldots \omega_r$, where $r \le m$ and $\omega_i \in E^1(Y \log D)$, such that at least p of the ω_i are of type (1,0).

Remark. (1) One can show that L_m only depends on the pair (U, e), and not on the embedding of U as $Y \setminus D$. As in the case of mixed Hodge structure on cohomology, to explicitly describe the Hodge and weight filtrations on L_m one usually embeds U as $Y \setminus D$ as above.
(2) $L_m(U, e)$ is functorial in (U, e).

2.5 De Rham lattice in $\mathcal{O}(\pi_1^{un})$ and periods of the fundamental group

Let K be a subfield of $\mathbb{C}$, U_0 be a smooth variety over K, $e \in U_0(K)$, and $U = U_0 \otimes_K \mathbb{C}$. We assume moreover that U_0 is affine. Let $\Omega^\cdot(U_0)$ (resp. $\Omega^\cdot(U)$) be the complex of global differential forms on U_0 (resp. U). Since U is affine, the complex $\Omega^\cdot(U)$ calculates the cohomology of U. More precisely, the natural map

$$\Omega^\cdot(U_0) \otimes_K \mathbb{C} = \Omega^\cdot(U) \to E^\cdot(U)$$

is a quasi-isomorphism. It follows that one has a natural isomorphism

$$H^0\overline{B}(\Omega^\cdot(U)) \cong H^0\overline{B}(E^\cdot(U))$$

strictly compatible with the filtrations. The de Rham fundamental group $\pi_1^{dR}(U_0, e)$ of U_0 with base point e is an affine group scheme over K with coordinate ring

$$\mathcal{O}(\pi_1^{dR}(U_0, e)) = H^0\overline{B}(\Omega^\cdot(U_0)).$$

We refer to the image of $\mathcal{B}_n H^0\overline{B}(\Omega^\cdot(U_0))$ under

$$\mathcal{B}_n H^0\overline{B}(\Omega^\cdot(U_0)) \subset \mathcal{B}_n H^0\overline{B}(\Omega^\cdot(U)) \cong \mathcal{B}_n H^0\overline{B}(E^\cdot(U)) \overset{\int}{\cong} L_n(U, e)_{\mathbb{C}}$$

as the *de Rham lattice* in $L_n(U, e)_{\mathbb{C}}$. We use the notation $L_n(U_0, e)$ to refer to $L_n(U, e)$ together with the data of the de Rham lattice. The space of *periods* of $L_n(U_0, e)$ is the space of numbers of the form $f(\gamma)$, where $\gamma \in \pi_1(U, e)$ and f belongs to the de Rahm lattice in $L_n(U, e)_{\mathbb{C}}$. It is a vector space over K. The space of periods of $\pi_1(U_0, e)$ is by definition the union of the spaces of periods of the $L_n(U_0, e)$.

2.6 Griffiths' Abel-Jacobi maps

For a detailed discussion of the material of this section, see [27] and [14].

2.6.1

Notation: Given a variety Y over a field K, by an algebraic cycle of dimension (resp. codimension) i we mean an element of the free abelian group $\mathcal{Z}_i(Y)$ (resp. $\mathcal{Z}^i(Y)$) generated by irreducible closed subsets of Y of dimension (resp. codimension) i. The Chow group $CH_i(Y)$ (resp. $CH^i(Y)$) is $\mathcal{Z}_i(Y)$ (resp. $\mathcal{Z}^i(Y)$) modulo rational equivalence.[†] As usual $\mathcal{Z}(Y) := \bigoplus \mathcal{Z}^i(Y)$ and $CH(Y) := \bigoplus CH^i(Y)$. Notation-wise, we do not distinguish between an algebraic cycle and its class in the corresponding Chow group. Given Y and Y' of dimensions d and d', the group of degree zero correspondences from Y to Y' is $\mathrm{Cor}(Y, Y') := \mathcal{Z}_d(Y \times Y')$. If $f : Y \to Y'$ is a morphism, the graph of f is denoted by Γ_f; it is an element of $\mathrm{Cor}(Y, Y')$. We use the standard notation (lower star) for push-forwards along morphisms. Given algebraic cycles $Z \in \mathcal{Z}_i(Y)$ and $Z' \in \mathcal{Z}_j(Y')$, $Z \times Z' \in \mathcal{Z}_{i+j}(Y \times Y')$ denotes the Cartesian product. Given $Z \in \mathcal{Z}_i(Y \times Y')$, ${}^tZ$ is the transpose of Z; it is an element of $\mathcal{Z}_i(Y' \times Y)$. Finally, if Y is a smooth variety over a subfield of $\mathbb{C}$, $\mathcal{Z}_i^{\mathrm{hom}}(Y)$ (resp. $CH_i^{\mathrm{hom}}(Y)$) refers to the subgroup of null-homologous cycles in $\mathcal{Z}_i(Y)$ (resp. $CH_i(Y)$).

2.6.2

Let Y be a smooth projective variety over $\mathbb{C}$. The n^{th} Abel-Jacobi map associated to Y is the map[‡]

$$AJ : \mathcal{Z}_n^{\mathrm{hom}}(Y) \to JH^{2n+1}(Y)^\vee$$

defined as follows. First note that the restriction map $\left(H_{\mathbb{C}}^{2n+1}(Y)\right)^\vee \to \left(F^{n+1}H^{2n+1}(Y)\right)^\vee$ gives an isomorphism

$$JH^{2n+1}(Y)^\vee \cong \frac{\left(F^{n+1}H^{2n+1}(Y)\right)^\vee}{H_{2n+1}(Y, \mathbb{Z})},$$

where an element of $H_{2n+1}(Y, \mathbb{Z})$ is considered as an element of $\left(F^{n+1}H^{2n+1}(Y)\right)^\vee$ via integration. Thus we can equivalently define AJ as a map into

$$\frac{\left(F^{n+1}H^{2n+1}(Y)\right)^\vee}{H_{2n+1}(Y, \mathbb{Z})}.$$

Given a null-homologous n-dimensional cycle Z on Y, there is a chain C such that $\partial C = Z$. Given $c \in F^{n+1}H^{2n+1}(Y)$, take a representative $\omega \in F^{n+1}E_{\mathbb{C}}^{2n+1}(Y)$, and set

$$\int_C c = \int_C \omega.$$

[†]Note that in our notation, $CH_i(Y)$ is merely an abelian group, and not a functor from K-schemes to abelian groups.

[‡]That our notation for this map does not incorporate Y or n should not lead to any confusion.

One can show that this is independent of the choice of ω. Then

$$\mathrm{AJ}(Z) \in \frac{\left(F^{n+1}H^{2n+1}(Y)\right)^{\vee}}{H_{2n+1}(Y,\mathbb{Z})}$$

is defined to be the class of the map

$$c \mapsto \int_C c.$$

The ambiguity in having to choose C is resolved by modding out by $H_{2n+1}(Y,\mathbb{Z})$. If one insists on having $\mathrm{AJ}(Z) \in JH^{2n+1}(Y)^{\vee}$, it is the class of any map $H^{2n+1}_{\mathbb{C}}(Y) \to \mathbb{C}$ whose restriction to $F^{n+1}H^{2n+1}(Y)$ is the map $\int_C$ above.

One can show that AJ factors through $\mathrm{CH}^{\mathrm{hom}}_n(Y)$. The induced map

$$\mathrm{CH}^{\mathrm{hom}}_n(Y) \to JH^{2n+1}(Y)^{\vee}$$

is also called Abel-Jacobi, and with abuse of notation we denote it by AJ as well.

2.6.3 Maps induces by correspondences on intermediate Jacobians

Let Y (resp. Y') be a smooth projective variety of dimension d (resp. d') over $\mathbb{C}$. Suppose $l \le d + d'$. One has natural isomorphisms

$$\begin{aligned}
H^{2l}(Y \times Y')^{\vee} &\cong \left(\bigoplus_r H^r(Y) \otimes H^{2l-r}(Y')\right)^{\vee} \\
&\cong \bigoplus_r H^r(Y)^{\vee} \otimes H^{2l-r}(Y')^{\vee} \\
&\cong \bigoplus_r \underline{\mathrm{Hom}}\left(H^r(Y), H^{2l-r}(Y')^{\vee}\right) \\
&\overset{\text{Poincare duality}}{\cong} \bigoplus_r \underline{\mathrm{Hom}}\left(H^{2d-r}(Y)^{\vee}(-d), H^{2l-r}(Y')^{\vee}\right) \\
&\cong \bigoplus_r \underline{\mathrm{Hom}}\left(H^{2d-r}(Y)^{\vee}, H^{2l-r}(Y')^{\vee}\right)(d).
\end{aligned}$$

Let $Z \in \mathrm{CH}_l(Y \times Y')$. Then the class $cl(Z)$ of Z is a Hodge class in

$$H^{2l}(Y \times Y')^{\vee},$$

which is given by integration over Z (or more precisely, the smooth locus of Z) if Z is an irreducible closed subset. In view of the isomorphisms above, $\mathrm{cl}(Z)$ decomposes as a sum of Hodge classes in

$$\underline{\mathrm{Hom}}\left(H^{2d-r}(Y)^\vee, H^{2l-r}(Y')^\vee\right).$$

It follows that for each r, $\mathrm{cl}(Z)$ gives a morphism of Hodge structures

$$H^{2d-r}(Y)^\vee(l-d) \to H^{2l-r}(Y')^\vee. \tag{2.12}$$

If r is odd, this induces a map

$$JH^{2d-r}(Y)^\vee = JH^{2d-r}(Y)^\vee(l-d) \to JH^{2l-r}(Y')^\vee. \tag{2.13}$$

With abuse of notation we denote the maps (2.12) and (2.13) also by $\mathrm{cl}(Z)$.

Let $m \leq d$. The push-forward map

$$Z_* : CH_m(Y) \to CH_{m+l-d}(Y')$$

restricts to a map

$$Z_* : CH_m^{\mathrm{hom}}(Y) \to CH_{m+l-d}^{\mathrm{hom}}(Y').$$

One has a commutative diagram

$$\begin{tikzcd}
CH_m^{\mathrm{hom}}(Y) \arrow[r, "AJ"] \arrow[d, "Z_*"'] & JH^{2m+1}(Y)^\vee \arrow[d, "\mathrm{cl}(Z)"] \\
CH_{m+l-d}^{\mathrm{hom}}(Y') \arrow[r, "AJ"] & JH^{2m+2l-2d+1}(Y')^\vee.
\end{tikzcd}$$

2.7 A construction of Gross and Schoen

In this section, we recall a construction of Gross and Shoen [17]. Let K be a subfield of $\mathbb{C}$ and X be a smooth projective geometrically connected curve over K. Let m be a positive integer and $e \in X(K)$. By convention, we set $X^0 = \mathrm{Spec}\, K$. For (possibly empty) $T \subset \{1, \ldots, m\}$, let $p_T : X^m \to X^{|T|}$ be the projection map onto the coordinates in T, and $q_T : X^{|T|} \to X^m$ be the embedding that is right inverse to p_T and fills the coordinates that are not in T by e. For instance, if $m = 3$ and $T = \{2, 3\}$,

$$(x_1, x_2, x_3) \overset{p_T}{\mapsto} (x_2, x_3) \qquad \text{and} \qquad (x_1, x_2) \overset{q_T}{\mapsto} (e, x_1, x_2).$$

In general, the composition $q_T \circ p_T : X^m \to X^m$ is the morphism that keeps the T coordinates unchanged, and replaces the rest by e. Let

$$P_e = \sum_T (-1)^{|T^c|} \Gamma_{q_T \circ p_T} \in \mathrm{Cor}(X^m, X^m),$$

where T^c denotes the complement of T. For the proof of the following result, see [17].

Theorem 2.7.1. If $i < m$, the map $(P_e)_*^h : H_i(X^m) \to H_i(X^m)$ induced by P_e on homology is zero.

Let $(P_e)_*$ be the push forward map $\mathcal{Z}(X^m) \to \mathcal{Z}(X^m)$ defined by the correspondence P_e. Then

$$(P_e)_* = \sum_T (-1)^{|T^c|} (q_T \circ p_T)_*.$$

In view of commutativity of the diagram

$$\begin{array}{ccc} \mathcal{Z}_i(X^m) & \xrightarrow{(P_e)_*} & \mathcal{Z}_i(X^m) \\ \downarrow & & \downarrow \\ H_{2i}(X^m, \mathbb{C}) & \xrightarrow{(P_e)_*^h} & H_{2i}(X^m, \mathbb{C}), \end{array}$$

where the vertical maps are class maps, it follows from the previous theorem that if $2i < m$, then

$$(P_e)_*(\mathcal{Z}_i(X^m)) \subset \mathcal{Z}_i^{\mathrm{hom}}(X^m).$$

This gives a way of constructing null-homologous cycles.

Example. For $m \geq 2$, denote by $\Delta^m(X)$ the diagonal copy of X in X^m, i.e.

$$\{(x, x, \dots, x) : x \in X\} \in \mathcal{Z}_1(X^m).$$

For $m \geq 3$, by the previous observation, the *modified diagonal cycle* $(P_e)_*(\Delta^m(X))$ is null-homologous. As it is pointed out in [17], this cycle has zero Abel-Jacobi image if $m > 3$. On the other hand, if $m = 3$, this cycle, which was first defined by Gross and

Kudla in [16] and then studied more by Gross and Schoen in [17], is well-known to be interesting. See the references just mentioned. It is easy to see from its definition that

$$\begin{aligned}(P_e)_*(\Delta^3(X)) &= \Delta^3(X) - \{(e,x,x) : x \in X\} - \{(x,e,x) : x \in X\} - \{(x,x,e) : x \in X\} \\ &\quad + \{(e,e,x) : x \in X\} + \{(e,x,e) : x \in X\} + \{(x,e,e) : x \in X\}.\end{aligned}$$

We denote this cycle by $\Delta_{KGS,e}$, the modified diagonal cycle of Kudla, Gross and Schoen.

Note that

$$(P_e)_*(\Delta^2(X)) = \Delta^2(X) - \{e\} \times X - X \times \{e\},$$

which is homologically nontrivial.

Chapter 3

Algebraic cycles and π_1 of a punctured curve

In this chapter, we will prove the geometric result of the thesis. Throughout, until the end of the thesis, X is a smooth (connected) projective curve over $\mathbb{C}$ of genus g, and $\infty, e \in X(\mathbb{C})$ are distinct points. Roughly speaking, our goal in this chapter is to relate the mixed Hodge structure on $\pi_1(X-\{\infty\}, e)$ to algebraic cycles on products of X. More precisely, for each n, we consider

$$\frac{L_n}{L_{n-2}}(X - \{\infty\}, e) \in \mathbf{MHS}.$$

We will see that the weight filtration on this is given by

$$W_{n-2} = 0, \quad , W_{n-1} = \frac{L_{n-1}}{L_{n-2}}(X - \{\infty\}, e), \quad \text{and} \quad W_n = \frac{L_n}{L_{n-2}}(X - \{\infty\}, e),$$

so that it gives rise to only one interesting extension. The goal of this chapter is to describe this extension in terms of algebraic cycles on a product of X (on X^{2n-1} to be precise). The result generalizes a work of Darmon, Rotger, and Sols [6].

3.1 Construction of certain elements in the Bar construction

In this section, given an augmented differential graded algebra satisfying certain properties, we give a procedure that constructs elements is $H^0\overline{B}$ with prescribed highest length terms. This construction will be particularly important in understanding the

Hodge theory of π_1 of a punctured curve.

We assume that $A^{\cdot}$ is an augmented differential graded algebra, and that

(i) $d(A^1) = (A^1)^2$,

(ii) for each pair (a, b) of elements of A^1, $s(a, b) \in A^1$ is such that $d(s(a, b)) = -ab$.

Let $a_1, \ldots, a_n \in A^1$ be closed. Our goal is to give a closed element of $\overline{B}^0(A^{\cdot})$ of the form

$$(a_1|\ldots|a_n) \; + \; \text{lower length terms.}$$

For this, it suffices to construct a closed element of $\oplus T^{-r,r}(A^{\cdot})$ of the form

$$[a_1|\ldots|a_n] \; + \; \text{lower length terms.}$$

Set $\lambda_n = [a_1|\ldots|a_n]$. Then $d_v(\lambda_n) = 0$, and $d_h(\lambda_n) \in T^{-n+1,n}$. The idea is to define, for each $r = n-1, \ldots, 1$, an element $\lambda_r \in T^{-r,r}$ such that $d_v(\lambda_r) = -d_h(\lambda_{r+1})$. The element

$$\lambda_n + \lambda_{n-1} + \ldots + \lambda_1$$

will then be closed.
For $r = n-1, \ldots, 1$, define λ_r to be the sum of all simple tensors in $T^{-r,r}$ of the form

$$[\ldots|\ldots|\ldots|\ldots \;\ldots\; \ldots|\ldots], \tag{3.1}$$

where each block is formed by (possibly 0) successions of $s(\;,\;)$, and such that when we remove the symbols "|" and "$s(\;,\;)$", we are left with

$$[a_1 \; a_2 \; \ldots \; a_n]. \tag{3.2}$$

For example,

$$\lambda_{n-1} = \sum_{i=1}^{n-1} [a_1|\ldots|s(a_i, a_{i+1})|\ldots|a_n],$$

and

$$\begin{aligned}
\lambda_{n-2} &= \sum_{1\le i<j-1\le n-2} [a_1|\ldots|s(a_i,a_{i+1})|\ldots|s(a_j,a_{j+1})|\ldots|a_n] \\
&+ \sum_{i=1}^{n-2}[a_1|\ldots|s(s(a_i,a_{i+1}),a_{i+2})|\ldots|a_n] \\
&+ \sum_{i=1}^{n-2}[a_1|\ldots|s(a_i,s(a_{i+1},a_{i+2}))|\ldots|a_n].
\end{aligned}$$

There will be much more variety for λ_{n-3}:

$$\begin{aligned}
\lambda_{n-3} = & \quad \sum[a_1|\ldots|s(a_i,a_{i+1})|\ldots|s(a_j,a_{j+1})|\ldots|s(a_k,a_{k+1})|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(a_i,a_{i+1})|\ldots|s(s(a_j,a_{j+1}),a_{j+2})|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(a_i,a_{i+1})|\ldots|s(a_j,s(a_{j+1},a_{j+2}))|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(s(a_i,a_{i+1}),a_{i+2})|\ldots|s(a_j,a_{j+1})|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(a_i,s(a_{i+1},a_{i+2}))|\ldots|s(a_j,a_{j+1})|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(s(s(a_i,a_{i+1}),a_{i+2}),a_{i+3})|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(s(a_i,s(a_{i+1},a_{i+2})),a_{i+3})|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(a_i,s(s(a_{i+1},a_{i+2}),a_{i+3}))|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(a_i,s(a_{i+1},s(a_{i+2},a_{i+3})))|\ldots|a_n] \\
&+ \sum[a_1|\ldots|s(s(a_i,a_{i+1}),s(a_{i+2},a_{i+3}))|\ldots|a_n].
\end{aligned}$$

Note that in every summand of λ_r, there are exactly $n-r$ occurrences of s.

Lemma 3.1.1. The element $\lambda_n+\ldots+\lambda_1$ is closed.

Proof. Note that $d_v(\lambda_n)=d_h(\lambda_1)=0$. It remains to check that for each r, $-d_h(\lambda_{r+1})=d_v(\lambda_r)$. But in view of the formulas (2.7) and (2.8), both $-d_h(\lambda_{r+1})$ and $d_v(\lambda_r)$ are the sum of all simple tensors in $T^{-r,r+1}$ of the form (3.1) where each block is formed by (possibly 0) successions of $s(\ ,\ )$, and such that when we remove the symbols "|" and "$s(\ ,\ )$", we are left with (3.2). That each a_i is closed is important to make sure $d_v(\lambda_r)$ is equal to the aforementioned sum. □

Remark. It is easy to see that if $s: A^1\times A^1\to A^1$ is bilinear, then the above construction gives a linear map $(A^1_{\text{closed}})^{\otimes n}\to\mathcal{B}_n H^0\overline{B}(A)$.

3.2 Hodge Theory of π_1- The case of a punctured curve

3.2.1

Let $S \subset X(\mathbb{C})$ be of finite cardinality $|S| \geq 1$, $U = X - S$, and $e \in U(\mathbb{C})$. Let $G = \pi_1(U, e)$ and $L_m = L_m(U, e)$. Our goal in this paragraph is to study $(L_m)_{\mathbb{C}}$ more closely.

It is well-known that in this case there are holomorphic differential forms α_i ($1 \leq i \leq 2g + |S| - 1$) on U whose classes form a basis of $H^1_{dR}(U)$. We can, and will, take these such that $\alpha_1, \ldots, \alpha_g$ are of first kind (i.e. holomorphic on X), $\alpha_{g+1}, \ldots \alpha_{2g}$ are of second kind (i.e. meromorphic on X with zero residue along S), and $\alpha_{2g+1}, \ldots, \alpha_{2g+|S|-1}$ are of third kind with simple poles at points in S. Let $R^{\cdot}$ be the sub-object of $E^{\cdot}_{\mathbb{C}}(U)$ given by $R^0 = \mathbb{C}$, $R^1 = \sum_{i=1}^{2g+|S|-1} \alpha_i \mathbb{C}$, and $R^2 = 0$. The inclusion map $R^{\cdot} \to E^{\cdot}_{\mathbb{C}}(U)$ is a quasi-isomorphism, so that in particular

$$\mathcal{B}_m H^0\overline{B}(R^{\cdot}) \cong \mathcal{B}_m H^0\overline{B}(E^{\cdot}_{\mathbb{C}}(U)) \qquad \text{and} \qquad H^0\overline{B}(R^{\cdot}) \cong H^0\overline{B}(E^{\cdot}_{\mathbb{C}}(U)).$$

It is easy to see that $H^0\overline{B}(R^{\cdot})$, as a vector space, is the (underlying vector space of the) tensor algebra on R^1, and the multiplication is the shuffle product. In other words, $H^0\overline{B}(R^{\cdot})$ is the shuffle algebra on the letters α_i ($1 \leq i \leq 2g + |S| - 1$). The filtration $\mathcal{B}_{\cdot}$ is the tensor length filtration. The following description of L_m is now immediate.

Proposition 3.2.1. The integration map $H^0\overline{B}(R^{\cdot}) \to \varinjlim \left(\frac{\mathbb{C}[G]}{I^{m+1}}\right)^{\vee}$ which maps

$$[\alpha_{i_1}|\ldots|\alpha_{i_r}] \mapsto \int \alpha_{i_1} \ldots \alpha_{i_r}$$

is an isomorphism, which maps $\mathcal{B}_m$ onto $(L_m)_{\mathbb{C}}$. In particular, any complex valued function on G that (after extending linearly to $\mathbb{C}[G]$) vanishes on I^{m+1} is given by a unique (linear combination of) iterated integral(s) of length $\leq m$ in the forms α_i.

3.2.2

From now on, let $S = \{\infty\}$. (Thus $U = X - \{\infty\}$ and $L_n = L_n(X - \{\infty\}, e)$.) The complex $F^1E^{\cdot}(X \log \infty)$ is exact in degree 2. For each $a, a' \in E^1(X \log \infty)$, let $s(a, a') \in F^1E^1(X \log \infty)$ be such that $d(s(a, a')) = -a \wedge a'$. If $a \wedge a' = 0$, we specifically take $s(a, a') = 0$.

The differential graded algebra $E^{\cdot}(X \log \infty)$ meets the condition of Section 3.1, and hence for $\omega_1, \ldots, \omega_n$ closed smooth 1-forms on X, the construction given in that section gives us a closed element of $\overline{B}^0 E^{\cdot}(X \log \infty)$ of the form

$$(\omega_1|\ldots|\omega_n) + \text{lower length terms},$$

and thus a closed iterated integral on $X - \{\infty\}$ of the form

$$\int \omega_1 \ldots \omega_n + \text{lower length terms}, \tag{3.3}$$

where all the 1-forms involved are in $E^1(X \log \infty)$. Moreover, by construction, in each term of length r above there are $n - r$ occurrences of s, and hence at most $n - r$ forms with a pole at ∞. In view of the description of the weight filtration given in Paragraph 2.4.4, this implies the following lemma.

Lemma 3.2.1. Given closed smooth 1-forms $\omega_1, \ldots, \omega_n$ on X, there is an element of $W_n(L_n)_{\mathbb{C}}$ of the form (3.3).

3.2.3

The Weight Filtration of L_m: We now show that the weight filtration on L_m coincides with the length filtration.

Proposition 3.2.2. For $n \leq m$, $W_n L_m = L_n$.

Proof. It is enough to show $W_n L_n = L_n$ for all n, for then, if $n \leq m$, we see in view of $W_n L_m \subset L_n$ that $W_n L_m = L_n$. We argue by induction on n. This is trivial for $n = 0$. Suppose $W_{n-1} L_{n-1} = L_{n-1}$. In view of Proposition 3.2.1, it suffices to show that

$$\int \alpha_{j_1} \ldots \alpha_{j_n} \in W_n(L_n)_{\mathbb{C}}.$$

For each i, let $\omega_i \in E^1_{\mathbb{C}}(X)$ be such that $\alpha_{j_i} = \omega_i + df_i$ on U, where f_i is a smooth function on U; this can be done because inclusion of U in X gives an isomorphism on the level of H^1. Thanks to the relations (2.10) satisfied by iterated integrals, we have

$$\int \alpha_{j_1} \ldots \alpha_{j_n} = \int \omega_1 \ldots \omega_n + \text{lower length terms}.$$

In view of Lemma 3.2.1 we can write

$$\begin{aligned}\int \alpha_{j_1} \ldots \alpha_{j_n} = \quad & \Big(\text{an element of } W_n(L_n)_{\mathbb{C}} \text{ of the form} \\ & \int \omega_1 \ldots \omega_n + \text{lower length terms}\Big) \\ & + \int \text{terms of length} \leq n-1.\end{aligned}$$

The left hand side and the first integral on the right are both closed, so that the second integral on the right also has to be closed, hence in $(L_{n-1})_{\mathbb{C}}$, and by the induction hypothesis in $W_{n-1}(L_{n-1})_{\mathbb{C}} \subset W_n(L_n)_{\mathbb{C}}$. The desired conclusion follows.

□

3.2.4

In this paragraph we review some facts from group theory and then apply them to our setting. Let Γ be a finitely generated group, $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$, and I be the augmentation ideal in $\mathbb{K}[\Gamma]$. Let $\Gamma^{ab} := \frac{\Gamma}{[\Gamma,\Gamma]}$. It is well-known that

$$\frac{I}{I^2} \to \Gamma^{ab} \otimes \mathbb{K} \qquad [\gamma - 1] \mapsto [\gamma] \tag{3.4}$$

is an isomorphism. For $n > 1$ however, the quotients $\frac{I^n}{I^{n+1}}$ become increasingly more complicated in general. (See Stallings [26].) On the other hand, if Γ is free, these quotients are easy to describe: One has an isomorphism

$$\frac{I^n}{I^{n+1}} \to \left(\frac{I}{I^2}\right)^{\otimes n} \tag{3.5}$$

given by

$$[(\gamma_1 - 1) \ldots (\gamma_n - 1)] \mapsto [\gamma_1 - 1] \otimes \ldots \otimes [\gamma_n - 1].$$

Let Γ be free. Then $\frac{I}{I^2}$, and hence $\frac{I^n}{I^{n+1}}$ for every n, is a free $\mathbb{K}$-module. (Of course, this is only interesting when $\mathbb{K} = \mathbb{Z}$.) One has for each n an obvious exact sequence (of $\mathbb{K}$-modules)

$$0 \longrightarrow \frac{I^n}{I^{n+1}} \longrightarrow \frac{\mathbb{K}[\Gamma]}{I^{n+1}} \longrightarrow \frac{\mathbb{K}[\Gamma]}{I^n} \longrightarrow 0.$$

We see by induction that each $\dfrac{\mathbb{K}[\Gamma]}{I^n}$ is free, and hence dualizing the previous sequence we get exact

$$0 \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^n}\right)^\vee \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^{n+1}}\right)^\vee \longrightarrow \left(\frac{I^n}{I^{n+1}}\right)^\vee \longrightarrow 0.$$

Via

$$\left(\frac{I^n}{I^{n+1}}\right)^\vee \overset{(3.5)}{\simeq} \left(\left(\frac{I}{I^2}\right)^{\otimes n}\right)^\vee \overset{(3.4)}{\simeq} \left(\left(\Gamma^{ab} \otimes \mathbb{K}\right)^{\otimes n}\right)^\vee,$$

we get a short exact sequence

$$0 \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^n}\right)^\vee \longrightarrow \left(\frac{\mathbb{K}[\Gamma]}{I^{n+1}}\right)^\vee \xrightarrow{q_{\mathbb{K}}} \left((\Gamma^{ab} \otimes \mathbb{K})^{\otimes n}\right)^\vee \longrightarrow 0. \tag{3.6}$$

Unwinding definitions, it is easy to see that $q_{\mathbb{K}}$ sends $f \in \left(\dfrac{\mathbb{K}[\Gamma]}{I^{n+1}}\right)^\vee$ to the map

$$[\gamma_1] \otimes \ldots \otimes [\gamma_n] \;\mapsto\; f\left([(\gamma_1 - 1)\ldots(\gamma_n - 1)]\right).$$

It is clear that (3.6) is compatible with extending $\mathbb{K}$.

We apply this to the group $G = \pi_1(U, e)$. In view of the definition of $(L_n)_{\mathbb{K}}$, the isomorphism $G^{ab} \otimes \mathbb{K} \simeq H_1(U, \mathbb{K})$ given by $[\gamma] \mapsto [\gamma]$, and

$$\left(H_1(U,\mathbb{K})^{\otimes n}\right)^\vee \cong \left(H_1(U,\mathbb{K})^\vee\right)^{\otimes n} \cong \left(H^1(U)_{\mathbb{K}}\right)^{\otimes n},$$

the sequence (3.6) reads

$$0 \longrightarrow (L_{n-1})_{\mathbb{K}} \overset{\text{inclusion}}{\longrightarrow} (L_n)_{\mathbb{K}} \xrightarrow{q_{\mathbb{K}}} \left(H^1(U)_{\mathbb{K}}\right)^{\otimes n} \longrightarrow 0. \tag{3.7}$$

Compatibility with extending $\mathbb{K}$ implies the maps in this sequence when $\mathbb{K} = \mathbb{C}$ are defined over $\mathbb{Z}$ (i.e. take integral lattices to integral lattices, and hence rationals to rationals). The sequence for $\mathbb{K} = \mathbb{Z}$ is the restriction of the sequence for $\mathbb{K} = \mathbb{C}$ to the integral lattices.

The inclusion $U \subset X$ gives an isomorphism $H^1(X) \to H^1(U)$. We will always identify the two Hodge structures via this map, and from now on simply write H^1 for

$H^1(U) = H^1(X)$. Unwinding definitions, in view of

$$\int_{(\gamma_1-1)\ldots(\gamma_n-1)} \omega_1 \ldots \omega_n + \text{lower length terms} = \int_{\gamma_1} \omega_1 \ \ldots \ \int_{\gamma_n} \omega_n, \tag{3.8}$$

we see that the map $q_{\mathbb{C}}$ sends

$$\int \omega_1 \ldots \omega_n + \text{lower length terms} \ \ \mapsto [\omega_1] \otimes \ldots \otimes [\omega_n], \tag{3.9}$$

where the integral on the left is closed, each ω_i is a closed smooth 1-form on U, and $[\omega_i]$ denotes the cohomology class of ω_i. Note that (3.8) is a consequence of (2.9).

It is clear from the description of the weight filtration on L_n given in Proposition 3.2.2 that the map $q_{\mathbb{C}}$ is compatible with the weight filtrations. We shall shortly see that it is also compatible with the Hodge filtrations, so that it gives an isomorphism of mixed Hodge structures

$$\frac{L_n}{L_{n-1}} \to (H^1)^{\otimes n}.^{\dagger}$$

We will not try to take the fastest route to this end. Rather, we will conclude this as a consequence of existence of a section of $q_{\mathbb{C}}$ respecting the Hodge filtrations. Over the next three paragraphs, we will construct a particular section s_F of $q_{\mathbb{C}}$. This map enjoys some nice properties and will play an important role later on.

3.2.5

In this paragraph, we review some basic facts about Green functions. For the proofs and further details, see [22].

Let φ be a real non-exact smooth form of type (1,1) on X, D be a nonzero divisor on X, and $\mathrm{supp}(D)$ be the support of D. Then φ is exact on $X - \mathrm{supp}(D)$. Indeed, one can prove that there is a unique (smooth) function $g_{D,\varphi} : X - \mathrm{supp}(D) \to \mathbb{R}$, called the Green function for φ relative to D, satisfying the following properties:

(1) If D is represented by a meromorphic function f on an open set (in analytic topol-

†This is not a new result. See Remark (3) of Paragraph 3.2.8.

ogy) V of X, then the function $V - \mathrm{supp}(D) \to \mathbb{R}$ defined by†

$$P \mapsto g_{D,\varphi}(P) + (\int_X \varphi)\ \log|f(P)|^2$$

extends smoothly to V.

(2) $dd^c g_{D,\varphi} = (\deg D)\varphi$ on $X - \mathrm{supp}(D)$, where $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$ with the ∂, $\bar{\partial}$ the usual operators.

(3) $\int_X g_{D,\varphi}\varphi = 0$.

One can show that a function satisfying (1) and (2) is unique up to a constant. Condition (3) is included to guarantee uniqueness. Conditions (1) and (2) are the important ones for us. Take $D = \infty$. It follows from (1) that locally near the point ∞, with a chart taken such that ∞ corresponds to $z = 0$, the function $g_{\infty,\varphi}$ looks like

$$-(\int_X \varphi)\ \log z\bar{z} + \text{a smooth function.}$$

It follows that $\partial g_{\infty,\varphi}$ near ∞ (again with $z = 0$ corresponding to the point ∞) is of the form

$$-(\int_X \varphi)\ \frac{dz}{z} + \text{a smooth 1-form,}$$

so that $\partial g_{\infty,\varphi}$ is in $E^1(X \log \infty)$. By condition (2), $d(\frac{1}{2\pi i}\partial g_{\infty,\varphi}) = \varphi$ on U. To sum up, given a a non-exact real two-form φ on X, we have a specific 1-form $\frac{1}{2\pi i}\partial g_{\infty,\varphi}$ of type (1,0) in $E^1(X \log \infty)$ with residue $-\dfrac{1}{2\pi i}\int_X \varphi$ at ∞ whose d is φ on U.

3.2.6

Throughout this paragraph, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{H}^1_{\mathbb{K}}(X)$ be the space of $\mathbb{K}$-valued harmonic 1-forms on X. One has a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^1_{\mathbb{R}}(X) & \cong & H^1_{\mathbb{R}} \\ \cap & & \cap \\ \mathcal{H}^1_{\mathbb{C}}(X) & \cong & H^1_{\mathbb{C}}. \end{array}$$

†The appearance of the extra factor $\int_X \varphi$ compared to Lang comes from the fact that φ is not normalized here.

Via the horizontal isomorphisms we get a pure real Hodge structure $\mathcal{H}^1(X)$ of weight one with $\mathbb{K}$-vector space $\mathcal{H}^1_{\mathbb{K}}(X)$. The subspace $F^1\mathcal{H}^1_{\mathbb{C}}(X)$ is the space of holomorphic 1-forms on X. Let $\wedge : \mathcal{H}^1_{\mathbb{K}} \otimes \mathcal{H}^1_{\mathbb{K}} \to E^2_{\mathbb{K}}(X)$ be the "wedge product" map, i.e. given by $\wedge(\omega_1 \otimes \omega_2) = \omega_1 \wedge \omega_2$. The following lemma combines some ideas of Pulte [25] and Darmon, Rotger and Sols [6].

Lemma 3.2.2. There is a $\mathbb{C}$-linear map

$$\nu : \mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X) \to E^1(X \log \infty)$$

such that

(i) for each $w \in \mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X)$, $d(\nu(w)) = - \wedge (w)$ on U,

(ii) ν respects the Hodge filtration F,

(iii) for each $w \in \mathcal{H}^1_{\mathbb{R}}(X) \otimes \mathcal{H}^1_{\mathbb{R}}(X)$, there is a smooth real 1-form $\nu_{\mathbb{R}} = \nu_{\mathbb{R}}(w)$ on U such that $\nu(w) - \nu_{\mathbb{R}}$ is exact on U,

(iv) for every $w \in \mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X)$, the residue of $\nu(w)$ at ∞ is $\dfrac{1}{2\pi i}\displaystyle\int_X \wedge(w)$.

Proof. The cup product $H^1 \otimes H^1 \xrightarrow{\smile} H^2(X)$ is a morphism of Hodge structures. Let K be its kernel. Ignoring the rational structures, we can think of K as a sub-Hodge structure of the real Hodge structure $H^1 \otimes H^1$. Let $\mathcal{K}$ be its copy in $\mathcal{H}^1(X) \otimes \mathcal{H}^1(X)$. Thus $\mathcal{K}_{\mathbb{K}}$ consists of those $w \in \mathcal{H}^1_{\mathbb{K}}(X) \otimes \mathcal{H}^1_{\mathbb{K}}(X)$ for which $\wedge(w) \in E^2_{\mathbb{K}}(X)$ is exact. One has a short exact sequence of real Hodge structures

$$0 \longrightarrow \mathcal{K} \xrightarrow{\text{inclusion}} \mathcal{H}^1(X) \otimes \mathcal{H}^1(X) \cong H^1 \otimes H^1 \xrightarrow{\smile} H^2(X) \longrightarrow 0.$$

The category of pure real Hodge structures is semisimple, so that there is

$$\phi \in \mathcal{H}^1_{\mathbb{R}}(X) \otimes \mathcal{H}^1_{\mathbb{R}}(X) \ \cap\ F^1(\mathcal{H}^1_{\mathbb{C}}(X) \otimes \mathcal{H}^1_{\mathbb{C}}(X))$$

giving rise to a decomposition of $\mathcal{H}^1(X) \otimes \mathcal{H}^1(X)$ as an internal direct sum

$$\mathcal{H}^1(X) \otimes \mathcal{H}^1(X) = \mathcal{K} \ \oplus\ \mathcal{L},$$

where $\mathcal{L}$ is the one dimensional sub-object of $\mathcal{H}^1(X) \otimes \mathcal{H}^1(X)$ generated by ϕ[†]. Because of the linear nature of the requirements, it suffices to define ν on $\mathcal{K}_{\mathbb{C}}$ and $\mathcal{L}_{\mathbb{C}}$ satisfying

[†]We could have instead worked over $\mathbb{Q}$ here, as the Mumford-Tate group of X is reductive. But this would not result in any major simplification.

(i)-(iv).

Definition of ν on $\mathcal{K}_{\mathbb{C}}$: This part is due to Pulte [25]. The operator d on X is strict with respect to the Hodge filtration, so that one can choose

$$\nu' : \mathcal{K}_{\mathbb{C}} \to E^1_{\mathbb{C}}(X)$$

respecting the Hodge filtration such that $d\nu'(w) = -\wedge(w)$ on X. Now recall that one has a decomposition $E^1_{\mathbb{K}}(X) = \mathcal{H}^1_{\mathbb{K}}(X) \oplus \mathcal{H}^1_{\mathbb{K}}(X)^{\perp}$, where $\mathcal{H}^1_{\mathbb{K}}(X)^{\perp}$ is the space of $\mathbb{K}$-valued 1-forms orthogonal to $\mathcal{H}^1_{\mathbb{K}}(X)$ with respect to the inner product defined using the Hodge $*$ operator. Recall also that the projections $E^1_{\mathbb{C}}(X) \to \mathcal{H}^1_{\mathbb{C}}(X)$ and $E^1_{\mathbb{C}}(X) \to \mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$ preserve type. Define ν to be the composition of ν' and the latter projection. Since harmonic forms are closed, we have $d\nu(w) = d\nu'(w) = -\wedge(w)$. Note that condition (iv) holds trivially. We claim that ν satisfies property (iii) as well. Let $w \in \mathcal{K}_{\mathbb{R}}$. Then $\wedge(w)$ is exact and real, so that there is $\nu'_{\mathbb{R}} \in E^1_{\mathbb{R}}(X)$ such that $d\nu'_{\mathbb{R}} = -\wedge(w)$. Let $\nu_{\mathbb{R}}$ be the component of $\nu'_{\mathbb{R}}$ in $\mathcal{H}^1_{\mathbb{R}}(X)^{\perp}$. Then $d\nu_{\mathbb{R}} = d\nu'_{\mathbb{R}} = -\wedge(w)$, so that $\nu(w) - \nu_{\mathbb{R}} \in \mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$ is closed. The desired conclusion follows from the general fact that a closed element of $\mathcal{H}^1_{\mathbb{K}}(X)^{\perp}$ is necessarily exact. Note that on the subspace $\mathcal{K}_{\mathbb{C}}$ the requirements of the lemma hold on all of X, not just U.

Definition of ν on $\mathcal{L}_{\mathbb{C}}$: Define ν on the subspace $\mathcal{L}_{\mathbb{C}} = \mathbb{C}\phi$ by $\nu(\phi) = -\frac{1}{2\pi i}\partial g_{\infty,\wedge(\phi)}$. Conditions (i), (ii) and (iv) hold by Paragraph 3.2.5. As for condition (iii), note that $-d^c g_{\infty,\wedge(\phi)}$ is real, and

$$-\frac{1}{2\pi i}\partial g_{\infty,\wedge(\phi)} + d_c g_{\infty,\wedge(\phi)} = -\frac{1}{4\pi i} d g_{\infty,\wedge(\phi)}.$$

□

If the point ∞ is not clear from the context, we will write ν_∞ instead of ν. Note that the map ν is not natural; it depends on the choices of ϕ and ν'.

3.2.7

In this paragraph, we use Lemma 3.2.2 to construct a section s_F of $q_{\mathbb{C}} : (L_n)_{\mathbb{C}} \to (H^1_{\mathbb{C}})^{\otimes n}$ that is compatible with the Hodge filtrations, and also such that its composition with $(L_n)_{\mathbb{C}} \to \left(\frac{L_n}{L_{n-2}}\right)_{\mathbb{C}}$ is defined over $\mathbb{R}$. This map is of crucial importance in the later parts of the chapter.

By exactness of $F^1E^{\cdot}(X\log\infty)$ in degree 2, one can (non-uniquely) extend the map ν of the previous paragraph to a map

$$\tilde{\nu} : E^1(X\log\infty)\otimes E^1(X\log\infty)\to E^1(X\log\infty)$$

respecting the Hodge filtrations and satisfying $d(\tilde{\nu}(w)) = -\wedge(w)$ for every $w\in E^1(X\log\infty)\otimes E^1(X\log\infty)$. The differential graded algebra $E^{\cdot}(X\log\infty)$ with the data of $s(a,a') = \tilde{\nu}(a\otimes a')$ for each $a,a'\in E^1(X\log\infty)$ satisfies the conditions of Section 3.1, and hence in particular for $\omega_1,\ldots,\omega_n\in\mathcal{H}^1_{\mathbb{C}}(X)$, we have a closed iterated integral on U of the form

$$\int \omega_1\ldots\omega_n + \sum_{i=1}^{n-1}\omega_1\ldots\nu(\omega_i\otimes\omega_{i+1})\ldots\omega_n + \text{terms of length at most } n-2. \quad (3.10)$$

(See the construction of Section 3.1.) In view of $(H^1_{\mathbb{C}})^{\otimes n}\cong(\mathcal{H}^1_{\mathbb{C}})^{\otimes n}$, we define the map $s_F : (H^1_{\mathbb{C}})^{\otimes n}\to(L_n)_{\mathbb{C}}$ by

$$[\omega_1]\otimes\ldots\otimes[\omega_n]\mapsto \text{the iterated integral described above,}$$

where $\omega_i\in\mathcal{H}^1_{\mathbb{C}}(X)$ and $[\omega_i]$ denotes the cohomology class of ω_i. This is well-defined and linear (see the final remark of Section 3.1), and in view of (3.9) it is a section of $q_{\mathbb{C}}$ (of Paragraph 3.2.4). Also, it is apparent from the construction of Section 3.1 that since $\tilde{\nu}$ preserves the Hodge filtration F, so does s_F. That s_F respects the weight filtration (over $\mathbb{C}$) is obvious from $W_n(L_n)_{\mathbb{C}} = (L_n)_{\mathbb{C}}$. We have proved parts (i)-(iii) of the following lemma.

Lemma 3.2.3. There is a $\mathbb{C}$-linear map $s_F : (H^1_{\mathbb{C}})^{\otimes n}\to(L_n)_{\mathbb{C}}$ that satisfies the following properties:

(i) Given $\omega_1,\ldots,\omega_n\in\mathcal{H}^1_{\mathbb{C}}(X)$, $s_F([\omega_1]\otimes\ldots\otimes[\omega_n])$ is of the form (3.10).

(ii) s_F is a section of $q_{\mathbb{C}} : (L_n)_{\mathbb{C}}\to(H^1_{\mathbb{C}})^{\otimes n}$.

(iii) s_F respects the Hodge and weight filtrations.

(iv) The composition

$$\mathfrak{s}_F :\ (H^1_{\mathbb{C}})^{\otimes n}\xrightarrow{s_F}(L_n)_{\mathbb{C}}\xrightarrow{\text{quotient}}\left(\frac{L_n}{L_{n-2}}\right)_{\mathbb{C}}$$

is defined over $\mathbb{R}$.

Proof. (of (iv)) We must show that if $w \in (H^1_{\mathbb{R}})^{\otimes n}$, then

$$\mathfrak{s}_F(w) \in (\frac{L_n}{L_{n-2}})_{\mathbb{R}} \subset (\frac{L_n}{L_{n-2}})_{\mathbb{C}},$$

or equivalently, $s_F(w) \in (L_n)_{\mathbb{R}} + (L_{n-2})_{\mathbb{C}}$. It suffices to consider $w = [\omega_1] \otimes \ldots \otimes [\omega_n]$, where the $\omega_i \in \mathcal{H}^1_{\mathbb{R}}(X)$. In view of Lemma 3.2.2(iii) and the relations (2.10) satisfied by iterated integrals, we have

$$s_F(w) = \int \omega_1 \ldots \omega_n + \sum_{i=1}^{n-1} \omega_1 \ldots \nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1}) \ldots \omega_n + \text{terms of length} \leq n-2.$$

Applying the construction of Section 3.1 to the differential graded algebra $E^{\cdot}_{\mathbb{R}}(U)$ with $s(-,-)$ chosen such that $s(\omega_i, \omega_{i+1}) = \nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1})$, we get a closed element of $\overline{B}^0(E^{\cdot}_{\mathbb{R}}(U))$ of the form

$$(\omega_1|\ldots|\omega_n) + \sum_{i=1}^{n-1} (\omega_1|\ldots|\nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1})|\ldots|\omega_n) + \text{terms of length} \leq n-2,$$

and hence an element of $(L_n)_{\mathbb{R}}$ of the form

$$\int \omega_1 \ldots \omega_n + \sum_{i=1}^{n-1} \omega_1 \ldots \nu_{\mathbb{R}}(\omega_i \otimes \omega_{i+1}) \ldots \omega_n + \text{terms of length} \leq n-2.$$

This differs from $s_F(w)$ by an element of $(L_{n-2})_{\mathbb{C}}$, giving the desired conclusion. □

3.2.8

Let $\overline{q}_{\mathbb{C}}$ be the isomorphism of vector spaces

$$\left(\frac{L_n}{L_{n-1}}\right)_{\mathbb{C}} \to (H^1_{\mathbb{C}})^{\otimes n}$$

induced by $q_{\mathbb{C}}$. Let $\overline{s}_F$ be the composition

$$(H^1_{\mathbb{C}})^{\otimes n} \xrightarrow{s_F} (L_n)_{\mathbb{C}} \xrightarrow{\text{quotient}} \left(\frac{L_n}{L_{n-1}}\right)_{\mathbb{C}}.$$

Then $\overline{s}_F$ is the inverse of $\overline{q}_{\mathbb{C}}$. By the discussion of Paragraph 3.2.4, $\overline{q}_{\mathbb{C}}$ restricts to an isomorphism of the integral lattices. It follows that the same is true for $\overline{s}_F$. Moreover, $\overline{s}_F$ is compatible with the Hodge and weight filtrations (because so is s_F), and hence is a morphism of mixed Hodge structures. In view of strictness of morphisms in **MHS** with respect to the Hodge filtration, $\overline{q}_{\mathbb{C}}$ is also compatible with the Hodge filtration. The following statement follows. (Compatibility of $\overline{q}_{\mathbb{C}}$ with the weight filtration is obvious.)

Proposition 3.2.3. The map $q_{\mathbb{C}}$ induces an isomorphism of mixed Hodge structures

$$\overline{q} : \frac{L_n}{L_{n-1}} \to (H^1)^{\otimes n}.$$

In the interest of keeping the notation simple we did not incorporate n in the notation for $\overline{q}$ here. This will not lead to any confusion in this chapter. Later, if there is a possibility of confusion, we will instead use the decorated notation $\overline{q}_n$ for the isomorphism given in Proposition 3.2.3.

Remark. (1) Note that in particular this says even though the mixed Hodge structure L_m may depend on the base point e, the quotient $\mathrm{Gr}_n^W L_m = \frac{L_n}{L_{n-1}}$ does not. In fact, it does not even depend on the point ∞ we removed from X. It is true in general that for any smooth connected complex variety the quotients $\frac{L_n}{L_{n-1}}$ are independent of the base point. See (3.22) Remark (iii) of [20].
(2) It follows from the above that the map $q_{\mathbb{C}}$ is also compatible with the Hodge filtration, and that (3.7) is a short exact sequence of mixed Hodge structures.
(3) We should clarify that Proposition 3.2.3 is not a new result. For instance, it can be deduced from the ideas behind Remark (iii) of Paragraph (3.22) of [20].

3.3 The extension $\mathbb{E}_{n,e}^{\infty}$

3.3.1

Let A be a mixed Hodge structure with torsion free $A_{\mathbb{Z}}$. The kernel of the surjective map

$$\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z})$$

induced by the natural quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})$. Putting this together with

$$\mathrm{Hom}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}),$$

we obtain

$$\frac{\mathrm{Hom}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R})}{\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})} \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}).$$

Now suppose A is pure of odd weight. Then so is $A^\vee$, and

$$JA^\vee \overset{(2.2)}{\cong} \frac{\mathrm{Hom}_{\mathbb{R}}(A_{\mathbb{R}}, \mathbb{R})}{\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})} \cong \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}).$$

Unwinding definitions, we see that given $f : A_{\mathbb{C}} \to \mathbb{C}$ defined over $\mathbb{R}$, the class of f in $JA^\vee$ corresponds under the identification to the composition

$$A_{\mathbb{Z}} \overset{\text{inclusion}}{\to} A_{\mathbb{R}} \overset{f}{\to} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \tag{3.11}$$

in $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z})$.

Remark. Here we make an observation that will be useful later on. Let A be of weight $2n-1$, and $f : A_{\mathbb{C}} \to \mathbb{C}$ be defined over $\mathbb{R}$. It follows from the above that $f(A_{\mathbb{Z}}) \subset \mathbb{Z}$ if and only if the restriction of f to $F^n A_{\mathbb{C}}$ is equal to that of an element of $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z}) \subset \mathrm{Hom}(A_{\mathbb{C}}, \mathbb{C})$. Indeed, the first statement is equivalent to that the composition (3.11) is trivial, which is equivalent to that the class of f is trivial in $JA^\vee$, i.e. $f \in F^{1-n}(A^\vee)_{\mathbb{C}} + \mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, \mathbb{Z})$, which, in view of

$$F^{1-n}(A^\vee)_{\mathbb{C}} = \{g : A_{\mathbb{C}} \to \mathbb{C} : g(F^n A_{\mathbb{C}}) = 0\},$$

is equivalent to the second statement. Note that the "only if" part of the statement is trivial.

3.3.2

Let $H_1 := (H^1)^\vee$. We identify $(H_1)_{\mathbb{Z}}$ with $H_1(X, \mathbb{Z})$ (the singular homology). One has an isomorphism of Hodge structures $H^1(1) \cong H_1$ given by Poincare duality

$$PD : H^1(1) \xrightarrow{\simeq} H_1, \qquad [\omega] \mapsto \int_X [\omega] \wedge -,$$

where ω is a smooth closed 1-form on X. This gives for each positive n an isomorphism

$$PD^{\otimes n} : (H^1)^{\otimes n}(n) \longrightarrow H_1^{\otimes n} \cong (H^1)^{\otimes -n},$$

given by

$$[\omega_1]\otimes\ldots\otimes[\omega_n] \mapsto PD([\omega_1])\otimes\ldots\otimes PD([\omega_n]) = \left([\omega_1'] \otimes \ldots \otimes [\omega_n'] \mapsto \prod_i \int_X [\omega_i] \wedge [\omega_i'] \right).$$

We have

$$
\begin{aligned}
\mathrm{Ext}\left((H^1)^{\otimes n},(H^1)^{\otimes n-1}\right) &\overset{\text{Carlson (Par. 2.2.2)}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n},(H^1)^{\otimes n-1}\right)\\
&\overset{\text{Lemma 2.1.1(a)}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n}\otimes(H^1)^{\otimes 1-n},\mathbb{Z}(0)\right)\\
&\overset{PD^{\otimes n-1}}{\cong} J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n}\otimes(H^1)^{\otimes n-1}(n-1),\mathbb{Z}(0)\right)\\
&\overset{\text{Lemma 2.1.1(f)}}{\cong} J((H^1)^{\otimes 2n-1})^{\vee}.
\end{aligned}
\tag{3.12}
$$

Let Ψ be the composition isomorphism

$$\mathrm{Ext}\left((H^1)^{\otimes n},(H^1)^{\otimes n-1}\right)\longrightarrow J((H^1)^{\otimes 2n-1})^{\vee}.$$

We denote by Φ the isomorphism

$$J((H^1)^{\otimes 2n-1})^{\vee}\longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left((H^1_{\mathbb{Z}})^{\otimes 2n-1},\mathbb{R}/\mathbb{Z}\right)$$

given by Paragraph 3.3.1. (To make the notation slightly simpler we did not include n as a part of the symbol for the maps Φ and Ψ. This should not cause any confusion as n will be clear from the context.)

3.3.3 Definition of $\mathbb{E}^{\infty}_{n,e}$

Let $n\geq 2$. In this paragraph, we use $\dfrac{L_n}{L_{n-2}}$ to define an element

$$\mathbb{E}^{\infty}_{n,e}\in \mathrm{Ext}((H^1)^{\otimes n},(H^1)^{\otimes n-1}).$$

It follows from Proposition 3.2.2 that the weight filtration on $\dfrac{L_n}{L_{n-2}}$ is given by

$$W_{n-2}=0,\qquad W_{n-1}=\frac{L_{n-1}}{L_{n-2}},\qquad \text{and}\quad W_n=\frac{L_n}{L_{n-2}}.$$

The filtration gives rise to the exact sequence

$$0\longrightarrow\frac{L_{n-1}}{L_{n-2}}\xrightarrow{\iota}\frac{L_n}{L_{n-2}}\xrightarrow{\text{quotient}}\frac{L_n}{L_{n-1}}\longrightarrow 0,$$

where ι is the inclusion map. Using the isomorphism of Proposition 3.2.3 to replace $\frac{L_{n-1}}{L_{n-2}}$ (resp. $\frac{L_n}{L_{n-1}}$) by $(H^1)^{\otimes n-1}$ (resp. $(H^1)^{\otimes n}$), we get the exact sequence

$$0 \longrightarrow (H^1)^{\otimes n-1} \xrightarrow{i} \frac{L_n}{L_{n-2}} \xrightarrow{q} (H^1)^{\otimes n} \longrightarrow 0. \tag{3.13}$$

Here $i = \iota \bar{q}^{\,-1}$, and q is the composition

$$\frac{L_n}{L_{n-2}} \overset{\text{quotient}}{\longrightarrow} \frac{L_n}{L_{n-1}} \xrightarrow{\bar{q}} (H^1)^{\otimes n}.$$

Let $\mathbb{E}^{\infty}_{n,e} \in \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1})$ be the extension defined by the sequence (3.13).

Remark. (1) For each $n \geq 2$, one gets a map (of sets)

$$X \times X\,(\mathbb{C}) - \text{diagonal} \to \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1})$$

given by $(e, \infty) \mapsto \mathbb{E}^{\infty}_{n,e}$.
(2) One can deduce from a theorem of Pulte [25] that the map

$$X(\mathbb{C}) - \{\infty\} \to \mathrm{Ext}((H^1)^{\otimes 2}, H^1)$$

defined by $e \mapsto \mathbb{E}^{\infty}_{2,e}$ is injective.

Our goal in the remainder of this section is to describe the images of $\mathbb{E}^{\infty}_{n,e}$ under Ψ and $\Phi \circ \Psi$. To this end, in view of Paragraph 3.3.1 and (ii) of Paragraph 2.2.2, we will define an integral retraction of i and a Hodge section of q defined over $\mathbb{R}$. (See (3.13).)

3.3.4

An integral retraction of i: In this paragraph, we define an integral retraction $r_{\mathbb{Z}}$ of i, i.e. a linear map

$$r_{\mathbb{Z}} : (\frac{L_n}{L_{n-2}})_{\mathbb{C}} \longrightarrow (H^1_{\mathbb{C}})^{\otimes n-1}$$

defined over $\mathbb{Z}$, that is left inverse to i.

Choose $\beta_1, \ldots, \beta_{2g} \in \pi_1(U, e)$ such that the $[\beta_j] \in H_1(X, \mathbb{Z})$ form a basis. To define an element of $(H^1_{\mathbb{C}})^{\otimes n-1}$, it suffices to specify how it pairs with the elements $[\beta_{j_1}] \otimes \ldots \otimes [\beta_{j_{n-1}}]$ of $H_1(X, \mathbb{Z})^{\otimes n-1}$. Moreover, an element of $(H^1_{\mathbb{C}})^{\otimes n-1}$ is in $(H^1_{\mathbb{Z}})^{\otimes n-1}$ if and only if it produces integer values when pairing with the $[\beta_{j_1}]\otimes\ldots\otimes[\beta_{j_{n-1}}]$. Given an element

$$f = \int \sum_{i\leq n} w_i \; + \; (L_{n-2})_{\mathbb{C}} \; \in \; (\frac{L_n}{L_{n-2}})_{\mathbb{C}},$$

where w_i is a sum of terms of length i and the iterated integral is closed, set $r_{\mathbb{Z}}(f)$ to be the element of $(H^1_{\mathbb{C}})^{\otimes n-1}$ satisfying

$$[\beta_{j_1}]\otimes\ldots\otimes[\beta_{j_{n-1}}](r_{\mathbb{Z}}(f)) = \int\limits_{(\beta_{j_1}-1)\ldots(\beta_{j_{n-1}}-1)} \sum_{i\leq n} w_i. \tag{3.14}$$

Note that

$$\int\limits_{(\beta_{j_1}-1)\ldots(\beta_{j_{n-1}}-1)} \sum_{i\leq n} w_i = \int\limits_{(\beta_{j_1}-1)\ldots(\beta_{j_{n-1}}-1)} w_n + w_{n-1}.$$

Since $(L_{n-2})_{\mathbb{C}}$ vanishes on I^{n-1}, $r_{\mathbb{Z}}$ is well-defined. Moreover, $r_{\mathbb{Z}}$ is defined over $\mathbb{Z}$, for if $f \in (\frac{L_n}{L_{n-2}})_{\mathbb{Z}}$, the iterated integral $\int \sum w_i$ can be chosen to be integer-valued on $\pi_1(U, e)$, and hence (3.14) is an integer. Finally, we check that $r_{\mathbb{Z}}$ is a retraction of i. In view of Lemma 3.2.1 and the formula (3.9) for $q_{\mathbb{C}}$, if $\omega_1, \ldots, \omega_{n-1}$ are smooth closed 1-forms on X, $i([\omega_1]\otimes\ldots\otimes[\omega_{n-1}])$ is of the form

$$\int \omega_1 \ldots \omega_{n-1} + \text{lower length terms} \mod (L_{n-2})_{\mathbb{C}},$$

where the iterated integral is closed. We have

$$\begin{aligned} [\beta_{j_1}]\otimes\ldots\otimes[\beta_{j_{n-1}}] \; \left(r_{\mathbb{Z}} \circ i([\omega_1]\otimes\ldots\otimes[\omega_{n-1}]) \right) &= \int\limits_{(\beta_{j_1}-1)\ldots(\beta_{j_{n-1}}-1)} \omega_1 \ldots \omega_{n-1} \\ &= \int\limits_{\beta_{j_1}} \omega_1 \ldots \int\limits_{\beta_{j_{n-1}}} \omega_{n-1}, \end{aligned}$$

which is the same as

$$[\beta_{j_1}]\otimes\ldots\otimes[\beta_{j_{n-1}}]\left([\omega_1]\otimes\ldots\otimes[\omega_{n-1}]\right),$$

as desired.

Remark. The retraction $r_{\mathbb{Z}}$ is by no means natural, as it depends on the choice of the β_j.

3.3.5

A real Hodge section of $\mathfrak{q}$: The first assertion of the following lemma is immediate from Lemma 3.2.3 (ii), (iii) and (iv). In view of the sequence (3.13) the second assertion follows immediately from the first.

Lemma 3.3.1. The map $\mathfrak{s}_F$ (defined in Lemma 3.2.3(iv)) is a section of $\mathfrak{q} : (\frac{L_n}{L_{n-2}})_{\mathbb{C}} \longrightarrow (H^1_{\mathbb{C}})^{\otimes n}$ defined over $\mathbb{R}$ that respects the Hodge and weight filtrations. In particular, it gives an isomorphism

$$\frac{L_n}{L_{n-2}} \simeq (H^1)^{\otimes n} \oplus (H^1)^{\otimes n-1}$$

as real mixed Hodge structures.

3.3.6

In this paragraph, we describe the images of the extension $\mathbb{E}^{\infty}_{n,e}$ under Ψ and $\Phi \circ \Psi$.

Proposition 3.3.1. (a) $\Psi(\mathbb{E}^{\infty}_{n,e})$ is the class of the map that given $c \in (H^1_{\mathbb{C}})^{\otimes n}$, $d \in (H^1_{\mathbb{C}})^{\otimes n-1}$, it sends $c \otimes d$ to $PD^{\otimes n-1}(d)(r_{\mathbb{Z}} \circ \mathfrak{s}_F(c))$. More explicitly, if $\beta_j \in \pi_1(U, e)$ ($1 \leq j \leq 2g$) are such that $\{[\beta_j]\}$ is a basis of $H_1(X, \mathbb{Z})$, and $\omega_1, \ldots, \omega_n \in \mathcal{H}^1_{\mathbb{C}}(X)$, $\Psi(\mathbb{E}^{\infty}_{n,e})$ is the class of the map that sends

$$[\omega_1] \otimes \ldots \otimes [\omega_n] \otimes (PD^{\otimes n-1})^{-1}([\beta_{j_1}] \otimes \ldots \otimes [\beta_{j_{n-1}}])$$

to

$$\int_{(\beta_{j_1}-1)\ldots(\beta_{j_{n-1}}-1)} \omega_1 \ldots \omega_n + \sum_i \omega_1 \ldots \nu(\omega_i \otimes \omega_{i+1}) \ldots \omega_n.$$

(b) $\Phi \circ \Psi(\mathbb{E}^{\infty}_{n,e})$ is the map that given $c \in (H^1_{\mathbb{Z}})^{\otimes n}$, $d \in (H^1_{\mathbb{Z}})^{\otimes n-1}$, it sends $c \otimes d$ to $PD^{\otimes n-1}(d)(r_{\mathbb{Z}} \circ \mathfrak{s}_F(c)) \mod \mathbb{Z}$. More explicitly, for $\gamma_j \in \pi_1(U, e)$ ($1 \leq j \leq n-1$), and $\omega_1, \ldots, \omega_n \in \mathcal{H}^1_{\mathbb{R}}(X)$ with integral periods, $\Phi \circ \Psi(\mathbb{E}^{\infty}_{n,e})$ sends

$$[\omega_1] \otimes \ldots \otimes [\omega_n] \otimes (PD^{\otimes n-1})^{-1}([\gamma_1] \otimes \ldots \otimes [\gamma_{n-1}])$$

to

$$\int_{(\gamma_1-1)\ldots(\gamma_{n-1}-1)} \omega_1 \ldots \omega_n + \sum_i \omega_1 \ldots \nu(\omega_i \otimes \omega_{i+1}) \ldots \omega_n \mod \mathbb{Z}.$$

Proof. (a) We track $\mathbb{E}^{\infty}_{n,e}$ through different steps of (3.12). In view of method (ii) of Paragraph 2.2.2, the element in $J\underline{\mathrm{Hom}}\left((H^1)^{\otimes n}, (H^1)^{\otimes n-1}\right)$ corresponding to $\mathbb{E}^{\infty}_{n,e}$ under

the isomorphism of Carlson is the class of $r_{\mathbb{Z}} \circ \mathfrak{s}_F$. (See Paragraph 3.3.3.) That the latter goes to the described element of $J((H^1)^{\otimes 2n-1})^\vee$ is clear. For the second assertion, define $r_{\mathbb{Z}}$ using the basis $\{[\beta_j]\}$, and then the assertion follows on noting that $r_{\mathbb{Z}} \circ \mathfrak{s}_F([\omega_1] \otimes \ldots \otimes [\omega_n])$, by its definition, pairs with the element $[\beta_{j_1}] \otimes \ldots \otimes [\beta_{j_{n-1}}] \in (H_1)_{\mathbb{Z}}^{\otimes n-1}$ in the desired fashion. (See (3.14) and Lemma 3.2.3(i),(v).)
(b) The section $\mathfrak{s}_F$ is defined over $\mathbb{R}$, and hence so is $r_{\mathbb{Z}} \circ \mathfrak{s}_F$. Thus the map

$$c \otimes d \mapsto PD^{\otimes n-1}(d)(r_{\mathbb{Z}} \circ \mathfrak{s}_F(c))$$

of Part (a) is also defined over $\mathbb{R}$. The first assertion follows. The explicit description of Part (a) implies that (with β_j as in Part (a)) $\Phi \circ \Psi(\mathbb{E}^\infty_{n,e})$ sends

$$[\omega_1] \otimes \ldots \otimes [\omega_n] \otimes (PD^{\otimes n-1})^{-1}([\beta_{j_1}] \otimes \ldots \otimes [\beta_{j_{n-1}}])$$

to

$$\int_{(\beta_{j_1}-1)\ldots(\beta_{j_{n-1}}-1)} \omega_1 \ldots \omega_n + \sum_i \omega_1 \ldots \nu(\omega_i \otimes \omega_{i+1}) \ldots \omega_n \mod \mathbb{Z}.$$

To get the basis-independent formula, in $H_1(X, \mathbb{Z})^{\otimes n-1}$ we write

$$[\gamma_1] \otimes \ldots \otimes [\gamma_{n-1}] = \sum_{j_1,\ldots,j_{n-1}} c_{j_1,\ldots,j_{n-1}} [\beta_{j_1}] \otimes \ldots \otimes [\beta_{j_{n-1}}],$$

where the coefficients are all integers. In view of the isomorphisms (3.4) and (3.5), the element

$$\lambda := (\gamma_1 - 1)\ldots(\gamma_{n-1} - 1) - \sum_{j_1,\ldots,j_{n-1}} c_{j_1,\ldots,j_{n-1}} (\beta_{j_1} - 1)\ldots(\beta_{j_{n-1}} - 1) \in I^{n-1},$$

where $I \in \mathbb{Z}[\pi_1(U, e)]$ is the augmentation ideal, actually belongs to I^n. Thus

$$\int_\lambda \omega_1 \ldots \omega_n + \sum_i \omega_1 \ldots \nu(\omega_i \otimes \omega_{i+1}) \ldots \omega_n = \int_\lambda \omega_1 \ldots \omega_n \in \mathbb{Z},$$

as $\lambda \in I^n$ and the ω_i have integer periods. This gives the desired conclusion. □

Remark. (1) The use of a basis in part (a) of the proposition is just to make the map well-defined.
(2) The map $\Phi \circ \Psi(\mathbb{E}^\infty_{n,e})$ can be thought of as an analog of the pointed harmonic volume

$$I_e \in \mathrm{Hom}(K_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z})$$

of B. Harris [21]. Pulte [25] showed that I_e corresponds under the isomorphisms

$$\mathrm{Ext}(K, H^1) \overset{\text{Carlson}}{\cong} J\underline{\mathrm{Hom}}(K, H^1) \overset{\text{Poincare duality}}{\cong} J\underline{\mathrm{Hom}}(K \otimes H^1, \mathbb{Z}(0)) \cong \mathrm{Hom}(K_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z})$$

to the extension given by the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{L_1}{L_0}(X, e) & \longrightarrow & \dfrac{L_2}{L_0}(X, e) & \longrightarrow & \dfrac{L_2}{L_1}(X, e) & \longrightarrow & 0. \\ & & \wr\| & & & & \wr\| & & \\ & & H^1 & & & & K & & \end{array} \tag{3.15}$$

3.4 Algebraic cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$

3.4.1

Let $n \geq 2$. In this paragraph, we use the construction of Gross and Schoen (see Section 2.7) to define a null-homologous cycle $\Delta_{n,e} \in \mathcal{Z}_{n-1}(X^{2n-1})$, which will play a crucial role in the remainder of the thesis. We use the notation of Section 2.7 with $m = 2n - 1$.

For $0 < i < n$, let $\delta_i : X^{n-1} \to X^n$ be the embedding

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_i, x_i, \ldots, x_{n-1}).$$

Then ${}^t\Gamma_{\delta_i} \in \mathcal{Z}_{n-1}(X^{2n-1})$, and thus $(P_e)_*({}^t\Gamma_{\delta_i})$ is null-homologous. We define

$$\Delta_{n,e} := (P_e)_* \left(\sum_i (-1)^{i-1}\, {}^t\Gamma_{\delta_i} \right) = \sum_i (-1)^{i-1} (P_e)_*({}^t\Gamma_{\delta_i}) \in \mathcal{Z}_{n-1}(X^{2n-1}).$$

It is of course null-homologous. (See Section 2.7.) It is clear from the definition that $\Delta_{2,e}$ is simply the modified diagonal cycle $\Delta_{KGS,e}$ of Gross, Kudla, and Schoen in X^3.

3.4.2

In this paragraph, we realize the cycle $\Delta_{n,e}$ as the boundary of a chain. This will be particularly important when later we study the image of $\Delta_{n,e}$ under the Abel-Jacobi map.

Let Λ_n be the closed subvariety

$$\{(x_1, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}) : x_i \in X\}$$

of X^{2n-1}, where each x_i ($i > 1$) is appearing in exactly two coordinates. It is a copy of X^{n-1} embedded in X^{2n-1} via the map

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, x_1, x_1, x_2, x_2, \ldots, x_{n-1}, x_{n-1}),$$

and can also be thought of as an element of $\mathcal{Z}_{n-1}(X^{2n-1})$. It is easy to see that

$$(P_e)_*(\Lambda_n) = \Delta_{2,e} \times \overbrace{(P_e)_*(\Delta^2(X)) \times \ldots \times (P_e)_*(\Delta^2(X))}^{n-2 \text{ factors}}.$$

Let $\partial^{-1}(\Delta_{2,e})$ be an integral 3-chain in X^3 whose boundary is $\Delta_{2,e}$. (See for instance the proof of Lemma 2.3 of [6] for such a 3-chain.) Then $(P_e)_*(\Lambda_n)$ is the boundary of

$$\partial^{-1}(\Delta_{2,e}) \times \overbrace{(P_e)_*(\Delta^2(X)) \times \ldots \times (P_e)_*(\Delta^2(X))}^{n-2 \text{ factors}} =: \partial^{-1}(P_e)_*(\Lambda_n).$$

It is clear that each ${}^t\Gamma_{\delta_i}$ is a copy of Λ_n. Specifically, ${}^t\Gamma_{\delta_i} = (\sigma_i)_*(\Lambda_n)$ where σ_i is the automorphism of X^{2n-1} that sends $(x_1, \ldots, x_{2n-1})$ to

$$(x_4, x_6, \ldots, x_{2i}, x_1, x_2, x_{2i+2}, x_{2i+4}, \ldots, x_{2n-2}, x_5, x_7, \ldots, x_{2i+1}, x_3, x_{2i+3}, x_{2i+5}, \ldots, x_{2n-1}).$$

Lemma 3.4.1. $(P_e)_*$ and $(\sigma_i)_*$ commute (as maps $\mathcal{Z}(X^{2n-1}) \to \mathcal{Z}(X^{2n-1})$).

Proof. With abuse of notation, suppose σ_i is the permutation of $1, 2, \ldots, 2n-1$ such that

$$\sigma_i(x_1, \ldots, x_{2n-1}) = (x_{\sigma_i^{-1}(1)}, x_{\sigma_i^{-1}(2)}, \ldots, x_{\sigma_i^{-1}(2n-1)}).$$

Then for each subset T of $\{1, 2, \ldots, 2n-1\}$, $q_T \circ p_T \circ \sigma_i = \sigma_i \circ q_{\sigma_i^{-1}T} \circ p_{\sigma_i^{-1}T}$. We have

$$\begin{aligned}
(P_e)_* \circ (\sigma_i)_* &= \left(\sum_T (-1)^{|T^c|}(q_T \circ p_T)_*\right)(\sigma_i)_* \\
&= \sum_T (-1)^{|T^c|}(q_T \circ p_T \circ \sigma_i)_* \\
&= \sum_T (-1)^{|T^c|}(\sigma_i \circ q_{\sigma_i^{-1}T} \circ p_{\sigma_i^{-1}T})_* \\
&= (\sigma_i)_* \left(\sum_T (-1)^{|T^c|}(q_{\sigma_i^{-1}T} \circ p_{\sigma_i^{-1}T})_*\right) \\
&= (\sigma_i)_* \circ (P_e)_*.
\end{aligned}$$

□

It follows from the lemma that

$$(\sigma_i)_* \left((P_e)_*(\Lambda_n)\right) = (P_e)_* \left({}^t\Gamma_{\delta_i}\right),$$

and hence

$$\partial \left((\sigma_i)_*(\partial^{-1}(P_e)_*(\Lambda_n))\right) = (P_e)_* \left({}^t\Gamma_{\delta_i}\right).$$

We set

$$\partial^{-1}\Delta_{n,e} := \sum_i (-1)^{i-1}(\sigma_i)_*(\partial^{-1}(P_e)_*(\Lambda_n)).$$

It is immediate from the above that the boundary of this chain is $\Delta_{n,e}$.

3.4.3

In this paragraph, we define another family of null-homologous cycles that will be used later on. Let $n \geq 2$. Given $y \in X(\mathbb{C})$, for $0 < i < n$, let $Z^y_{n,i} \in \mathcal{Z}_{n-1}(X^{2n-1})$ be

$$\{(x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_{n-1}, x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{n-1}) : x_1, \ldots, x_{n-1} \in X\}.$$

Here each x_j appears in exactly two coordinates. There are different ways of thinking about this cycle. For instance,

$$Z^y_{n,i} = (\pi_{n+i,y})_* {}^t\Gamma_{\delta_i},$$

where $\pi_{n+i,y}$ is the map $X^{2n-1} \to X^{2n-1}$ that replaces the $(n+i)$-th coordinate by y, and keeps the other coordinates unchanged.

It is clear that the cycle $Z^{\infty}_{n,i} - Z^{e}_{n,i}$ is null-homologous. For future reference, here we explicitly define a chain whose boundary is $Z^{\infty}_{n,i} - Z^{e}_{n,i}$. Choose a path γ^{∞}_{e} in X from e to ∞, and let

$$C^{\infty}_{n,e} := \Delta^2(X)^{n-1} \times \gamma^{\infty}_{e} = \{(x_1, x_1, \ldots, x_{n-1}, x_{n-1}, \gamma^{\infty}_{e}(t)) : x_i \in X, t \in [0,1]\}.$$

One clearly has

$$\partial C^{\infty}_{n,e} = \Delta^2(X)^{n-1} \times \{\infty\} - \Delta^2(X)^{n-1} \times \{e\}.$$

For $0 < i < n$, let τ_i be the automorphism of X^{2n-1} that maps $(x_1, \ldots, x_{2n-1})$ to

$$(x_1, x_3, \ldots, x_{2(i-1)-1}, x_{2i-1}, x_{2i}, x_{2(i+1)-1}, \ldots, x_{2(n-1)-1}, x_2, x_4, \ldots, x_{2(i-1)}, x_{2n-1}, x_{2(i+1)}, \ldots, x_{2(n-1)}),$$

which is designed so that

$$(\tau_i)_* \left(\Delta^2(X)^{n-1} \times \{y\}\right) = Z^{y}_{n,i}$$

for every y. Then

$$\partial(\tau_i)_*(C^{\infty}_{n,e}) = Z^{\infty}_{n,i} - Z^{e}_{n,i}. \tag{3.16}$$

We put together all the $Z^{\infty}_{n,i} - Z^{e}_{n,i}$ and define

$$Z^{\infty}_{n,e} := \sum_{i=1}^{n-1} (-1)^{i-1} (Z^{\infty}_{n,i} - Z^{e}_{n,i}) \ \in \mathcal{Z}^{\hom}_{n-1}(X^{2n-1}).$$

3.4.4

Remark. While we worked over $\mathbb{C}$ in this section, it is clear that the constructions of $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ remain valid over any field K that can be embedded into $\mathbb{C}$. More precisely, if X_0 is a geometrically connected smooth projective curve over K, and $e, \infty \in X_0(K)$, the above constructions give null-homologous cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ in $\mathcal{Z}_{n-1}(X_0^{2n-1})$ (or in $\mathrm{CH}_{n-1}(X_0^{2n-1})$).

3.5 Statement of the main theorem

Our goal in this section is to state the main result of the chapter, which expresses the extension $\mathbb{E}^{\infty}_{n,e}$ in terms of the cycles $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$. Its proof will take the remainder of this chapter.

3.5.1

Notation. We adopt the following notation for the Kunneth decomposition of cohomology. Given manifolds M and N, we think of $H^i(M) \otimes H^j(N)$ (singular or de Rham cohomology) as a subspace of $H^{i+j}(M \times N)$. Given $c \in H^i(M)$, $d \in H^j(N)$, the element $c \otimes d$ of $H^{i+j}(M \times N)$ is $pr_1^*(c) \wedge pr_2^*(d)$, where pr_i the the projection of $M \times N$ onto its i^{th} factor. We adopt a similar notation for differential forms: given ω and ϕ differential forms on M and N, we refer to the differential form $pr_1^*(\omega) \wedge pr_2^*(\phi)$ on $M \times N$ by $\omega \otimes \phi$. Similar notation is used for more than two factors.

3.5.2

For $n \geq 1$, let h_n be the composition of the Abel-Jacobi map

$$CH^{\mathrm{hom}}_{n-1}(X^{2n-1}) \longrightarrow JH^{2n-1}(X^{2n-1})^{\vee}$$

with the map

$$JH^{2n-1}(X^{2n-1})^{\vee} \longrightarrow J((H^1)^{\otimes 2n-1})^{\vee}$$

induced by the Kunneth inclusion $(H^1)^{\otimes 2n-1} \subset H^{2n-1}(X^{2n-1})$. It is easy to see from definitions that if $Z \in \mathcal{Z}^{\mathrm{hom}}_{n-1}(X^{2n-1})$ and C is an integral chain in X^{2n-1} whose boundary is Z, $h_n(Z)$ is the class of the map that, given harmonic 1-forms $\omega_1, \ldots, \omega_{2n-1}$ on X, it sends

$$[\omega_1] \otimes \ldots \otimes [\omega_{2n-1}] \mapsto \int_C \omega_1 \otimes \ldots \otimes \omega_{2n-1}. \tag{3.17}$$

Note that h_1 is just the "classical" Abel-Jacobi map $CH^{\mathrm{hom}}_0(X) \to J(H^1)^{\vee}$.

If Z and C are as above, since the map (3.17) is defined over $\mathbb{R}$,

$$\Phi(h_n(Z)) : (H^1_{\mathbb{Z}})^{\otimes 2n-1} \to \mathbb{R}/\mathbb{Z}$$

is the map that, given harmonic forms $\omega_1, \ldots, \omega_{2n-1}$ on X with integral periods, it maps

$$[\omega_1] \otimes \ldots \otimes [\omega_{2n-1}] \mapsto \int_C \omega_1 \otimes \ldots \otimes \omega_{2n-1} \quad \mod \mathbb{Z}.$$

(See Paragraph 3.3.1 and Paragraph 3.3.2.)

3.5.3

Now we are ready to state the main result.

Theorem 3.5.1. Let $n \geq 2$. We have

$$\Psi(\mathbb{E}^{\infty}_{n,e}) = (-1)^{\frac{n(n-1)}{2}} h_n \left(\Delta_{n,e} - Z^{\infty}_{n,e}\right). \tag{3.18}$$

When $n = 2$, a slightly weaker of this is due to Darmon, Rotger, and Sols [6]. (See the next section.)

3.6 $n = 2$ case - A formula of Darmon et al revisited

3.6.1 Independence of $-\Psi(\mathbb{E}^{\infty}_{2,e}) + h_2(Z^{\infty}_{2,e})$ from ∞

Lemma 3.6.1. The element $-\Psi(\mathbb{E}^{\infty}_{2,e}) + h_2(Z^{\infty}_{2,e})$ is independent of the point $\infty \neq e$, i.e. if $\infty_1, \infty_2 \neq e$, then

$$-\Psi(\mathbb{E}^{\infty_1}_{2,e}) + h_2(Z^{\infty_1}_{2,e}) = -\Psi(e^{\infty_2}_{2,e}) + h_2(Z^{\infty_2}_{2,e}).$$

Proof. Let $\infty_1, \infty_2 \neq e$ be distinct. After passing to $\operatorname{Hom}\left((H^1_{\mathbb{Z}})^{\otimes 3}, \mathbb{R}/\mathbb{Z}\right)$ via Φ, in view of Proposition 3.3.1, we need to show that if ω, ρ, η are harmonic forms with integral periods on X, and $\gamma_\eta \in \pi_1(X - \{\infty_1, \infty_2\}, e)$ is such that its homology class in $H_1(X, \mathbb{Z})$ is $PD([\eta])$, then

$$-\int_{\gamma_\eta} \omega\rho + \nu_{\infty_1}(\omega \otimes \rho) \; + \; \int_X \omega \wedge \rho \int_e^{\infty_1} \eta \overset{\mathbb{Z}}{\equiv} -\int_{\gamma_\eta} \omega\rho + \nu_{\infty_2}(\omega \otimes \rho) \; + \; \int_X \omega \wedge \rho \int_e^{\infty_2} \eta,$$

or equivalently

$$-\int_{\gamma_\eta} \nu_{\infty_1}(\omega \otimes \rho) - \nu_{\infty_2}(\omega \otimes \rho) \; + \; \int_X \omega \wedge \rho \int_{\infty_2}^{\infty_1} \eta \in \mathbb{Z}, \tag{3.19}$$

where the integrals of η are over any path in X with the specified end points. Fix ω and ρ. For brevity we write ν_i for $\nu_{\infty_i}(\omega \otimes \rho)$. Note that if $\omega \wedge \rho$ is exact on X, then the statement clearly holds, as then $\nu_i \in \mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$ and $\nu_1 - \nu_2$, being a closed element of $\mathcal{H}^1_{\mathbb{C}}(X)^{\perp}$, is exact, so that the number above is simply zero. (See the proof of Lemma 3.2.2.) So we may assume $\omega \wedge \rho$ is not exact on X. Then the 1-form $\nu_1 - \nu_2$ satisfies the following properties:

(i) It is meromorphic on X, holomorphic on $X - \{\infty_1, \infty_2\}$, with logarithmic poles at ∞_1 and ∞_2 with residues $\frac{a}{2\pi i}$ and $-\frac{a}{2\pi i}$ respectively for some integer $a \neq 0$.

(ii) Its cohomology class in $H^1(X - \{\infty_1, \infty_2\})$ is real, i.e. it can be written on $X - \{\infty_1, \infty_2\}$ as the sum of an exact form and a real closed form.

Indeed, (i) follows from that both ν_1 and ν_2 are of type (1,0), and $d\nu_1 = d\nu_2 = -\omega \wedge \rho$ on $X - \{\infty_1, \infty_2\}$, so that $\nu_1 - \nu_2$ is holomorphic on $X - \{\infty_1, \infty_2\}$. For the behavior at ∞_i, note that $\nu_i \in E^1(X \log \infty_i)$. The statement about the residues is immediate from Lemma 3.2.2(iv) ($a = \int_X \omega \wedge \rho$). Statement (ii) follows from that each form ν_i can be written as a real form on $X - \{\infty_i\}$ plus an exact form on the same space.

The statement (3.19) now follows from the following lemma. $\square$

Lemma 3.6.2. Let $\infty_1, \infty_2 \neq e$, and ζ be any 1-form satisfying conditions (i) and (ii) above. Then for any harmonic 1-form η on X with integral periods,

$$-\int_{\gamma_\eta} \zeta + a \int_{\infty_2}^{\infty_1} \eta \in \mathbb{Z},$$

where $\gamma_\eta \in \pi_1(X - \{\infty_1, \infty_2\}, e)$ satisfies $\mathrm{PD}([\eta]) = [\gamma_\eta]$.

Proof. First note that the integral $\int_X \zeta \wedge \eta$ converges for any $\eta \in \mathcal{H}^1_{\mathbb{C}}(X)$, as the integral of $\frac{dz d\bar{z}}{z}$ converges on the unit disk in $\mathbb{C}$. Thus one gets a map $h : H^1_{\mathbb{C}} \to \mathbb{C}$ given by $[\eta] \mapsto \int_X \zeta \wedge \eta$. We claim that this map takes integer values on $H^1_{\mathbb{Z}}$. Note that since h vanishes on $F^1 H^1$, by the remark in Paragraph 3.3.1, it suffices to show that it is defined over $\mathbb{R}$. Suppose $\eta \in \mathcal{H}^1_{\mathbb{R}}(X)$ has integer periods. The claim is established if we show $h([\eta])$ is real. We may assume that the map

$$\int \eta : H_1(X, \mathbb{Z}) \to \mathbb{Z} \tag{3.20}$$

is surjective, and that $\gamma_\eta \in \pi_1(X - \{\infty_1, \infty_2\}, e)$ (Poincare dual to $[\eta]$ in $H_1(X, \mathbb{Z})$) has a simple representative loop, which we also denote by γ_η. One can show that there is a Riemann surface $\tilde{X}$, a covering projection $\pi : \tilde{X} \to X$, and a deck transformation T of π such that

- $\pi^*\eta = df$ for a real function f on $\tilde{X}$.
- $fT - f$ is the constant function 1.
- There is a lift $\tilde{\gamma}_\eta$ of γ_η, and a submanifold with boundary $X^{(0)}$ of $\tilde{X}$ such that $\partial X^{(0)} = T\tilde{\gamma}_\eta - \tilde{\gamma}_\eta$, and the restriction of π to $X^{(0)} - \partial X^{(0)}$ is an isomorphism of Riemann surfaces onto $X - \gamma_\eta$.[†]

Now let for each i, D_i be an open disk around ∞_i in X, small enough so that $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\overline{D_i} \cap \gamma_\eta = \emptyset$ (bar denoting closure). Denote by $\tilde{\infty}_i$ and $\tilde{D}_i$ the lift of ∞_i and D_i in $X^{(0)}$. Then we have

$$\begin{aligned}
\int_{X - D_1 \cup D_2} \zeta \wedge \eta = \int_{X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2} \pi^*\zeta \wedge \pi^*\eta &= \int_{X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2} -df \wedge \pi^*\zeta \\
&= \int_{X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2} -d(f\pi^*\zeta) \\
&= -\int_{\partial(X^{(0)} - \tilde{D}_1 \cup \tilde{D}_2)} f\pi^*\zeta \\
&= \int_{\tilde{\gamma}_\eta - T\tilde{\gamma}_\eta + \partial\tilde{D}_1 + \partial\tilde{D}_2} f\pi^*\zeta \\
&= \int_{\tilde{\gamma}_\eta - T\tilde{\gamma}_\eta} f\pi^*\zeta + \int_{\partial\tilde{D}_1 + \partial\tilde{D}_2} f\pi^*\zeta.
\end{aligned}$$

[†]Such a covering projection is obtained by taking a copy $X^{(i)}$ of X for each integer i, "cutting" the $X^{(i)}$ along γ_η, and then gluing $X^{(i)}$ to $X^{(i+1)}$ appropriately along γ_η. The deck transformation simply sends a point in $X^{(i)}$ to its counterpart in $X^{(i+1)}$.

It follows that

$$\int_{X-D_1\cup D_2} \zeta\wedge\eta = -\int_{\gamma_\eta}\zeta + \int_{\partial\tilde{D}_1+\partial\tilde{D}_2} f\pi^*\zeta\,.$$

We would like to know what happens as $D_i \to \{\infty_i\}$. Write

$$\int_{\partial\tilde{D}_i} f\pi^*\zeta = \int_{\partial\tilde{D}_i} f(\tilde{\infty}_i)\pi^*\zeta + \int_{\partial\tilde{D}_i} (f-f(\tilde{\infty}_i))\,\pi^*\zeta.$$

Since ζ is holomorphic on $\tilde{D}_i - \tilde{\infty}_i$ with a pole of order 1 at ∞_i, and $f - f(\tilde{\infty}_i)$ is smooth and vanishes at $\tilde{\infty}_i$, the second term goes to zero as $D_i \to \{\infty_i\}$. The first term is equal to $2\pi i f(\tilde{\infty}_i)\mathrm{res}_{\infty_i}(\zeta)$. Thus

$$\int_X \zeta\wedge\eta = -\int_{\gamma_\eta}\zeta + a(f(\tilde{\infty}_1) - f(\tilde{\infty}_2)). \tag{3.21}$$

The second term on the right is real as a and f are real. The first term is also real because the cohomology class of ζ in $H^1(X - \{\infty_1, \infty_2\})$ is real. Thus the claim is established.

Now it is easy to conclude the lemma. Let η be as described in the statement. Without loss of generality we may assume that (3.20) is surjective, and that γ_η has a simple representative loop. Then we know (3.21), and hence

$$\int_X \zeta\wedge\eta \overset{\mathbb{Z}}{\equiv} -\int_{\gamma_\eta}\zeta + a\int_{\infty_2}^{\infty_1}\eta.$$

The left hand side (which is $h(\eta)$) is an integer. □

3.6.2

When $n = 2$, Theorem 3.5.1 asserts that

$$\Psi(\mathbb{E}^{\infty}_{2,e}) = h_2(-\Delta_{2,e} + Z^{\infty}_{2,e}). \tag{3.22}$$

This is a slightly stronger version of a theorem of Darmon, Rotger, and Sols [6, Theorem 2.5]. Their result can be stated as to assert that, for every Hodge class ξ of $(H^1)^{\otimes 2}$, one has

$$\xi^{-1}\left(\Psi(\mathbb{E}^{\infty}_{2,e})\right) = \xi^{-1}\left(h_2(-\Delta_{2,e} + Z^{\infty}_{2,e})\right), \tag{3.23}$$

where $\xi^{-1} : J((H^1)^{\otimes 3})^{\vee} \to J(H^1)^{\vee}$ is the map that sends $[f] \mapsto [f(\xi \otimes -)]$ for any $f \in \left((H^1_{\mathbb{C}})^{\otimes 3}\right)^{\vee}$. (This is well-defined because ξ is a Hodge class.)

Let $\{\beta_j\}_j \subset \pi_1(U, e)$ be such that $\{[\beta_j]\}_j$ forms a basis of $H_1(X, \mathbb{Z})$. For each j, let η_j be the harmonic form on X such that $PD([\eta_j]) = [\beta_j]$. In view of our description of $\Psi(\mathbb{E}^{\infty}_{2,e})$ given in Proposition 3.3.1, (3.23) is equivalent to that if $\xi = \sum[\omega_i]\otimes[\rho_i]$ with ω_i and ρ_i harmonic forms on X with integral periods, then the two maps $H^1_{\mathbb{C}} \to \mathbb{C}$ given by

$$[\eta_j] \mapsto \int_{\partial^{-1}\Delta_{2,e}} \sum_i \omega_i \otimes \rho_i \otimes \eta_j$$

and

$$[\eta_j] \mapsto -\left(\int_{\beta_j} \sum \omega_i\rho_i + \nu(\xi)\right) + \int_{\Delta^2(X)} \xi \int_{\gamma^{\infty}_e} \eta_j,$$

represent the same class in $J(H^1)^{\vee}$. For this it suffices to verify that the restrictions of the two maps to $F^1H^1_{\mathbb{C}}$ differ by (the restriction of) an element of $(H_1)_{\mathbb{Z}}$, and this is what Darmon, Rotger and Sols do in [6].

The argument given in [6] combined with Lemma 3.6.1 indeed implies (3.22). To see this, let us start with an obvious observation. Suppose A, B, and C are abelian groups. Then a map $f : A \otimes B \to C$ is zero if and only if, for every $a \in A$, the map $B \to C$ defined by $b \mapsto f(a \otimes b)$ is zero. Now suppose we have a map $f : (H^1_{\mathbb{C}})^{\otimes 3} \to \mathbb{C}$, defined over $\mathbb{R}$. Then $[f]$ is trivial in $J((H^1)^{\otimes 3})^{\vee}$ if and only if $\Phi([f]) = 0$, and since f is defined over the reals, the latter amounts to that $\pi \circ f|_{(H^1_{\mathbb{Z}})^{\otimes 3}} = 0$, where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the natural map. This is equivalent to that for every $\xi \in (H^1_{\mathbb{Z}})^{\otimes 2}$, the map $H^1_{\mathbb{Z}} \to \mathbb{R}/\mathbb{Z}$ given by $c \mapsto \pi \circ f(\xi \otimes c)$ is zero, or equivalently, the map $\xi^{-1}f : H^1_{\mathbb{C}} \to \mathbb{C}$ defined by $c \mapsto f(\xi \otimes c)$ is integer-valued on $H^1_{\mathbb{Z}}$. The latter by the remark in Paragraph 3.3.1 is equivalent to that the restriction of $\xi^{-1}f$ to F^1H^1 coincides with that of an element of $(H_1)_{\mathbb{Z}}$.

In view of the above observation, (3.22) is equivalent to that, for every $\xi = [\omega] \otimes [\rho] \in (H^1_{\mathbb{Z}})^{\otimes 2}$, where the ω and ρ are harmonic forms on X with integral periods, the restriction to $F^1H^1_{\mathbb{C}}$ of the map $H^1_{\mathbb{C}} \to \mathbb{C}$ defined by

$$[\eta_j] \mapsto \int_{\partial^{-1}\Delta_{2,e}} \omega \otimes \rho \otimes \eta_j \ + \ \left(\int_{\beta_j} \omega\rho + \nu(\xi) \right) \ - \int_{\Delta^2(X)} \xi \int_{\gamma_e^\infty} \eta_j$$

is equal to that of of an element of $(H_1)_{\mathbb{Z}}$. This is exactly Theorem 2.5 of [6], except that here ξ is not necessarily a Hodge class, but rather merely an integral class. However, the argument in [6] works just as well here too, as long as one can replace the point ∞ by a point at which certain technical conditions† hold. Lemma 3.6.1 allows one to do this‡.

3.6.3

We close this section by noting that applying the map Φ to (3.22), we get that, if ω, ρ, η are harmonic forms on X with integral periods, and $\gamma_\eta \in \pi_1(U, e)$ is such that $PD([\eta]) = [\gamma_\eta]$ in homology of X, then

$$\int_{\partial^{-1}\Delta_{2,e}} \omega \otimes \rho \otimes \eta \overset{\mathbb{Z}}{\equiv} - \int_{\gamma_\eta} (\omega\rho + \nu(\omega \otimes \rho)) \ + \ \int_X \omega \wedge \rho \int_e^\infty \eta. \tag{3.24}$$

(See Proposition 3.3.1(b).)

3.7 Proof of the general case

In this section we use the contents of the previous sections to prove Theorem 3.5.1 in $n \geq 3$ case. We will equivalently show that the two sides of (3.18) have equal images under Φ. Let $\omega_1, \dots, \omega_n$ and $\eta_1, \dots, \eta_{n-1}$ be harmonic forms on X with integral periods, and for each i, $\gamma_i \in \pi_1(U, e)$ be such that $[\gamma_i] = PD([\eta_i])$ in $H_1(X, \mathbb{Z})$. All equalities below take place in $\mathbb{R}/\mathbb{Z}$. We use the notation $[\dots|\dots]$ for $\dots \otimes \dots$, and for brevity denote $\frac{(n-3)(n-2)}{2}$ by m. The reader can refer to Section 3.4 to recall the definition of the chains and permutations that appear in the calculations.

†on the "positioning" of ∞ relative to $\partial^{-1}\Delta_{2,e}$

‡In [6], a similar task is performed by Lemma 1.3, which asserts that our Lemma 3.6.1 holds after applying ξ^{-1}.

We have

$$\begin{aligned}\Phi(h_n(\Delta_{n,e}))[\omega_1|\ldots|\omega_n|\eta_1|\ldots|\eta_{n-1}] &= \int_{\partial^{-1}\Delta_{n,e}} [\omega_1|\ldots|\omega_n|\eta_1|\ldots|\eta_{n-1}] \\ &= \sum_{i=1}^{n-1}(-1)^{i-1} \int_{(\sigma_i)_*(\partial^{-1}(P_e)_*(\Lambda_n))} [\omega_1|\ldots|\omega_n|\eta_1|\ldots|\eta_{n-1}].\end{aligned}$$

We also have

$$\int_{(\sigma_i)_*(\partial^{-1}(P_e)_*(\Lambda_n))} [\omega_1|\ldots|\omega_n|\eta_1|\ldots|\eta_{n-1}] = \int_{\partial^{-1}(P_e)_*(\Lambda_n)} (\sigma_i)^*([\omega_1|\ldots|\omega_n|\eta_1|\ldots|\eta_{n-1}]),$$

and recalling how σ_i permutes coordinates of X^{2n-1}, we see this is

$$\begin{aligned} &= (-1)^{n+i-1+m} \int_{\partial^{-1}(P_e)_*(\Lambda_n)} [\omega_i|\omega_{i+1}|\eta_i|\omega_1|\eta_1|...|\omega_{i-1}|\eta_{i-1}|\omega_{i+2}|\eta_{i+1}|\ldots|\omega_n|\eta_{n-1}] \\ &= (-1)^{n+i-1+m} \int_{(\partial^{-1}\Delta_{2,e})\times((P_e)_*\Delta^2(X))^{n-2}} [\omega_i|\omega_{i+1}|\eta_i|\omega_1|\eta_1|...|\omega_{i-1}|\eta_{i-1}|\omega_{i+2}|\eta_{i+1}|\ldots|\omega_n|\eta_{n-1}] \\ &= (-1)^{n+i-1+m} \int_{\partial^{-1}\Delta_{2,e}} [\omega_i|\omega_{i+1}|\eta_i] \prod_{j=1}^{i-1} \int_{(P_e)_*\Delta^2(X)} [\omega_j|\eta_j] \prod_{j=i+2}^{n} \int_{(P_e)_*\Delta^2(X)} [\omega_j|\eta_{j-1}] \\ &= (-1)^{n+i-1+m} \int_{\partial^{-1}\Delta_{2,e}} [\omega_i|\omega_{i+1}|\eta_i] \prod_{j=1}^{i-1} \int_{\Delta^2(X)} [\omega_j|\eta_j] \prod_{j=i+2}^{n} \int_{\Delta^2(X)} [\omega_j|\eta_{j-1}],\end{aligned}$$

as the other summands in $(P_e)_*\Delta^2(X)$ do not contribute to the integrals. In view of (3.24), the last expression is

$$\begin{aligned} &= (-1)^{n+i-1+m} \left(-\int_{\gamma_i} \omega_i\omega_{i+1} + \nu([\omega_i|\omega_{i+1}]) + \int_X \omega_i \wedge \omega_{i+1} \int_e^{\infty} \eta_i\right) \prod_{j=1}^{i-1} \int_X \omega_j \wedge \eta_j \prod_{j=i+2}^{n} \int_X \omega_j \wedge \eta_{j-1} \\ &= (-1)^{i-1+m} \left(-\int_{\gamma_i} \omega_i\omega_{i+1} + \nu([\omega_i|\omega_{i+1}]) + \int_X \omega_i \wedge \omega_{i+1} \int_e^{\infty} \eta_i\right) \prod_{j=1}^{i-1} \int_{\gamma_j} \omega_j \prod_{j=i+2}^{n} \int_{\gamma_{j-1}} \omega_j.\end{aligned}$$

It follows that

$$(-1)^m\Phi(h_n(\Delta_{n,e}))[\omega_1|\ldots|\omega_n|\eta_1|\ldots|\eta_{n-1}] = -(\mathrm{I}) + (\mathrm{II}), \tag{3.25}$$

where

$$(\mathrm{I}) \;=\; \sum_{i=1}^{n-1}\left(\int_{\gamma_i}\omega_i\omega_{i+1}+\nu([\omega_i|\omega_{i+1}])\right)\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j$$

and

$$(\mathrm{II}) \;=\; \sum_{i=1}^{n-1}\int_X\omega_i\wedge\omega_{i+1}\int_e^{\infty}\eta_i\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j.$$

In view of (2.9),

$$\begin{aligned}(\mathrm{I}) &= \sum_{i=1}^{n-1}\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\int_{\gamma_i}\omega_i\omega_{i+1}\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j+\sum_{i=1}^{n-1}\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\int_{\gamma_i}\nu([\omega_i|\omega_{i+1}])\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j\\ &= \int_{(\gamma_1-1)\dots(\gamma_{n-1}-1)}\omega_1\dots\omega_n+\sum_{i=1}^{n-1}\int_{(\gamma_1-1)\dots(\gamma_{n-1}-1)}\omega_1\dots\omega_{i-1}\nu([\omega_i|\omega_{i+1}])\omega_{i+2}\dots\omega_n\\ &= \Phi(\Psi(\mathbb{E}^{\infty}_{n,e}))([\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]),\end{aligned} \tag{3.26}$$

by Proposition 3.3.1(b).

On the other hand, for $1\le i\le n-1$, in view of (3.16),

$$\begin{aligned}\Phi(h_n(Z^{\infty}_{n,i}-Z^{e}_{n,i}))([\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]) &= \int_{(\tau_i)_*(C^{\infty}_{n,e})}[\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]\\ &= \int_{C^{\infty}_{n,e}}(\tau_i)^*[\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}],\end{aligned}$$

which, in view of the definition of $C^{\infty}_{n,e}$ and on recalling how τ_i permutes the coordinates of X^{2n-1}, is

$$\begin{aligned}&= (-1)^{m+n+i-1}\int_X\omega_i\wedge\omega_{i+1}\int_{\gamma_e^{\infty}}\eta_i\prod_{j=1}^{i-1}\int_X\omega_j\wedge\eta_j\prod_{j=i+2}^{n}\int_X\omega_j\wedge\eta_{j-1}\\ &= (-1)^{m+i-1}\int_X\omega_i\wedge\omega_{i+1}\int_{\gamma_e^{\infty}}\eta_i\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j.\end{aligned}$$

Thus

$$\begin{aligned}
\Phi(h_n(Z^{\infty}_{n,e}))([\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]) &= \sum_{i=1}^{n-1}(-1)^{i-1}\Phi(h_n(Z^{\infty}_{n,i}-Z^{e}_{n,i}))([\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]) \\
&= \sum_{i=1}^{n-1}(-1)^{m}\int_X \omega_i\wedge\omega_{i+1}\int_{\gamma^{\infty}_e}\eta_i\prod_{j=1}^{i-1}\int_{\gamma_j}\omega_j\prod_{j=i+2}^{n}\int_{\gamma_{j-1}}\omega_j \\
&= (-1)^m\,(II). \qquad (3.27)
\end{aligned}$$

Finally, combining equations (3.25), (3.26), and (3.27), we have

$$\begin{aligned}
(-1)^m\Phi(h_n(\Delta_{n,e}))[\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}] &= -\Phi(\Psi(\mathbb{E}^{\infty}_{n,e}))([\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]) \\
&+ (-1)^m\Phi(h_n(Z^{\infty}_{n,e}))([\omega_1|\dots|\omega_n|\eta_1|\dots|\eta_{n-1}]),
\end{aligned}$$

giving the desired conclusion.

Chapter 4

Hodge Theory of $\pi_1(X - \{\infty\}, e)$ and Rational Points on the Jacobian

Fix a subfield $K \subset \mathbb{C}$. From now on, we assume that the curve X and the points e, ∞ (notation as in the previous chapter) are defined over K. More precisely, suppose $X = X_0 \times_K \operatorname{Spec}(\mathbb{C})$, where X_0 is a projective curve over K, and that $e, \infty \in X_0(K)$. Let $\mathrm{Jac} = \mathrm{Jac}(X_0)$ be the Jacobian of X_0.

4.1 Associating points in $\mathrm{Jac}(K)$ to $\mathbb{E}^{\infty}_{n,e}$

In this section, we show that one can associate to the extension $\mathbb{E}^{\infty}_{n,e}$ a family of points in $\mathrm{Jac}(K)$. The approach follows the ideas leading to Theorem 1 and Corollary 1 of [6], and generally speaking, is in line with Darmon's philosophy of trying to construct rational points on Jacobian varieties using higher dimensional varieties.

Throughout, we identify

$$\mathrm{Jac}(\mathbb{C}) = \mathrm{CH}_0^{\mathrm{hom}}(X) \overset{\mathrm{AJ}}{\cong} J(H^1)^{\vee}.$$

Thus in particular, $\mathrm{Jac}(K)$ is identified as a subgroup of $J(H^1)^{\vee}$. For a Hodge class ξ in $(H^1)^{\otimes 2n-2}$, let

$$\xi^{-1} : J((H^1)^{\otimes 2n-1})^{\vee} \to J(H^1)^{\vee}$$

be the map $[f] \mapsto [f(\xi \otimes -)]$. For an algebraic cycle $Z \in \mathrm{CH}_{n-1}(X_0^{2n-2})$, we denote by ξ_Z

the $(H^1)^{\otimes 2n-2}$ Kunneth component of

$$cl(Z) \in H^{2n-2}_{\mathbb{C}}(X^{2n-2})^{\vee} \overset{\text{Poincare duality}}{\cong} H^{2n-2}_{\mathbb{C}}(X^{2n-2}).$$

The main result of this chapter is the following theorem.

Theorem 4.1.1. Let $Z \in CH_{n-1}(X_0^{2n-2})$. Then

$$\xi_Z^{-1}(\Psi(\mathbb{E}^{\infty}_{n,e})) \in Jac(K).$$

We should point out that this is not a priori obvious, as to get the extension $\mathbb{E}^{\infty}_{n,e}$ one first extends the scalars to $\mathbb{C}$. Note that varying Z, we get a family of points in $Jac(K)$ associated to $\mathbb{E}^{\infty}_{n,e}$ parametrized by $CH_{n-1}(X_0^{2n-2})$. In other words, the weight filtration on (the mixed Hodge structure associated to) $\pi_1(X-\{\infty\},e)$ is giving rise to families of points in $Jac(K)$ parametrized by algebraic cycles on powers of X_0.

With abuse of notation, we denote the compositions

$$CH^{hom}_{n-1}(X_0^{2n-1}) \overset{\text{natural map}}{\to} CH^{hom}_{n-1}(X^{2n-1}) \overset{AJ}{\to} JH^{2n-1}(X^{2n-1})^{\vee}$$

and

$$CH^{hom}_{n-1}(X_0^{2n-1}) \overset{\text{natural map}}{\to} CH^{hom}_{n-1}(X^{2n-1}) \overset{h_n}{\to} J((H^1)^{\otimes 2n-1})^{\vee}$$

by AJ and h_n respectively. In view of Theorem 3.5.1 and the fact that both $\Delta_{n,e}$ and $Z^{\infty}_{n,e}$ are defined over K (see Paragraph 3.4.4), Theorem 4.1.1 follows immediately from the following lemma.

Lemma 4.1.1. Let $Z \in CH_{n-1}(X_0^{2n-2})$. Then the image of the composition

$$CH^{hom}_{n-1}(X_0^{2n-1}) \overset{h_n}{\to} J((H^1)^{\otimes 2n-1})^{\vee} \overset{\xi_Z^{-1}}{\to} J(H^1)^{\vee}$$

lies in the subgroup $Jac(K)$.

Proof. Denote the diagonal of X_0 by $\Delta(X_0)$. Let $Z' \in CH_n(X_0^{2n})$ be such that its class in

$$H^{2n}(X^{2n})^{\vee}$$

is the $((H^1)^{\otimes 2n})^{\vee}$ Kunneth component of

$$cl(Z \times \Delta(X_0)) \in H^{2n}(X^{2n})^{\vee}.$$

Such Z' can be explicitly constructed using the fact that the Kunneth components of the class of the diagonal $\Delta(X_0) \in CH_1(X_0^2)$ are algebraic. We will show that the diagram

$$\begin{tikzcd} CH^{hom}_{n-1}(X_0^{2n-1}) \arrow[r, "h_n"] \arrow[d, "Z'_*"'] & J((H^1)^{\otimes 2n-1})^\vee \arrow[d, "\xi_Z^{-1}"] \\ CH^{hom}_0(X_0) \arrow[r, "h_1 = AJ"] & J(H^1)^\vee \end{tikzcd} \tag{4.1}$$

commutes. This will prove the assertion, as h_1 is the map that identifies $Jac(K) = CH_0^{hom}(X_0)$ as a subgroup of $J(H^1)^\vee$.

By functoriality of the Abel-Jacobi maps with respect to correspondences, one has a commutative diagram

$$\begin{tikzcd} CH^{hom}_{n-1}(X_0^{2n-1}) \arrow[r, "AJ"] \arrow[d, "Z'_*"'] & JH^{2n-1}(X^{2n-1})^\vee \arrow[d, "cl(Z')"] \\ CH^{hom}_0(X_0) \arrow[r, "h_1"] & J(H^1)^\vee. \end{tikzcd} \tag{4.2}$$

Thus to establish commutativity of (4.1), it suffices to show that

$$\begin{tikzcd} JH^{2n-1}(X^{2n-1})^\vee \arrow[r, "\text{natural}", "\text{projection}"'] \arrow[d, "cl(Z')"'] & J((H^1)^{\otimes 2n-1})^\vee \arrow[dl, "\xi_Z^{-1}"] \\ J(H^1)^\vee & \end{tikzcd} \tag{4.3}$$

commutes. This in turn will be established if we verify the commutativity of

$$\begin{tikzcd} H^{2n-1}(X^{2n-1})^\vee \arrow[r, "\text{natural}", "\text{projection}"'] \arrow[d, "cl(Z')"'] & ((H^1)^{\otimes 2n-1})^\vee \arrow[dl, "\xi_Z^{-1}"] \\ (H^1)^\vee, & \end{tikzcd} \tag{4.4}$$

where with abuse of notation ξ_Z^{-1} denotes the map $f \mapsto f(\xi_Z \otimes -)$. Note that since

$$\mathrm{cl}(Z') \in ((H^1)^{\otimes 2n})^\vee \subset H^{2n}(X^{2n})^\vee,$$

we only need to verify commutativity on the direct summand

$$((H^1)^{\otimes 2n-1})^\vee \subset H^{2n-1}(X^{2n-1})^\vee.$$

Let $f \in ((H^1_{\mathbb{C}})^{\otimes 2n-1})^\vee$. Suppose f is the Poincare dual of $\alpha \in H^{2n-1}_{\mathbb{C}}(X^{2n-1})$, i.e.

$$f(-) = \int_{X^{2n-1}} \alpha \wedge -.$$

Then α lies in the Kunneth component $(H^1_{\mathbb{C}})^{\otimes 2n-1}$. Let $\beta \in H^1_{\mathbb{C}}$. Unwinding definitions, in view of the fact that $\mathrm{cl}(Z')$ is the $((H^1)^{\otimes 2n})^\vee$ component of $\mathrm{cl}(Z \times \Delta(X_0))$, we have

$$\mathrm{cl}(Z')(f)(\beta) = \mathrm{cl}(Z')(\alpha \otimes \beta) = \mathrm{cl}(Z \times \Delta(X_0))(\alpha \otimes \beta).$$

Let

$$\alpha = \sum_i \alpha_1^{(i)} \otimes \ldots \otimes \alpha_{2n-1}^{(i)}.$$

Then

$$\begin{aligned}
\mathrm{cl}(Z')(f)(\beta) &= \sum_i \mathrm{cl}(Z \times \Delta(X_0))(\alpha_1^{(i)} \otimes \ldots \otimes \alpha_{2n-1}^{(i)} \otimes \beta) \\
&= \sum_i \mathrm{cl}(Z)(\alpha_1^{(i)} \otimes \ldots \otimes \alpha_{2n-2}^{(i)}) \int_X \alpha_{2n-1}^{(i)} \wedge \beta \\
&= \sum_i \int_{X^{2n-2}} \xi_Z \wedge (\alpha_1^{(i)} \otimes \ldots \otimes \alpha_{2n-2}^{(i)}) \int_X \alpha_{2n-1}^{(i)} \wedge \beta \\
&= \sum_i \int_{X^{2n-1}} \left(\xi_Z \wedge (\alpha_1^{(i)} \otimes \ldots \otimes \alpha_{2n-2}^{(i)})\right) \otimes (\alpha_{2n-1}^{(i)} \wedge \beta) \\
&= \int_{X^{2n-1}} \alpha \wedge (\xi_Z \otimes \beta) \\
&= f(\xi_Z \otimes \beta).
\end{aligned}$$

Thus $\mathrm{cl}(Z')(f) = \xi_Z^{-1}(f)$ as desired. $\square$

From now on, in the interest of simplifying the notation, for a Hodge class $\xi \in (H^1)^{\otimes 2n-2}$, we write P_ξ for $\xi^{-1}(\Psi(\mathbb{E}^\infty_{n,e}))$. For $Z \in CH_{n-1}(X_0^{2n-2})$, we simply write P_Z for P_{ξ_Z}.

Remark. The idea of constructing points on the Jacobian of X_0 using Hodge classes in $H^2(X^2)$ first arose in the work [30] of W. Yuan, S. Zhang, and W. Zhang in the setting of modular curves.

4.2 An analytic description of P_Z

Recall that g is the genus of X.

4.2.1

Proposition 3.3.1(a) gives us a description of $\Psi(\mathbb{E}^\infty_{n,e})$, and hence can be used to give an analytic description of points of the form P_Z, or more generally P_ξ. The issue with this description will be that it involves the forms ν. More precisely, to do computations with it one needs to know $\nu(\omega_1 \otimes \omega_2)$ for harmonic forms ω_1, ω_2 on X. In this section, we try to give a different description of $\Psi(\mathbb{E}^\infty_{n,e})$, and hence P_Z and P_ξ, which does not have this issue, as it uses differentials of the second kind as opposed to harmonic forms.

Recall that in view of Carlson's theorem (see Paragraph 2.2.2), a Hodge section of q and an integral retraction of i (see (3.13)) will give us a description of

$$\mathbb{E}^\infty_{n,e} \in \mathrm{Ext}((H^1)^{\otimes n}, (H^1)^{\otimes n-1}) \cong J\underline{\mathrm{Hom}}((H^1)^{\otimes n}, (H^1)^{\otimes n-1}).$$

We will use the same retraction $r_{\mathbb{Z}}$ of i as in Chapter 3, but seek for a different, rather more simple, Hodge section of q.

From now on (to the end of the thesis), we fix the following data:

(i) $\alpha_1, \dots, \alpha_{2g}$ as in Paragraph 3.2.1: For $1 \le i \le g$, α_i is holomorphic on X, and for $g+1 \le i \le 2g$, α_i is meromorphic on X and holomorphic on $X-\{\infty\}$, and the cohomology classes of the α_i form a basis of $H^1_{\mathbb{C}}$.

(ii) a basis $d_1, \dots, d_{2g}$ of $H^1_{\mathbb{Z}}$

(iii) $\beta_1, \ldots, \beta_{2g} \in \pi_1(X-\{\infty\}, e)$ such that $[\beta_i] = \mathrm{PD}(d_i)$, i.e.

$$\int_{\beta_i} - = \int_X d_i \wedge -.$$

As in Paragraph 3.2.1, let

$$R^1 = \sum_i \mathbb{C}\alpha_i \subset \Omega^1_{\mathrm{hol}}(X - \{\infty\}),$$

where for any Riemann surface M, by $\Omega^1_{\mathrm{hol}}(M)$ we denote the space of holomorphic 1-forms on M.

The map

$$(H^1)^{\otimes n}_{\mathbb{C}} \to (L_n)_{\mathbb{C}}$$

defined by

$$[\alpha_{i_1}] \otimes \ldots \otimes [\alpha_{i_n}] \mapsto \int \alpha_{i_1} \ldots \alpha_{i_n},$$

or equivalently by

$$[\omega_{i_1}] \otimes \ldots \otimes [\omega_{i_n}] \mapsto \int \omega_{i_1} \ldots \omega_{i_n} \quad (\omega_i \in R^1), \tag{4.5}$$

is a section of $q_{\mathbb{C}}$; this is clear from (3.9). Thus the composition

$$\sigma_F : (H^1)^{\otimes n}_{\mathbb{C}} \overset{(4.5)}{\to} (L_n)_{\mathbb{C}} \overset{\text{quotient}}{\to} \left(\frac{L_n}{L_{n-2}}\right)_{\mathbb{C}}$$

is a section of q.

Hypothesis $\star(n)$: We say that the α_i satisfy *Hypothesis* $\star(n)$ if the map σ_F above is compatible with the Hodge filtrations.

Remark. It is possible that Hypothesis $\star(n)$ is always true. Note that the α_i certainly satisfy Hypothesis $\star(n)$ if $F^p(L_n)_{\mathbb{C}}$ is the span of iterated integrals of the form

$$\int \alpha_{i_1} \ldots \alpha_{i_l} \qquad (l \leq n)$$

with at least p of the α_{i_r} of the first kind.[†]

Recall from Paragraph 3.3.4 that our choice of the β_i leads to an integral retraction $r_{\mathbb{Z}}$ of i given by (3.14). In view of Carlson's theorem, if the α_i satisfy Hypothesis $\star(n)$, the extension $\mathbb{E}^{\infty}_{n,e} \in J\underline{\mathrm{Hom}}((H^1)^{\otimes n},(H^1)^{\otimes n-1})$ is represented by the map $r_{\mathbb{Z}} \circ \sigma_F$. Thus we have the following description of $\Psi(\mathbb{E}^{\infty}_{n,e})$. (See the argument for Proposition 3.3.1(a).)

Proposition 4.2.1. If the α_i satisfy Hypothesis $\star(n)$, then $\Psi(\mathbb{E}^{\infty}_{n,e})$ is represented by the map

$$(H^1_{\mathbb{C}})^{\otimes 2n-1} \to \mathbb{C}$$

given by

$$[\omega_1] \otimes \ldots \otimes [\omega_n] \otimes d_{i_1} \otimes \ldots \otimes d_{i_{n-1}} \mapsto \int_{(\beta_{i_1}-1)\ldots(\beta_{i_{n-1}}-1)} \omega_1 \ldots \omega_n \qquad (\omega_i \in R^1).$$

Let

$$s : H^1_{\mathbb{C}} \to \Omega^1_{\mathrm{hol}}(X-\{\infty\})$$

be the map that sends $c \in H^1_{\mathbb{C}}$ to the unique element of R^1 representing c. From now on, $\omega_i := s(d_i)$; it is in particular a linear combination of the α_i with integral periods. For $c \in H^1_{\mathbb{C}}$, we write

$$c = \sum_i p_i(c) d_i,$$

which is equivalent to

$$s(c) = \sum_i p_i(c)\omega_i.$$

For a *multi-index*

$$I = (i_m, \ldots, i_m) \subset \{1, \ldots, 2g\}^m,$$

let

$$d_I = d_{i_1} \otimes \ldots \otimes d_{i_m} \in (H^1)^{\otimes m}_{\mathbb{Z}}.$$

For a Hodge class $\xi \in (H^1)^{\otimes 2n-2}$, we write

$$\xi = \sum_{I \subset \{1,\ldots,2g\}^{2n-2}} \lambda_I(\xi) d_I.$$

[†]More generally, one may hope that $F^p L_n(X-S,e)$, where S is any finite nonempty subset of $X(\mathbb{C})$, has a similar description (now counting the number of differentials of first or third kind).

Note that the $\lambda_I(\xi)$ are integers. For $Z \in CH_{n-1}(X_0^{2n-2})$, let $\lambda_I(Z) = \lambda_I(\xi_Z)$.

Denote the map of the previous proposition tentatively by f. If the α_i satisfy Hypothesis $\star(n)$, by definition, P_ξ is the class of the map

$$f_\xi : H^1_{\mathbb{C}} \to \mathbb{C} \qquad \text{defined by} \qquad c \mapsto f(\xi \otimes c).$$

We have

$$\begin{aligned} f(\xi \otimes c) &= \sum_j p_j(c) f(\xi \otimes d_j) \\ &= \sum_j \sum_I p_j(c)\lambda_I(\xi) f(d_I \otimes d_j) \\ &= \sum_j \sum_I p_j(c)\lambda_I(\xi) \int\limits_{(\beta_{i_{n+1}}-1)\ldots(\beta_{i_{2n-2}}-1)(\beta_j-1)} \omega_{i_1}\ldots\omega_{i_n}, \end{aligned}$$

where in all summations $1 \le j \le 2g$ and $I = (i_1, \ldots, i_{2n-2}) \in \{1, \ldots, 2g\}^{2n-2}$. We record the conclusion as a proposition.

Proposition 4.2.2. Suppose the α_i satisfy Hypothesis $\star(n)$. Then P_ξ is the class of the map $f_\xi : H^1_{\mathbb{C}} \to \mathbb{C}$ defined by

$$f_\xi(c) = \sum_j \sum_I p_j(c)\lambda_I(\xi) \int\limits_{(\beta_{i_{n+1}}-1)\ldots(\beta_{i_{2n-2}}-1)(\beta_j-1)} \omega_{i_1}\ldots\omega_{i_n}.$$

In particular, for $Z \in CH_{n-1}(X_0^{2n-2})$, P_Z is the class of f_{ξ_Z}.

We finish this paragraph by rewriting the formula for f_ξ in a form that will be useful in the next chapter. For future reference, we record it as a proposition.

Proposition 4.2.3. For $i, j, k \le 2g$, let

$$\mu'_{ijk}(\xi; c) = \sum_{r=1}^{n-1} \sum_{\substack{i_1,\ldots,i_{2n-1}\le 2g \\ (i_r,i_{r+1},i_{r+n})=(i,j,k)}} \lambda_{(i_1,\ldots,i_{2n-2})}(\xi)\, p_{i_{2n-1}}(c) \prod_{l=1}^{r-1} \int\limits_{\beta_{i_{l+n}}} \omega_{i_l} \prod_{l=r+2}^{n} \int\limits_{\beta_{i_{l+n-1}}} \omega_{i_l}.$$

Then

$$f_\xi(c) = \sum_{i,j,k\le 2g} \mu'_{ijk}(\xi; c) \int\limits_{\beta_k} \omega_i\omega_j.$$

Proof. This follows from the previous formula for f_ξ on noting that by (2.9),

$$\int_{(\beta_{i_{n+1}}-1)\dots(\beta_{i_{2n-2}}-1)(\beta_{i_{2n-1}}-1)} \omega_{i_1}\dots\omega_{i_n} = \sum_{r=1}^{n-1} \prod_{l=1}^{r-1} \int_{\beta_{i_{l+n}}} \omega_{i_l} \int_{\beta_{i_{r+n}}} \omega_{i_r}\omega_{i_{r+1}} \prod_{l=r+2}^{n} \int_{\beta_{i_{l+n-1}}} \omega_{i_l}.$$

□

For $Z \in CH_{n-1}(X_0^{2n-2})$, to simplify the notation we simply write f_Z for f_{ξ_Z}.

4.2.2 More on Hypothesis $\star(n)$

In this paragraph, we show that in the case of elliptic curves, one can indeed take the α_i such that they satisfy Hypothesis $\star(n)$ for all n. Note that when $g = 1$, by assumption α_2 has a pole of order ≥ 2 at ∞ and is holomorphic elsewhere. The form α_1 is holomorphic on X.

Proposition 4.2.4. Let $g = 1$. Suppose the order of ∞ as a pole of α_2 is 2. Then the α_i satisfy Hypothesis $\star(n)$.

Before we prove the proposition, we state an easy lemma.

Lemma 4.2.1. Let D be the open unit disc in $\mathbb{C}$. Suppose α is a holomorphic 1-form on $D - \{0\}$ with a pole of order 2 at 0, and η is a smooth closed 1-form on D. Let f be a smooth function on $D - \{0\}$ such that $df = \alpha - \eta$ on $D - \{0\}$. Then $z^2 f(z) \to 0$ as $z \to 0$.

Proof. Write $\alpha = (\frac{C}{z^2} + h)dz$, where $C \neq 0$ is a constant and h is a holomorphic function on D. Let F be a smooth function on D such that $dF = \eta$. Then

$$df = (\frac{C}{z^2} + h)dz + dF = d\left(-\frac{C}{z} + H + F\right),$$

where H is an anti-derivative of h. Thus

$$f = -\frac{C}{z} + H + F + \text{constant}.$$

The desired conclusion follows. □

Proof of Proposition 4.2.4: For convenience, we adopt the following temporary notation. For $i = 1, 2$, η_i denotes the harmonic 1-form on X whose cohomology class coincides with that of α_i. In particular, $\eta_1 = \alpha_1$. For each i, we write $\alpha_i = \eta_i + df_i$, where f_i is a smooth function on $X-\{\infty\}$ satisfying $f_i(e)=0$. (Thus f_1 is just 0.) Let $a_i=[\alpha_i]$. Note that $a_1\in F^1H^1_{\mathbb{C}}$. We will be using the multi-index notation

$$a_I = a_{i_1}\otimes\ldots\otimes a_{i_n}$$

for $I=(i_1,\ldots,i_n)\in\{1,2\}^n$.

To verify Hypothesis $\star(n)$, we need to show that σ_F is compatible with the Hodge filtrations. Since $\mathfrak{s}_F$ is known to be compatible with the Hodge filtrations (see Lemma 3.2.3), we can equivalently show that $\sigma_F-\mathfrak{s}_F$ respects the Hodge filtration. In view of the fact that both σ_F and $\mathfrak{s}_F$ are sections of q, we see that $\sigma_F-\mathfrak{s}_F$ actually maps into the subspace

$$\left(\frac{L_{n-1}}{L_{n-2}}\right)_{\mathbb{C}} = \ker(q)\subset\left(\frac{L_n}{L_{n-2}}\right)_{\mathbb{C}}$$

(see Paragraph 3.3.3). Thus we need to show that

$$(\sigma_F-\mathfrak{s}_F)(F^p(H^1_{\mathbb{C}})^{\otimes n})\subset F^p\left(\frac{L_n}{L_{n-2}}\right)_{\mathbb{C}}\cap\left(\frac{L_{n-1}}{L_{n-2}}\right)_{\mathbb{C}} = F^p\left(\frac{L_{n-1}}{L_{n-2}}\right)_{\mathbb{C}}.$$

Equivalently, in view of Proposition 3.2.3, we will be done if we show that

$$\overline{q}_{n-1}\circ(\sigma_F-\mathfrak{s}_F)(F^p(H^1_{\mathbb{C}})^{\otimes n})\subset F^p(H^1_{\mathbb{C}})^{\otimes n-1}.$$

(Here $\overline{q}_{n-1}$ is the isomorphism $\dfrac{L_{n-1}}{L_{n-2}}\cong(H^1_{\mathbb{C}})^{\otimes n-1}$ given by Proposition 3.2.3.)
Let

$$I=(i_1,\ldots,i_n)\in\{1,2\}^n$$

be such that at least p of the i_r are 1. It suffices to show that

$$\overline{q}_{n-1}\circ(\sigma_F-\mathfrak{s}_F)(a_I)\in F^p(H^1_{\mathbb{C}})^{\otimes n-1}.$$

By Lemma 3.2.3,

$$\begin{aligned}\mathfrak{s}_F(a_I) &= \int\eta_{i_1}\ldots\eta_{i_n}+\sum_{r=1}^{n-1}\eta_{i_1}\ldots\nu(\eta_{i_r}\otimes\eta_{i_{r+1}})\ldots\eta_{i_n}\\ &+\ \text{terms of length at most } n-2 \mod L_{n-2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma_F(a_I) &= \int \alpha_{i_1} \dots \alpha_{i_n} \mod L_{n-2} \\ &= \int (\eta_{i_1} + df_{i_1}) \dots (\eta_{i_n} + df_{i_n}) \mod L_{n-2}.\end{aligned}$$

The integral above expands as the integral of

$$\begin{aligned}&\eta_{i_1} \dots \eta_{i_n} + \sum_r \eta_{i_1} \dots (df_{i_r}) \dots \eta_{i_n} + \sum_{r<s} \eta_{i_1} \dots (df_{i_r}) \dots (df_{i_s}) \dots \eta_{i_n} \\ + \quad &\text{terms with three or more appearances of } df.\end{aligned}$$

In view of the relations (2.10) satisfied by iterated integrals, every summand in which two factors df_{i_r} and df_{i_s} with $s > r + 1$ appear, can be replaced by terms of length at most $n - 2$. (In particular, this can be done for terms with three or more appearances of df.) We get

$$\begin{aligned}\sigma_F(a_I) &= \int \eta_{i_1} \dots \eta_{i_n} + \underbrace{\sum_r \eta_{i_1} \dots (df_{i_r}) \dots \eta_{i_n}}_{(I)} \\ &+ \underbrace{\sum_{r<n} \eta_{i_1} \dots (df_{i_r})(df_{i_{r+1}}) \dots \eta_{i_n}}_{(II)} \\ &+ \text{terms of length at most } n-2 \qquad \mod L_{n-2}.\end{aligned}$$

On recalling $f_j(e) = 0$, straightforward computations using (2.10) show

$$\int (I) = \int \sum_{r<n} \eta_{i_1} \dots \eta_{i_{r-1}} (f_{i_r} \eta_{i_{r+1}} - f_{i_{r+1}} \eta_{i_r}) \eta_{i_{r+2}} \dots \eta_{i_n}$$

and

$$\begin{aligned}\int (II) &= \int \sum_{r<n} \eta_{i_1} \dots \eta_{i_{r-1}} (f_{i_r} df_{i_{r+1}}) \eta_{i_{r+2}} \dots \eta_{i_n} \\ &+ \text{terms of length at most } n-2.\end{aligned}$$

Thus

$$\begin{aligned}(\sigma_F-\mathfrak{s}_F)(a_I) &= \int \sum_{r<n} \eta_{i_1}\dots\eta_{i_{r-1}}\left(f_{i_r}\eta_{i_{r+1}} - f_{i_{r+1}}\eta_{i_r} - \nu(\eta_{i_r}\otimes\eta_{i_{r+1}}) \right)\eta_{i_{r+2}}\dots\eta_{i_n} \\ &+ \sum_{r<n}\eta_{i_1}\dots\eta_{i_{r-1}}(f_{i_r}df_{i_{r+1}})\eta_{i_{r+2}}\dots\eta_{i_n} \\ &+ \text{terms of length at most } n-2 \qquad \operatorname{mod} L_{n-2}.\end{aligned}$$

Note that each term on the right appearing on the first two lines has length $n-1$. The integral on the right (which is closed) lives in L_{n-1}. We claim that both $f_{i_r}\eta_{i_{r+1}} - f_{i_{r+1}}\eta_{i_r} - \nu(\eta_{i_r}\otimes\eta_{i_{r+1}})$ and $f_{i_r}df_{i_{r+1}}$ are closed. This is clear for the latter element. As for the former, if $i_r = i_{r+1}$, then

$$d\left(f_{i_r}\eta_{i_{r+1}} - f_{i_{r+1}}\eta_{i_r} - \nu(\eta_{i_r}\otimes\eta_{i_{r+1}}) \right) = -d\nu(\eta_{i_r}\otimes\eta_{i_{r+1}}) = -\eta_{i_r}\wedge\eta_{i_r} = 0.$$

On the other hand, if $i_r \neq i_{r+1} = 1$, then on recalling $f_1 = 0$, one has

$$f_{i_r}\eta_{i_{r+1}} - f_{i_{r+1}}\eta_{i_r} - \nu(\eta_{i_r}\otimes\eta_{i_{r+1}}) = f_2\eta_1 - \nu(\eta_2\otimes\eta_1),$$

the latter easily seen to be closed. The case $i_r \neq i_{r+1} = 2$ is similar.

It follows that

$$\begin{aligned}\overline{q}_{n-1}(\sigma_F-\mathfrak{s}_F)(a_I) &= \sum_{r<n} a_{i_1}\otimes\dots\otimes a_{i_{r-1}}\otimes b_r\otimes a_{i_{r+2}}\otimes\dots\otimes a_{i_n} \\ &+ \sum_{r<n} a_{i_1}\otimes\dots\otimes a_{i_{r-1}}\otimes [f_{i_r}df_{i_{r+1}}]\otimes a_{i_{r+2}}\otimes\dots\otimes a_{i_n},\end{aligned}$$

where

$$b_r = [f_{i_r}\eta_{i_{r+1}} - f_{i_{r+1}}\eta_{i_r} - \nu(\eta_{i_r}\otimes\eta_{i_{r+1}})].$$

To complete the proof, it suffices to show that every term in the expansion of $\overline{q}_{n-1}(\sigma_F - \mathfrak{s}_F)(a_I)$ above belongs to $F^p(H^1_{\mathbb{C}})^{\otimes n-1}$. The element

$$a_{i_1}\otimes\dots\otimes a_{i_{r-1}}\otimes[f_{i_r}df_{i_{r+1}}]\otimes a_{i_{r+2}}\otimes\dots\otimes a_{i_n} \tag{4.6}$$

is zero if i_r or i_{r+1} is 1. If both i_r and i_{r+1} are 2, then by assumption at least p of

$$i_1,\dots,i_{r-1},i_{r+2},\dots i_n \tag{4.7}$$

are 1, and hence (4.6) belongs to $F^p(H^1_{\mathbb{C}})^{\otimes n-1}$. We show that

$$a_{i_1}\otimes\ldots\otimes a_{i_{r-1}}\otimes b_r\otimes a_{i_{r+2}}\otimes\ldots\otimes a_{i_n} \tag{4.8}$$

is also in $F^p(H^1_{\mathbb{C}})^{\otimes n-1}$. If $i_r=i_{r+1}=1$, then $b_r=0$. (In fact, the differential $f_{i_r}\eta_{i_{r+1}}-f_{i_{r+1}}\eta_{i_r}-\nu(\eta_{i_r}\otimes\eta_{i_{r+1}})$ is zero, see Lemma 3.2.2(ii).) If $i_r=i_{r+1}=2$, then again by assumption as least p of (4.7) are 1. Finally, suppose $i_r\neq i_{r+1}$. Then at least $p-1$ of (4.7) are 1, so that it is enough to show that $b_r\in F^1H^1_{\mathbb{C}}$. We consider the case $i_r=1$. (The other case is similar.) The 1-form

$$f_{i_r}\eta_{i_{r+1}}-f_{i_{r+1}}\eta_{i_r}-\nu(\eta_{i_r}\otimes\eta_{i_{r+1}})=-f_2\eta_1-\nu(\eta_1\otimes\eta_2) \tag{4.9}$$

on $X-\{\infty\}$ is of type (1,0), as ν preserves the Hodge filtration. It is also closed, and hence is holomorphic on $X-\{\infty\}$. By the previous lemma and the fact that ν takes values in $E^1(X\log\infty)$, (4.9) (is meromorphic at ∞ and) has a pole of order at most 1 at ∞. It follows from the residue theorem that indeed (4.9) is holomorphic on X, and hence $b_r\in F^1H^1_{\mathbb{C}}$, as desired. □

We close this chapter with a remark.

Remark. Note that by Riemann-Roch, there exists a meromorphic form on X with divisor $\geq -2\infty$, so that by the previous proposition there always exist α_1,α_2 satisfying Hypothesis $\star(n)$. More explicitly, if X_0 is given by the affine equation

$$y^2=4x^3-g_2x-g_3,$$

and ∞ is the point at infinity, we can take $\alpha_2=\frac{xdx}{y}$. If ∞ in not the point at infinity, we can take α_2 to be the pullback of $\frac{xdx}{y}$ along a translation. (Meanwhile, α_1 can be taken to be any nonzero holomorphic form on X.)

Chapter 5

Application to Periods

5.1 Some elementary remarks

For $c \in H^1_{\mathbb{C}}$, define the space of periods of X corresponding to c to be

$$\mathrm{Per}_{\mathbb{Q}}(c) := (H_1)_{\mathbb{Q}}(c) = \sum_{i \leq 2g} \mathbb{Q} \int_{\beta_i} c.$$

It is easy to see that

$$\mathrm{Per}_{\mathbb{Q}}(c) = \sum_{i \leq 2g} \mathbb{Q} p_i(c). \tag{5.1}$$

Indeed, if $B = (b_{ij})$ where

$$b_{ij} = \int_{\beta_i} \omega_j,$$

then

$$B \begin{pmatrix} p_1(c) \\ \vdots \\ p_{2g}(c) \end{pmatrix} = \begin{pmatrix} \int_{\beta_1} c \\ \vdots \\ \int_{\beta_{2g}} c \end{pmatrix}.$$

Since the Poincare pairing is non-degenerate, B is invertible. Hence (5.1) follows.

From now on we assume that the α_i belong to $\Omega^1(X_0)$, i.e. are (global) algebraic 1-forms on X_0. Recall that the space of periods of X_0 is the K-span of the numbers

$$\int_{\beta_i} \alpha_j.$$

We denote this space by $\mathrm{Per}(X_0)$. For $1 \leq i, j \leq 2g$, let

$$p_{ij} = p_j([\alpha_i]),$$

so that

$$\alpha_i = \sum_j p_{ij}\omega_j.$$

It follows from (5.1) that $\mathrm{Per}(X_0)$ is spanned (over K) by the numbers p_{ij} $(i, j \leq 2g)$.

It is easy to see that the de Rham lattice (see Section 2.5) in $L_n(X_0 - \{\infty\}, e)$ is the K-span of the iterated integrals

$$\int \alpha_{i_1} \dots \alpha_{i_r} \quad (r \leq n,\ i_1, \dots, i_r \leq 2g),$$

and thus the space of periods of $L_n(X_0 - \{\infty\}, e)$ is the K-span of the numbers

$$\int_\gamma \alpha_{i_1} \dots \alpha_{i_r} \quad (i_1, \dots, i_r \leq 2g) \tag{5.2}$$

with $r \leq n$ and $\gamma \in \pi_1(X - \{\infty\}, e)$.

Let $\mathbb{Q}(\mathrm{Per}(X_0))$ be the field generated over $\mathbb{Q}$ by the periods of X_0. One can easily see that for any $\gamma \in \pi_1(X - \{\infty\}, e)$ and r, the $\mathbb{Q}(\mathrm{Per}(X_0))$-span of the numbers

$$\int_\gamma \omega_{i_1} \dots \omega_{i_r} \quad (i_1, \dots, i_r \leq 2g) \tag{5.3}$$

is equal to the $\mathbb{Q}(\mathrm{Per}(X_0))$-span of the numbers (5.2). Indeed, each number in (5.2) (resp. (5.3)) can be written as a linear combination of the elements of the other set with coefficients being explicit polynomials (resp. rational functions) in the p_{ij}.

5.2 Methodology

5.2.1

It is well-known that algebraic cycles on products of X, or rather Hodge classes in tensor powers of H^1, give rise to algebraic relations between periods of X_0. In short, this is because these Hodge classes cut down the dimension of the Mumford-Tate group of X, which in turn cuts down the transcendence degree over K of the field obtained by adjoining the periods of X_0 to K†. A very elementary and explicit version of this explanation goes as follows: The subspace $F^p(H^1_{\mathbb{C}})^{\otimes 2p}$ is spanned by certain elements of the form

$$\sum_{I \subset \{1,\ldots,2g\}^{2p}} P_I d_I,$$

where each P_I is a product of p of the p_{ij} with $i \le g$. It follows from basic linear algebra that if $\{\lambda_I\}_I$ are integers, the element

$$\sum_I \lambda_I d_I \in (H^1)^{\otimes 2p}_{\mathbb{Z}}$$

is a Hodge class if and only if the p_{ij} ($i \le g$) satisfy certain polynomial relations with integer coefficients.

Example. Let $g = 1$. Then $F^1(H^1)^{\otimes 2}$ is spanned by the four elements

$$[\alpha_1] \otimes d_i,\ d_i \otimes [\alpha_1] \qquad (i = 1, 2),$$

which can be written as

$$p_{11} d_{1i} + p_{12} d_{2i},\ p_{11} d_{i1} + p_{12} d_{i2} \qquad (i = 1, 2).$$

An element

$$\xi = \sum_{ij} \lambda_{ij} d_{ij} \in (H^1_{\mathbb{Z}})^{\otimes 2}$$

is a Hodge class (or equivalently, belongs to $F^1(H^1)^{\otimes 2}$) if and only if p_{11}, p_{12} satisfy

$$\lambda_{22} p_{11}^2 - (\lambda_{12} + \lambda_{21}) p_{11} p_{12} + \lambda_{11} p_{12}^2 = 0. \tag{5.4}$$

It is easy to see

$$\xi_{\Delta(X)} = \pm^{\ddagger}(-d_{12} + d_{21}),$$

so that (5.4) becomes trivial. The statements (i)-(iv) below are equivalent.

†It is known that the transcendence degree over K of the field obtained by adjoining the periods of X_0 to K is less than or equal to the dimension of the Mumford-Tate group of X. It is conjectured that the two quantities are indeed equal.

‡depending on whether $\int_X \omega_1 \wedge \omega_2 = \pm 1$

(i) dim $(H^1_{\mathbb{Z}})^{\otimes 2} \cap F^1(H^1_{\mathbb{C}})^{\otimes 2} = 2$

(ii) There is a Hodge class $\xi = \sum_{ij} \lambda_{ij} d_{ij}$ with $\lambda_{22} \neq 0$.

(ii) The periods p_{11}, p_{12} satisfy a non-trivial quadratic relation given by (5.4).

(iv) X has complex multiplication.

The Hodge class ξ in (ii) corresponds to a complex endomorphism of X.

5.2.2

The main objective of the current chapter is to show that Hodge classes in tensor powers of H^1, and hence algebraic cycles on products of X, in addition to giving rise to algebraic relations between the periods of X_0 (as discussed in the previous paragraph), *may* also give rise to non-trivial relations among the periods of the fundamental group of $X_0 - \{\infty\}$ that lie deeper in the weight filtration, at least among the periods of $L_2(X_0 - \{\infty\}, e)$ (i.e. iterated integrals of length ≤ 2 in the forms α_i). In this paragraph, we explain how a Hodge class might give rise to such relations.

Throughout, to simplify the notation, we identify

$$\Omega^1_{\text{hol}}(X) = H^{1,0}$$

via the distinguished isomorphism between them.

In the previous chapter for each Hodge class $\xi \in (H^1)^{\otimes 2n-2}$ we defined a point

$$P_\xi = \xi^{-1}(\Psi(\mathbb{E}^\infty_{n,e})) \in J(H^1)^\vee = \text{Jac}(\mathbb{C}).$$

We identify

$$J(H^1)^\vee \cong \frac{\Omega^1_{\text{hol}}(X)^\vee}{H_1(X, \mathbb{Z})}$$

via the isomorphism given by

$$[f] \mapsto [f\,|_{\Omega^1_{\text{hol}}(X)}].$$

If the α_i satisfy Hypothesis $\star(n)$, the point

$$P_\xi \in \frac{\Omega^1_{\text{hol}}(X)^\vee}{H_1(X, \mathbb{Z})}$$

is $[f_\xi\, |_{\Omega^1_{\mathrm{hol}}(X)}]$. (See Proposition 4.2.2.)

Lemma 5.2.1. Suppose the α_i satisfy Hypothesis $\star(n)$.
(a) If P_ξ is torsion, then for every $\alpha \in \Omega^1_{\mathrm{hol}}(X)$, $f_\xi(\alpha) \in \mathrm{Per}_{\mathbb{Q}}(\alpha)$.
(b) If $g = 1$, then P_ξ is torsion if and only if for all (or equivalently for some nonzero) $\alpha \in \Omega^1_{\mathrm{hol}}(X)$, $f_\xi(\alpha) \in \mathrm{Per}_{\mathbb{Q}}(\alpha)$.

Proof. Both parts are immediate from that P_ξ is torsion if and only if $f_\xi\, |_{\Omega^1_{\mathrm{hol}}(X)}$ coincides with an element of $H_1(X, \mathbb{Q})$. □

Suppose the α_i satisfy Hypothesis $\star(n)$, and a Hodge class $\xi \in (H^1)^{\otimes 2n-2}$ is such that P_ξ is torsion. Then by the previous lemma and Proposition 4.2.3, for every $\alpha \in \Omega^1_{\mathrm{hol}}(X)$ one has

$$\sum_{i,j,k \leq 2g} \mu'_{(i,j,k)}(\xi; \alpha) \int_{\beta_k} \omega_i \omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha). \tag{5.5}$$

The μ' are integral linear combinations of the $p_l(\alpha)$, and by (5.1) they belong to $\mathrm{Per}_{\mathbb{Q}}(\alpha)$. Setting $\alpha = \alpha_1, \ldots, \alpha_g$, we get relations between

$$1, \int_{\beta_k} \omega_i \omega_j \quad (i, j, k \leq 2g) \tag{5.6}$$

with coefficients in $\mathrm{Per}(X_0)$.

One has the formal relations among (5.6) of the form

$$\int_{\beta_k} \omega_i \int_{\beta_k} \omega_j = \int_{\beta_k} \omega_i \omega_j + \int_{\beta_k} \omega_j \omega_i, \tag{5.7}$$

which come from the shuffle product property of iterated integrals. These will enable us to write the relations (5.5) in fewer "variables". For $\alpha \in \Omega^1_{\mathrm{hol}}(X)$ and a Hodge class $\xi \in (H^1)^{\otimes 2n-2}$, and i, j, k such that

$$i, j, k \leq 2g, i < j,$$

let

$$\mu_{(i,j,k)}(\xi; \alpha) = \mu'_{(i,j,k)}(\xi; \alpha) - \mu'_{(j,i,k)}(\xi; \alpha).$$

Proposition 5.2.1. Suppose a Hodge class $\xi \in (H^1)^{\otimes 2n-2}$ is such that P_ξ is torsion. If the α_i satisfy Hypothesis $\star(n)$, then for every $\alpha \in \Omega^1_{\text{hol}}(X)$,

$$\sum_{\substack{i,j,k\leq 2g\\ i<j}} \mu_{(i,j,k)}(\xi;\alpha)\int_{\beta_k}\omega_i\omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha).$$

Proof. We know (5.5) is true. Now note that by (5.7),

$$\begin{aligned}\sum_{i,j,k\leq 2g} \mu'_{(i,j,k)}(\xi;\alpha)\int_{\beta_k}\omega_i\omega_j &= \sum_{\substack{i,j,k\leq 2g\\ i<j}} \mu'_{(i,j,k)}(\xi;\alpha)\int_{\beta_k}\omega_i\omega_j\\ &+ \sum_{i,k\leq 2g} \frac{1}{2}\mu'_{(i,i,k)}(\xi;\alpha)(\int_{\beta_k}\omega_i)^2\\ &+ \sum_{\substack{i,j,k\leq 2g\\ i>j}} \mu'_{(i,j,k)}(\xi;\alpha)\left(\int_{\beta_k}\omega_i\int_{\beta_k}\omega_j - \int_{\beta_k}\omega_j\omega_i\right)\\ &\overset{\mathrm{Per}_{\mathbb{Q}}(\alpha)}{\equiv} \sum_{\substack{i,j,k\leq 2g\\ i<j}} \mu_{(i,j,k)}(\xi;\alpha)\int_{\beta_k}\omega_i\omega_j\ ,\end{aligned}$$

since the μ' belong to $\mathrm{Per}_{\mathbb{Q}}(\alpha)$. □

Suppose the α_i satisfy Hypothesis $\star(n)$, and that P_ξ is torsion. Taking $\alpha = \alpha_1, \ldots, \alpha_g$, we get g linear relations between

$$1, \int_{\beta_k}\omega_i\omega_j \quad (i,j,k\leq 2g, i<j) \tag{5.8}$$

with coefficients in $\mathrm{Per}(X_0)$. In view of the last comment in Section 5.1 and the shuffle product property of iterated integrals, each of these relations can be rewritten as a linear relation in

$$1, \int_{\beta_k}\alpha_i\alpha_j \quad (i,j,k\leq 2g, i<j) \tag{5.9}$$

with coefficients in $\mathbb{Q}(\mathrm{Per}(X_0))$.

Remark. (1) Suppose the α_i satisfy Hypothesis $\star(n)$. Recall that if $\xi = \xi_Z$ for an algebraic cycle $Z \in CH_{n-1}(X_0^{2n-2})$, then P_ξ is in $\mathrm{Jac}(K)$. (See Theorem 4.1.1.) If the Mordell-Weil group $\mathrm{Jac}(K)$ is finite, then P_ξ will automatically be torsion, and hence in view of Proposition 5.2.1 we get relations among (5.8). We will pursue this further in the next section.

(2) As it was mentioned earlier, it may be the case that Hypothesis $\star(n)$ in fact always holds. Recall that at least we know it does hold if $g = 1$ and α_2 has order 2 at ∞. (See Proposition 4.2.4.)

5.3 Examples

In this section, we carry out the ideas of the previous section in some cases. In order to simplify the calculations, we will assume from now on that the cohomology classes d_i are chosen in such a way that

$$\int_X d_i \wedge d_j = 1 \quad \text{if} \quad i < j.$$

5.3.1 Relations coming from the diagonal of X_0

In this paragraph, we show that interestingly, the diagonal $\Delta(X_0) \in CH_1(X_0^2)$ can give rise to relations between (5.8) that do not seem to be trivial. This is in particular interesting, because $\Delta(X_0)$ does not give rise to a relation between the periods of X_0 itself (as we discussed for $g = 1$ case in Section 5.1). We begin by finding $\xi_{\Delta(X_0)}$. Recall that in our notation $\xi_Z = \sum_I \lambda_I(Z) d_I$.

Lemma 5.3.1. We have

$$\lambda_{ij}(\Delta(X_0)) = \begin{cases} (-1)^{i+j} & \text{if } i < j \\ 0 & \text{if } i = j \\ (-1)^{i+j+1} & \text{if } i > j. \end{cases}$$

Proof. Let $\{c_i\}$ be the basis of $H^1_{\mathbb{Z}}$ that is dual to $\{d_i\}$ with respect to Poincare duality, i.e.

$$\int_X c_i \wedge d_j = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We use the multi-index notation for the c_i as well: c_{ij} means $c_i \otimes c_j$. For simplicity, write λ_{ij} for $\lambda_{ij}(\Delta(X_0))$. One has for each i, j,

$$\int_{\Delta(X)} c_{ij} = \int_{X^2} \sum_{k,l} \lambda_{kl} d_{kl} \wedge c_{ij},$$

which can be rewritten as

$$\int_X c_i \wedge c_j = -\sum_{k,l} \lambda_{kl} \int_{X^2} (d_k \wedge c_i) \otimes (d_l \wedge c_j),$$

the latter being clearly equal to

$$-\sum_{k,l} \lambda_{kl} \int_X (d_k \wedge c_i) \int_X (d_l \wedge c_j).$$

It follows that

$$\lambda_{ij} = \int_X c_j \wedge c_i. \tag{5.10}$$

Let $A = (a_{ij})$, where

$$a_{ij} = \int_X d_i \wedge d_j,$$

so that A is a $2g$ by $2g$ skew-symmetric matrix with the entries above the diagonal all equal to 1. For each i, let

$$c_i = \sum_j b_{ij} d_j.$$

Let $B = (b_{ij})$. One has

$$\delta_{ij} = \int_X c_i \wedge d_j = \int_X \sum_k b_{ik} d_k \wedge d_j = \sum_k b_{ik} a_{kj},$$

so that BA is identity, $B = A^{-1}$. It follows that

$$b_{ij} = \begin{cases} (-1)^{i+j} & \text{if } i < j \\ 0 & \text{if } i = j \\ (-1)^{i+j+1} & \text{if } i > j. \end{cases}$$

On the other hand, by (5.10),

$$\lambda_{ij} = \sum_{k,l} b_{jk} b_{il} a_{kl},$$

which is the ij-entry of the matrix $B(BA)^t = B$. The result follows. $\square$

Let $\alpha \in \Omega^1_{\mathrm{hol}}(X)$. One has

$$\mu'_{ijk}(\xi_{\Delta(X_0)};\alpha) = \lambda_{ij}(\Delta(X_0))p_k(\alpha),$$

and hence for $i < j$,

$$\begin{aligned}\mu_{ijk}(\xi_{\Delta(X_0)};\alpha) &= p_k(\alpha)\left(\lambda_{ij}(\Delta(X_0)) - \lambda_{ji}(\Delta(X_0))\right)\\ &= 2(-1)^{i+j}p_k(\alpha).\end{aligned}$$

Proposition 5.3.1. Suppose the α_i satisfy Hypothesis $\star(2)$. If $P_{\Delta(X_0)}$ is torsion, then

$$\sum_{\substack{i,j,k\leq 2g\\ i<j}} (-1)^{i+j} p_{lk} \int_{\beta_k} \omega_i\omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) \qquad (l = 1,\ldots,g). \tag{5.11}$$

Moreover, these, as linear relations among (5.8) with coefficients in $\mathbb{Q}(\mathrm{Per}(X_0))$, are independent.

Proof. The first assertion is a special case of Proposition 5.2.1. As for the independence of the relations, note that the $g \times 2g$ matrix whose lk-entry is the coefficient in the relation corresponding to α_l of

$$\int_{\beta_k} \omega_1\omega_2,$$

is minus the top half of the matrix $(p_{ij})_{i,j\leq 2g}$ of periods. The latter matrix is invertible and hence the former has rank g.

□

By Theorem 4.1.1, $P_{\Delta(X_0)}$ is in $\mathrm{Jac}(K)$, so that the torsion condition automatically holds if the Mordell-Weil group $\mathrm{Jac}(K)$ is finite. In particular, one obtains:

Corollary 5.3.1. Let $K = \mathbb{Q}$ and X_0 be either the hyper-elliptic curve given by the affine equation

$$y^2 = x(x-3)(x-4)(x-6)(x-7),$$

or the Fermat curve given by the affine equation

$$x^p + y^p = 1,$$

where p is an odd prime ≤ 7. Suppose (in each case) the α_i satisfy Hypothesis $\star(2)$. Then one has g (the genus of X_0 in each case) independent relations as in (5.11).

Indeed, in each of these situations $\mathrm{Jac}(\mathbb{Q})$ is known to be finite. See [15] for the hyper-elliptic curve and and [13] for the given Fermat curves. Note that the points e, ∞ must be in $X_0(\mathbb{Q})$.

Of course, if X_0 is an elliptic curve over K with finite $X_0(K)$, then again a relation of the form (5.11) is guaranteed. The reason we did not include this as a part of the corollary is that one may not need $X_0(K)$ to be finite in this case, as we will see in the next paragraph.

5.3.2 More on the genus one case

In this paragraph, we prove the following result.

Theorem 5.3.1. Let $g = 1$. Suppose the α_i satisfy Hypothesis $\star(2)$ (which is guaranteed for instance if α_2 has order 2 at ∞). Then

$$p_{11} \int_{\beta_1} \omega_1\omega_2 + p_{12} \int_{\beta_2} \omega_1\omega_2 \in \mathrm{Per}_{\mathbb{Q}}(\alpha_1) \tag{5.12}$$

if and only if $\infty - e$ is torsion in $\mathrm{CH}_0^{\mathrm{hom}}(X_0)$ (or equivalently, in $X_0(K)$).

Before we prove this statement, let us make a remark. The relation (5.12) is just what (5.11) reads when $g = 1$. The “only if” part of the result is particularly satisfying, as it confirms that the relation is not just a consequence of the formal properties of iterated integrals, and that it cannot be proved using methods that are not sensitive to whether or not $\infty - e$ is torsion.

As for the proof of the theorem, since (5.12) can be rewritten as

$$f_{\Delta(X_0)}(\alpha_1) \in \mathrm{Per}_{\mathbb{Q}}(\alpha_1),$$

in view of Lemma 5.2.1(b), the result follows if we show that

$$\text{when } g = 1,\ P_{\Delta(X_0)} \text{ is torsion if and only if } \infty - e \text{ is torsion.} \tag{5.13}$$

We first prove a lemma.

Lemma 5.3.2. Let $g = 1$. Then we have:
(a) $h_2(\Delta_{2,e}) \in J((H^1)^{\otimes 3})^\vee$ is torsion.
(b) $\xi^{-1}_{\Delta(X_0)}(h_2(Z^{\infty}_{2,e})) = -2(\infty - e)$.

Proof. (a) This is immediate on recalling that when $g = 1$,

$$6\Delta_{2,e} = 0 \in \mathrm{CH}_1^{\mathrm{hom}}(X^3).$$

(See [17, Corollary 4.7].) Here we proceed to give an alternative argument that shows that in fact, $2h_2(\Delta_{2,e}) = 0$. We will show equivalently that

$$2\Phi(h_2(\Delta_{2,e})) \in \mathrm{Hom}((H^1_{\mathbb{Z}})^{\otimes 3}, \mathbb{R}/\mathbb{Z})$$

is zero. For a permutation $\sigma \in S_3$, denote the map

$$X^3 \longrightarrow X^3 \qquad (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$$

also by σ. It is easy to see that $\sigma_*(\Delta_{2,e}) = \Delta_{2,e}$. Let $\partial^{-1}(\Delta_{2,e})$ be as in Chapter 3, i.e. a chain whose boundary is $\Delta_{2,e}$. Then

$$\partial\sigma_*(\partial^{-1}(\Delta_{2,e})) = \sigma_*\partial(\partial^{-1}(\Delta_{2,e})) = \sigma_*(\Delta_{2,e}) = \Delta_{2,e},$$

so that $\sigma_*\partial^{-1}(\Delta_{2,e})$ can also be used to calculate $\Phi(h_2(\Delta_{2,e}))$.

Let η_1, η_2 be harmonic 1-forms on X with integral periods whose images in cohomology form a basis of $H^1_{\mathbb{Z}}$. Then

$$\begin{aligned}
\int_{\partial^{-1}(\Delta_{2,e})} \eta_{i_1} \otimes \eta_{i_2} \otimes \eta_{i_3} &\overset{\mathbb{Z}}{\equiv} \int_{\sigma_*\partial^{-1}(\Delta_{2,e})} \eta_{i_1} \otimes \eta_{i_2} \otimes \eta_{i_3} \\
&= \int_{\partial^{-1}(\Delta_{2,e})} \sigma^*(\eta_{i_1} \otimes \eta_{i_2} \otimes \eta_{i_3}) \\
&= \mathrm{sgn}(\sigma) \int_{\partial^{-1}(\Delta_{2,e})} \eta_{i_{\sigma(1)}} \otimes \eta_{i_{\sigma(2)}} \otimes \eta_{i_{\sigma(3)}}.
\end{aligned}$$

So far σ was arbitrary. Now given a triple (i_1, i_2, i_3), take σ to be a transposition that fixes the triple. (Such transposition exists because $g = 1$.) Then it follows from the above that

$$\int_{\partial^{-1}(\Delta_{2,e})} \eta_{i_1} \otimes \eta_{i_2} \otimes \eta_{i_3} \in \frac{1}{2}\mathbb{Z}.$$

Thus the image of $\Phi(h_2(\Delta_{2,e})$ lies in $(\frac{1}{2}\mathbb{Z})/\mathbb{Z}$, i.e. $2\Phi(h_2(\Delta_{2,e}) = 0$.

(b) Fix a path γ_e^∞ from e to ∞ in X. Then

$$\xi^{-1}_{\Delta(X_0)}(h_2(Z^\infty_{2,e})) \in \frac{\Omega^1_{\mathrm{hol}}(X)^\vee}{H_1(X,\mathbb{Z})}$$

is represented by the map

$$\alpha \mapsto \int\limits_{\Delta(X)} \xi_{\Delta(X_0)} \int\limits_{\gamma_e^\infty} \alpha.$$

If follows that

$$\xi^{-1}_{\Delta(X_0)}(h_2(Z^\infty_{2,e})) = (\int\limits_{\Delta(X)} \xi_{\Delta(X_0)})\ (\infty - e).$$

A straightforward calculation (say using Lemma 5.3.1) shows the coefficient of $\infty - e$ on the right is -2. □

We are now ready to conclude (5.13), and hence Theorem 5.3.1. Suppose $g = 1$. By Theorem 3.5.1 and (a) of the previous lemma, the element

$$-\Psi(\mathbb{E}^\infty_{2,e}) + h_2(Z^\infty_{2,e}) \in J((H^1)^{\otimes 3})^\vee$$

is torsion. Thus

$$P_{\Delta(X_0)} = \xi^{-1}_{\Delta(X_0)}(\Psi(\mathbb{E}^\infty_{2,e})) \overset{\text{torsion}}{\equiv} \xi^{-1}_{\Delta(X_0)}(h_2(Z^\infty_{2,e})).$$

By (b) of the previous lemma, the element on the right is just $-2(\infty - e)$, establishing (5.13).

Remark. (1) A straightforward computation shows

$$p_{11}\int\limits_{\beta_1}\omega_1\omega_2 + p_{12}\int\limits_{\beta_2}\omega_1\omega_2 = \frac{1}{p_{11}p_{22}-p_{12}p_{21}}\left(p_{11}\int\limits_{\beta_1}\alpha_1\alpha_2 + p_{12}\int\limits_{\beta_2}\alpha_1\alpha_2 - \frac{1}{2}p_{11}p_{12}(p_{21}+p_{22})\right),$$

so that Theorem 5.3.1 asserts that if α_2 has order 2 at ∞, then

$$\frac{1}{p_{11}p_{22}-p_{12}p_{21}}\left(p_{11}\int\limits_{\beta_1}\alpha_1\alpha_2 + p_{12}\int\limits_{\beta_2}\alpha_1\alpha_2 - \frac{1}{2}p_{11}p_{12}(p_{21}+p_{22})\right) \in \mathrm{Per}_{\mathbb{Q}}(\alpha_1) \qquad (5.14)$$

if and only if $\infty - e$ is torsion.
(2) Suppose X_0 is given by the affine equation

$$y^2 = 4x^3 - g_2x - g_3.$$

Let ∞ be the point at infinity. Take $\alpha_1 = \frac{dx}{y}$ and $\alpha_2 = \frac{xdx}{y}$. One then has the *Legendre relation*

$$p_{11}p_{22} - p_{12}p_{21} = 2\pi i.$$

In view of the previous remark,

$$p_{11}\int_{\beta_1}\alpha_1\alpha_2 + p_{12}\int_{\beta_2}\alpha_1\alpha_2 - \frac{1}{2}p_{11}p_{12}(p_{21}+p_{22}) \in 2\pi i \mathrm{Per}_{\mathbb{Q}}(\alpha_1) \tag{5.15}$$

if and only if $e \in X_0(K) - \{\infty\}$ is torsion. Using

$$p_{l1} = -\int_{\beta_2}\alpha_l, \quad p_{l2} = \int_{\beta_1}\alpha_l \quad (l = 1, 2)$$

and the shuffle product property of iterated integrals, (5.15) can be equivalently written as

$$\int_{\beta_1}\alpha_1\int_{\beta_2}(\alpha_1\alpha_2 - \alpha_2\alpha_1) \;-\; \int_{\beta_2}\alpha_1\int_{\beta_1}(\alpha_1\alpha_2 - \alpha_2\alpha_1) \;\in 2\pi i\mathrm{Per}_{\mathbb{Q}}(\alpha_1).$$

(3) Let $\xi \in (H^1)^{\otimes 2}$ be any Hodge class. Suppose P_ξ is torsion. It is easy to see that the relation induced by ξ is just a multiple of (5.12).

5.3.3 Relations coming from the diagonal of X_0^2

So far in this section we considered relations that arise from a Hodge class in $(H^1)^{\otimes 2}$ (namely, the class of the diagonal of X_0), and hence only used $n = 2$ case of Theorem 3.5.1 and Theorem 4.1.1. In fact, we did not even need the full machinery of the former: We only needed (3.23) of Darmon, Rotger, and Sols. Our goal in this paragraph is to provide evidence for that, applying the method of this chapter to Hodge classes in higher tensor powers of H^1, or algebraic cycles in higher powers of X, and hence using the results of the previous chapters in $n > 2$ setting, one may indeed obtain new information about the periods. To this end, we will study the relations that can arise from $\Delta(X_0^2)$, where

$$\Delta(X_0^2) = \{(x_1, x_2, x_1, x_2) : x_i \in X\} \in CH_2(X_0^4)$$

is the diagonal of X_0^2. We will then show that at least in $g = 2$ case, these relations are not the same as the ones arising from $\Delta(X_0)$.

Throughout, for simplicity, we write λ_{ij} for $\lambda_{ij}(\Delta(X_0))$ (calculated in Lemma 5.3.1).

Lemma 5.3.3. Let $\alpha \in \Omega^1_{hol}(X)$. Then for $i, j, k \le 2g$, $i < j$,

$$\mu_{ijk}(\xi_{\Delta(X_0^2)}; \alpha) = \lambda_{jk} p_i(\alpha) - \lambda_{ik} p_j(\alpha) - 2(-1)^{i+j} p_k(\alpha).$$

The proof of this lemma is a fairly long computation. We postpone it to the end of the section.

Suppose the α_i satisfy Hypothesis $\star(3)$ and $P_{\Delta(X_0^2)}$ is torsion. (The latter for instance will automatically hold if Jac(K) is finite, e.g. in the cases as in Corollary 5.3.1.) Then by Proposition 5.2.1,

$$\sum_{\substack{i,j,k \\ i<j}} \left(\lambda_{jk} p_{li} - \lambda_{ik} p_{lj} - 2(-1)^{i+j} p_{lk} \right) \int_{\beta_k} \omega_i \omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) \qquad (l \le g). \tag{5.16}$$

Proposition 5.3.2. The relations (5.16) are independent (as linear relations among (5.8) with coefficients in $\mathbb{Q}(\mathrm{Per}(X_0))$.

Proof. Let A be the matrix formed by the coefficients of

$$\int_{\beta_1} \omega_1 \omega_2, \quad \text{and} \qquad \int_{\beta_j} \omega_1 \omega_j \quad (1 < j \le 2g)$$

in the relations. (In other words, the $l1$-entry of A is the coefficient of $\int_{\beta_1} \omega_1 \omega_2$ in the relation corresponding to α_l, and for $j > 1$, its lj-entry is the coefficient of $\int_{\beta_j} \omega_1 \omega_j$ in the relation corresponding to α_l.) It is enough to show that A has rank g. But this is clear, since one has

$$\mu_{1,2,1}(\Delta(X_0^2); \alpha_l) = 3p_{l1}$$

and

$$\mu_{1,j,j}(\Delta(X_0^2); \alpha_l) = 3(-1)^j p_{lj},$$

so that the j^{th} column of A is ± 3 the j^{th} column of the top half of the period matrix $(p_{ij})_{i,j \le 2g}$. $\square$

Suppose both $P_{\Delta(X_0)}$ and $P_{\Delta(X_0^2)}$ are torsion, and that the α_i satisfy Hypothesis $\star(n)$ for $n = 2, 3$. Then one has two sets of g independent relations as in (5.11) and (5.16). In $g = 1$ case, the two relations are trivially dependent. On the other hand, one has:

Proposition 5.3.3. Let $g = 2$. Among relations (5.11) and (5.16) there are at least 3 (i.e. $g + 1$) independent ones.

Proof. In view of (5.11), we can replace (5.16) by

$$\sum_{\substack{i,j,k \\ i<j}} \Big(\lambda_{jk} p_{li} - \lambda_{ik} p_{lj} \Big) \int_{\beta_k} \omega_i \omega_j \in \mathrm{Per}_{\mathbb{Q}}(\alpha_l) \qquad (l = 1, 2). \tag{5.17}$$

We refer to the relations (5.11) by R_1, R_2 (R_l for the one corresponding to α_l), and to the relations (5.17) by R_1', R_2'. Suppose both $\{R_1, R_2, R_1'\}$ and $\{R_1, R_2, R_2'\}$ are dependent. We claim that for all distinct $i, j, k \leq 4$, $i < j$, and $l \leq 2$,

$$(\lambda_{jk} p_{li} - \lambda_{ik} p_{lj} + \lambda_{ij} p_{lk})(p_{1i} p_{2j} - p_{1j} p_{2i}) = 0. \tag{5.18}$$

Indeed, given i, j, k, l as above, form the 3×3 matrix whose columns are the coefficients of

$$\int_{\beta_i} \omega_i \omega_j, \quad \int_{\beta_j} \omega_i \omega_j, \quad \int_{\beta_k} \omega_i \omega_j$$

in E_1, E_2, E_l'. One easily calculates its determinant to be

$$(\lambda_{jk} p_{li} - \lambda_{ik} p_{lj} + \lambda_{ij} p_{lk})(p_{1i} p_{2j} - p_{1j} p_{2i}),$$

so that the claim follows.

Next, we show that the equations (5.18) contradict the fact that the matrix

$$P := (p_{ij})_{i \leq g, j \leq 2g}$$

has rank g. This will be done in two steps. Let P_j be the j^{th} column of P.

Step 1: Consider the following situations:

$$(i) \det \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \neq 0, \quad (ii) \det \begin{pmatrix} p_{11} & p_{14} \\ p_{21} & p_{24} \end{pmatrix} \neq 0,$$

$$(iii) \det \begin{pmatrix} p_{13} & p_{14} \\ p_{23} & p_{24} \end{pmatrix} \neq 0, \quad (iv) \det \begin{pmatrix} p_{12} & p_{13} \\ p_{22} & p_{23} \end{pmatrix} \neq 0.$$

Suppose (i) holds. Then in view of (5.18),

$$\lambda_{2k}P_1 - \lambda_{1k}P_2 + \lambda_{12}P_k = 0 \quad (k = 3, 4). \tag{5.19}$$

Setting $k = 3, 4$ it follows $P_3 = -P_4$. On the other hand, (5.19) gives $P_3 = -(P_1 + P_2)$, so that

$$P = \begin{pmatrix} p_{11} & p_{12} & -(p_{11} + p_{12}) & p_{11} + p_{12} \\ p_{21} & p_{22} & -(p_{21} + p_{22}) & p_{21} + p_{22} \end{pmatrix}.$$

It follows that (ii) holds.

Similarly, one can check that

- (ii) implies (iii) and that $P_2 = -P_3$,
- (iii) implies (iv) and that $P_1 = -P_2$, and finally
- (iv) implies (i) and that $P_1 = P_4$.

Since P has rank 2, it follows none of $(i) - (iv)$ hold, i.e.

$$\det \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \det \begin{pmatrix} p_{11} & p_{14} \\ p_{21} & p_{24} \end{pmatrix} = \det \begin{pmatrix} p_{13} & p_{14} \\ p_{23} & p_{24} \end{pmatrix} = \det \begin{pmatrix} p_{12} & p_{13} \\ p_{22} & p_{23} \end{pmatrix} = 0.$$

Step 2: Since 3rd and 4th column of P are linearly dependent and P has rank 2, one of the first two columns must be nonzero. We assume the first column is not zero; the other case is similar. By the previous step, P must look like

$$\begin{pmatrix} p_{11} & 0 & p_{13} & 0 \\ p_{21} & 0 & p_{23} & 0 \end{pmatrix}.$$

Indeed, P_2 and P_4 are scalar multiples of P_1, so that $\mathrm{rank}(P) = 2$ forces P_1, P_3 to be linearly independent. Each of P_2, P_4 is a scalar multiple of both P_1 and P_3, and hence is zero. Now taking $(i, j, k) = (1, 3, 2)$ in (5.18) we see $P_1 = -P_3$, contradicting $\mathrm{rank}(P) = 2$. □

We finish by proving Lemma 5.3.3.

Proof of Lemma 5.3.3: For the moment, let $\xi \in (H^1)^{\otimes 4}$ be an arbitrary Hodge class. For simplicity, we write λ_{ijkl} for $\lambda_{ijkl}(\xi)$. Let $\alpha \in \Omega^1_{\mathrm{hol}}(X)$. We will simply write p_j for

$p_j(\alpha)$. One easily sees

$$\mu'_{ijk}(\xi;\alpha) = \sum_{l,m} p_m \lambda_{ijlk} a_{ml} + p_k \lambda_{lijm} a_{ml},$$

where $a_{ml} = \int_{\beta_m} \omega_l$. Thus for $i < j$,

$$\begin{aligned}\mu_{ijk}(\xi;\alpha) &= \sum_{l,m} a_{ml} \left(p_m \lambda_{ijlk} + p_k \lambda_{lijm} - p_m \lambda_{jilk} - p_k \lambda_{ljim} \right) \\ &= \sum_{l,m} a_{ml} \left(p_m \left(\lambda_{ijlk} - \lambda_{jilk}\right) + p_k \left(\lambda_{lijm} - \lambda_{ljim}\right) \right).\end{aligned}$$

In view of $a_{ml} = -a_{lm}$ and $a_{ml} = 1$ if $m < l$, this can be rewritten as

$$\sum_{m<l} \left(p_m \left(\lambda_{ijlk} - \lambda_{jilk}\right) + p_k \left(\lambda_{lijm} - \lambda_{ljim} - \lambda_{mijl} + \lambda_{mjil}\right) + p_l \left(\lambda_{jimk} - \lambda_{ijmk}\right) \right),$$

which can again be rewritten as

$$\sum_m p_m \left(\sum_{l=m+1}^{2g} \left(\lambda_{ijlk} - \lambda_{jilk}\right) + \sum_{l=1}^{m-1} (\lambda_{jilk} - \lambda_{ijlk}) \right) + p_k \sum_{m<l} \left(\lambda_{lijm} - \lambda_{ljim} - \lambda_{mijl} + \lambda_{mjil}\right). \tag{5.20}$$

Now let $\xi = \xi_{\Delta(X_0^2)}$. We will simply write μ_{ijk} for $\mu_{ijk}(\xi;\alpha)$, and continue to write λ_{ij} (resp. λ_{ijkl}) for $\lambda_{ij}(\Delta(X_0)$ (resp. $\lambda_{ijkl}(\Delta(X_0^2))$. Since $\Delta(X_0^2)$ is obtained from $\Delta(X_0) \times \Delta(X_0)$ by switching the 2nd and 3rd coordinates, one has

$$\lambda_{ijkl} = -\lambda_{ik}\lambda_{jk}.$$

In view of $\lambda_{ij} = -\lambda_{ji}$, (5.20) simplifies to

$$\sum_m p_m \left(\sum_{l=m+1}^{2g} \left(\lambda_{ijlk} - \lambda_{jilk}\right) + \sum_{l=1}^{m-1} (\lambda_{jilk} - \lambda_{ijlk}) \right) + 2p_k \sum_{m<l} (\lambda_{lijm} - \lambda_{ljim}),$$

Thus so far we know

$$\mu_{ijk} = \sum_m a_m p_m + 2p_k \sum_{m<l} (\lambda_{lijm} - \lambda_{ljim}),$$

where

$$a_m = \sum_{l=m+1}^{2g} (\lambda_{ijlk} - \lambda_{jilk}) + \sum_{l=1}^{m-1} (\lambda_{jilk} - \lambda_{ijlk})$$
$$= \left(\sum_{l=m+1}^{2g} - \sum_{l=1}^{m-1} \right) (\lambda_{ijlk} - \lambda_{jilk}).$$

Thus we will be done if we show

$$\sum_{m<l} (\lambda_{lijm} - \lambda_{ljim}) = (-1)^{i+j+1} \qquad \text{(for all } i < j) \tag{5.21}$$

and

$$\left(\sum_{l=m+1}^{2g} - \sum_{l=1}^{m-1} \right) \lambda_{ijlk} = \begin{cases} \lambda_{jk} & \text{if } m = i \\ 0 & \text{if } m \neq i. \end{cases} \qquad \text{(for all distinct } i, j)$$

The latter is equivalent to that for all i and m,

$$\left(\sum_{l=m+1}^{2g} - \sum_{l=1}^{m-1} \right) \lambda_{li} = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{if } m \neq i. \end{cases} \tag{5.22}$$

Before we try to verify these, note that for any fixed i and r, one has:

(i) If $r < i$, then

$$\sum_{l \le r} \lambda_{li} = \begin{cases} \lambda_{1i} = \lambda_{ri} & (r \overset{2}{\not\equiv} 0) \\ 0 & (r \overset{2}{\equiv} 0) \end{cases}.$$

(ii) If $r \geq i$, then

$$\sum_{i<l\le r} \lambda_{li} = \begin{cases} \lambda_{(i+1)i} = \lambda_{ri} & (r \overset{2}{\not\equiv} i) \\ 0 & (r \overset{2}{\equiv} i) \end{cases}.$$

For $r \geq i$, writing

$$\sum_{l \le r} \lambda_{li} = \left(\sum_{l \le i-1} + \sum_{i<l\le r} \right) \lambda_{li},$$

we see that for any r, i,

$$\sum_{l \le r} \lambda_{li} = \begin{cases} \lambda_{1i} = (-1)^{i+1} & (r < i, r \overset{2}{\not\equiv} 0) \\ 0 & (r < i, r \overset{2}{\equiv} 0) \\ \lambda_{1i} = (-1)^{i+1} & (r \ge i, r \overset{2}{\equiv} i \overset{2}{\equiv} 0) \\ \lambda_{1i} + \lambda_{ri} = 0 & (r \ge i, r \overset{2}{\not\equiv} i \overset{2}{\equiv} 0) \\ 0 & (r \ge i, r \overset{2}{\equiv} i \overset{2}{\not\equiv} 0) \\ \lambda_{ri} = (-1)^{i+1} & (r \ge i, r \overset{2}{\not\equiv} i \overset{2}{\not\equiv} 0), \end{cases}$$

or in short,

$$\sum_{l \le r} \lambda_{li} = \begin{cases} (-1)^{i+1} & (r < i, r \overset{2}{\not\equiv} 0) \text{ or } (r \ge i, r \overset{2}{\equiv} 0) \\ 0 & (r < i, r \overset{2}{\equiv} 0) \text{ or } (r \ge i, r \overset{2}{\not\equiv} 0). \end{cases} \tag{5.23}$$

Now we verify (5.21) and (5.22). Writing

$$\left(\sum_{l=m+1}^{2g} - \sum_{l=1}^{m-1} \right) \lambda_{li} = -\lambda_{mi} + \left(\sum_{l \le 2g} -2 \sum_{l \le m-1} \right) \lambda_{li},$$

a straightforward computation using (5.23) gives (5.22).

Turning our attention to (5.21), start by breaking the sum as

$$\sum_{m<l} (\lambda_{lijm} - \lambda_{ljim}) = \sum_{m<l} \lambda_{lijm} \; - \; \sum_{m<l} \lambda_{ljim}.$$

We have

$$\begin{aligned} \sum_{m<l} \lambda_{lijm} &= \sum_{l} \lambda_{lj} \sum_{m=1}^{l-1} \lambda_{mi} \\ &= (-1)^j \left(\overbrace{\sum_{l<j}}^{(I)} - \overbrace{\sum_{l>j}}^{(II)} \right) (-1)^l \sum_{m=1}^{l-1} \lambda_{mi}. \end{aligned}$$

Before we proceed any further, it is convenient to use the following notation. Given a subset $S \subset \mathbb{R}$, we denote by $E(S)$ (resp. $O(S)$) the number of even (resp. odd) numbers in S. In view of (5.23),

$$(I) = (-1)^{i+1} \left(E((0, i]) - O((i, j)) \right)$$

and

$$(II) = (-1)^{i} O((j, 2g]).$$

(since $i < j$). Thus

$$\sum_{m<l} \lambda_{lijm} = (-1)^{i+j+1} \left(E((0, i]) - O((i, j)) + O((j, 2g]) \right). \tag{5.24}$$

Similarly,

$$\begin{aligned} \sum_{m<l} \lambda_{ljim} &= \sum_{l} \lambda_{li} \sum_{m=1}^{l-1} \lambda_{mj} \\ &= (-1)^{i} \left(\sum_{l<i} - \sum_{l>i} \right) (-1)^{l} \sum_{m=1}^{l-1} \lambda_{mj}. \end{aligned}$$

In view of (5.23), keeping in mind $i < j$, we get

$$\sum_{m<l} \lambda_{ljim} = (-1)^{i+j+1} \left(E((0, i)) - E((i, j]) + O((j, 2g]) \right). \tag{5.25}$$

Now (5.21) follows from (5.24) and (5.25) on noting that

$$E((0, i]) - O((i, j)) - E((0, i))) + E((i, j]) = E([i, j]) - O((i, j)) = 1.$$

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