Periods in number theory and algebraic geometry

Payman Eskandari

University of Winnipeg

PIMS-UBC, December 2022

1. Periods and period relations I (following Kontsevich and Zagier)

2. Periods and period relations II

3. Philosophy of motives and Grothendieck's period conjecture

4. Joint work with K. Murty

Periods I

Definition (Kontsevich-Zagier)

A period is a complex number whose real and imaginary parts are absolutely convergent integrals of rational functions with rational* coefficients over subsets of* $\mathbb{R}^n$ *given by polynomial inequalities with rational* coefficients.*

Examples:

- Algebraic numbers, e.g. $\sqrt{2} = \int_{x^2-2\leq 0\,,\,x\geq 0} dx$

$$\mathbb{Q} \subset \text{algebraic numbers} \subset \textit{periods} \subset \mathbb{C}$$

- $\pi = \int\limits_{x^2+y^2\leq 1} dxdy$
- logarithms of algebraic numbers, e.g.

$$\log(a) = \int\limits_{1\leq x\leq a} \frac{dx}{x} \quad (a \in \mathbb{Q},\ a \geq 1)$$

- Special values of the Riemann zeta function:
$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \int\limits_{0\leq x_1\leq x_2\leq\cdots\leq x_s\leq 1} \frac{dx_1 dx_2 \cdots dx_s}{(1-x_1)x_2\cdots x_s}$$
- Multiple zeta values:

$$\zeta(s_1, \ldots, s_r) \ := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$$

Periods form a countable set. The number e is not expected to be a period.

Period relations I

Obvious sources of relations between periods:

- linearity properties of integration (in the integrand and the domain of int.)
- change of variables
- Stoke's theorem

Conjecture (Kontsevich-Zagier, 2001)

Any two integral representations (as above) for a period can be transformed to each other by a sequence of the above operations, in which all functions are algebraic and all domains of integration are semi-algebraic with coef. in $\overline{Q}$.

Periods II

$$\int_{\underbrace{|z|=1}_{\text{a basis of } H_1(\mathbb{C}^*,\mathbb{Q})}} \overbrace{\frac{dz}{z}}^{\text{a basis of } H^1_{dR}(\overbrace{\mathbb{C}^*}^{\mathbb{G}_m\,(/\mathbb{Q})})} = \underbrace{2\pi i}_{\text{(by def) a period of } \mathbb{G}_m}$$

$$\int : \underbrace{H_1(\mathbb{C}^*,\mathbb{C})}_{H_1(\mathbb{C}^*,\mathbb{Q})\otimes\mathbb{C}} \otimes \underbrace{H^1_{dR,sm,\mathbb{C}}(\mathbb{C}^*)}_{H^1_{dR}(\mathbb{G}_m)\otimes\mathbb{C}} \xrightarrow{\text{nondeg.}} \mathbb{C} \quad \rightsquigarrow \quad H^1_{dR,sm,\mathbb{C}}(\mathbb{C}^*) \cong H^1(\mathbb{C}^*,\mathbb{C})$$

Restrict integration to

$$\int : H_1(\mathbb{C}^*,\mathbb{Q}) \otimes H^1_{dR}(\mathbb{G}_m) \longrightarrow \mathbb{C} \qquad \text{Image} = 2\pi i \cdot \mathbb{Q}.$$

$$\begin{cases} X \text{ (smooth) variety } /\mathbb{Q} \\ n \in \mathbb{Z}_{\geq 0} \end{cases} \rightsquigarrow \begin{cases} H_n(X^{an}, \mathbb{Q}) = \text{sing. hom. of } X^{an} \\ H^n_{dR}(X) = \text{alg. dR coh.} \\ \text{both vec. spaces } /\mathbb{Q} \text{ with} \\ \int : \underbrace{H_n(X^{an}, \mathbb{Q})}_{\substack{\cap \\ H_n(X^{an}, \mathbb{C})}} \otimes \underbrace{H^n_{dR}(X)}_{\substack{\cap \\ H^n_{dR,sm,\mathbb{C}}(X^{an})}} \to \mathbb{C} \end{cases}$$

Definition (Periods, II)

Periods of $H^n(X)$ are the integrals of elements of $H^n_{dR}(X)$ over elements of $H_n(X^{an}, \mathbb{Q})$. That is, numbers in the image of the integration pairing above (restricted to the rational lattices).

Equivalently, periods of $H^n(X)$ are the numbers than can appear in a change of basis matrix for the period (or de Rham) comparison isomorphism

$$H^n_{dR}(X)\otimes\mathbb{C} \longrightarrow H^n(X^{an},\mathbb{Q})\otimes\mathbb{C}$$

for bases taken in the rational lattices.

Fixing bases for the rational lattices $\rightsquigarrow$ the period matrix (wrt the bases).

- Similar story for the relative groups $H^n(X,A)$.
- When all possibilities included, same set of numbers as with the KZ definition.

Period relations II

Geometry gives rise to period relations.

Example: X = a compact Riemann surface of genus g; $\dim H^1_{dR,sm}(X) = 2g$

- $\omega_1, \ldots, \omega_{2g}$ = a basis of $H^1_{dR}(X)$, $\omega_1, \ldots, \omega_g$ = holomorphic on X
- $\eta_1, \ldots, \eta_{2g}$ = a basis of $H^1_{dR,sm}(X)$ in $H^1(X, \mathbb{Q})$

Write

$$\omega_i = \sum_j p_{ij}\eta_j \quad \text{so} \quad (p_{ij}) = \text{period martix.}$$

Let $Z \subset X \times X$ be a subvariety of dimension 1 (e.g. Z is the graph of a map $X \longrightarrow X$).
For any r, s,

$$\int_Z \omega_r(x_1)\omega_s(x_2) = \sum_{i,j} p_{ri}p_{sj} \overbrace{\int_Z \eta_i(x_1)\eta_j(x_2)}^{\text{rational number}}$$

For each $r, s \leq g$, we get a period relation

$$\sum_{i,j} [\int_Z \eta_i(x_1)\eta_j(x_2)]\, p_{ri}p_{sj} = 0\,.$$

Grothendieck's period conjecture, rough version

All period relations should come from geometry.

Grothendieck's period conjecture

- The first mention of a possible conjecture is in a footnote of a letter from Grothendieck to Atiyah in 1966. In the letter he only alludes to the existence of a conjecture.
- Grothendieck formulated a precise conjecture using the language of motives and motivic Galois groups, also ideas of his ("Grothendieck's dream'').
- At the time there was no formal theory of motives, but nonetheless Grothendieck gave a precise formulation of the conjecture in this language (his unpublished notes from mid-late 60's, now available).

$$\begin{cases} X \text{ var.}/\mathbb{Q} \\ n \in \mathbb{Z} \end{cases} \rightsquigarrow \overbrace{\begin{cases} H^n(X^{an},\mathbb{Q}) =: H^n_B(X) \ + \ \text{extra str. (MHS)} \\ H^n_{dR}(X) \ + \ \text{extra str. (wt \& Hod. fil.)} \\ H^n_\ell(X) \ + \ \text{extra str. } (Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ action}) \\ + \text{ comparison iso.'s} \\ \ldots \end{cases}}^{\text{a system of realization}}$$

Have a functor

$$\text{Varieties}/\mathbb{Q} \longrightarrow \text{Realizations.}$$

Grothendieck's idea: There should be an intermediate category $\text{Mot}_\mathbb{Q}$ of motives over $\mathbb{Q}$, which is "linear" but yet of geometric nature.

$$\text{Varieties}/\mathbb{Q} \xrightarrow{H^n(-)} \text{Mot}_\mathbb{Q} \xrightarrow{\text{realization functor}} \text{Realizations}$$

- "linear" = Tannakian (i.e. having properties of the category of f.d. lin. representations of a group)
- Grothendieck's own attempt: Case of smooth projective (semisimple/pure case). Idea: Use correspondences (algebraic cycles in $X \times Y$) as morphisms from mot. of X to mot. of Y; get a nice category assuming the Standard Conjectures
- Grothendieck already thought the picture should also hold in the non-projective case (mixed case)
- $H^n(X)$ for sm. proj. X of weight n (pure); in general a wt. filt. (mixed)
- Morphisms in $\mathrm{Mot}_{\mathbb{Q}}$ should have a geometric nature.
- Other mentions: Serre, Deligne, Jannsen, André, Levine, Voevodsky, until:
- Now we finally have genuinely geometric candidates for the category of (mixed) motives, due to Nori (2000's) and Ayoub (2014). Their constructions are different, but the categories are equivalent.)

Grothendieck's period conjecture

$$M \in \mathrm{Mot}_{\mathbb{Q}} \overset{\text{Tan. formalism}}{\rightsquigarrow} \begin{array}{l} G^{mot}(M) \\ \text{the motivic Galois group of } M \\ \text{a subgr. of } GL(M_B)\ (/\mathbb{Q})) \end{array}$$

Grothendieck's period conjecture (GPC), precise version*

For any motive M over $\mathbb{Q}$ (e.g. $H^n(X)$),

$$\text{tr. deg } \mathbb{Q}(\text{periods of } M) = \dim\, G^{mot}(M).$$

- The $\leq$ assertion is known (and is not hard).
- $\mathbb{Q}$ can be replaced with $\overline{\mathbb{Q}}$.
- Many conjectures in tr. number theory are special cases of GPC.

- Part of a web of conjectures about motives and their realizations (various formulations of Hodge, Mumford-Tate)
- KZ conj. $\iff$ GPC (as stated here) + ... (Kontsevich, Huber - Müller-Stach)
- There is a generalized version due to André for over arb. subfields of $\mathbb{C}$.

Examples:

- $M = H^1(\mathbb{G}_m) = H^2(\mathbb{P}^1) =: \mathbb{Q}(-1)$: $G^{mot} = GL_1$; GPC for M $\iff$ transcendence of π. (Known, a classical theorem of Lindemann.)
- $M = H^1$ of an elliptic curve E: Then

$$\dim(G^{mot}) = \begin{cases} 2 & \text{CM case} \\ 4 & \text{non-CM} \end{cases}$$

GPC for M $\iff$ tr. deg. of the field gen. by periods is 2 (resp. 4) in the CM (resp. non-CM) case.
The CM case is known (Chudnovsky); the non-CM case is open.

Mixed examples

- Motives of logarithm values (Kummer motives, extentions of $\mathbb{1}$ by $\mathbb{Q}(1)$): Let $a \in \mathbb{Q}_{>0} \setminus \{1\}$. There is a motive with period matrix $\begin{pmatrix} (2\pi i)^{-1} & (2\pi i)^{-1}\log(a) \\ 0 & 1 \end{pmatrix}$ and motivic Galois group $\{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\} \subset GL_2$.
  In this case, GPC $\iff$ $\{\pi, \log(a)\}$ is algebraically independent. (Open.)
- (Deligne) Motives of ζ values (ext. of $\mathbb{1}$ by $\mathbb{Q}(n)$, $n > 1$ odd): Let $n > 1$ be odd. There is a motive with period matrix $\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n) \\ 0 & 1 \end{pmatrix}$ and the same motivic Galois group as in the previous example. In this case,

$$\text{GPC} \iff \{\zeta(n), \pi\} \text{ is algebraically independent. (Open.)}$$

Report on recent work on unipotent radicals (joint with K. Murty)

- M = a mixed motive; M_B Betti realization; $G^{mot}(M) \subset GL(M_B)$
- Weight filtration:
  $0 = W_{-N \ll 0}M \subset \cdots \subset W_{n-1}M \subset W_nM \subset \cdots \subset W_{N \gg 0}M = M$
  $gr(M) = \bigoplus gr_n(M)$, $gr_n^W(M) = W_nM/W_{n-1}M$; $gr(M)$ semisimple
- $G^{mot}(M) \subset$ subgr. of $GL(M_B)$ preserving $W_\bullet$
  $U^{mot}(M) :=$ subgr. of $G^{mot}(M)$ act. trivially on $gr(M_B)$
- $U^{mot}(M) \leftrightarrow$ extension data in the cat. gen. by M; studied earlier by Deligne, Bertrand, Bertolin, Hardouin, Jossen, etc.
- $P^{-1}(M) :=$ subgr. of $GL(M_B)$ resp. $W_\bullet$ and act. trivially on $gr(M_B)$

Definition

$U^{mot}(M)$ is maximal if it is equal to $P^{-1}(M)$.

- Why interesting?

$$1 \longrightarrow U^{mot}(M) \longrightarrow G^{mot}(M) \longrightarrow G^{mot}(gr(M)) \longrightarrow 1$$

GPC $\Rightarrow$ Periods of motives with max. U^{mot} have max. tr. deg. between all motives with the same gr.

- Idea: Form larger motives with max. U^{mot} from smaller ones.

Question

Suppose L has max. U^{mot} and $0 \longrightarrow L \longrightarrow M \longrightarrow \mathbb{1} \longrightarrow 0$ is totally nonsplit. Does is follows that M has a max. U^{mot}?

- Ans: Yes if L is semisimple (e.g. pure); this is a theorem of Hardouin 2006 building on Bertrand 2001.
  But no for non-semisimple L!

Theorem, rough statement (E. - K. Murty)

Suppose

$$0 \longrightarrow W_pM \longrightarrow M \longrightarrow \mathbb{1} \longrightarrow 0 \qquad (p < 0).$$

There are conditions that guarantee that if W_pM and $M/W_{p-1}M$ both have max. U^{mot}, then so does M.

- Special case: Back to the previous Question, if L has neg. weights $-n < -m$ with $n \neq 2m$, the answer is yes.
- This theorem doesn't say if W_pM and $M/W_{p-1}M$ can be "patched together". There is a nice homological criterion for that.
- The theorem + ideas from hom. algebra $\rightsquigarrow$ homological classification results for motives with max. U^{mot}.
  For example, get a homological classification of all motives with ass. graded iso. to $\mathbb{Q}(n) \oplus A \oplus \mathbb{1}$ with A pure of weight $-p < 0$ with $-2n < -p < 0$, $n \neq p$.

Example

- n odd >1, $1 \neq a \in \mathbb{Q}_{>0}$: There is a unique motive M over $\mathbb{Q}$ with

$$gr(M) = \mathbb{Q}(n+1) \oplus \mathbb{Q}(1) \oplus \mathbb{1},$$

$$W_{-2}M \text{ "=" motive of } \zeta(n)\ , \quad M/W_{-3} = \text{motive of } \log(a).$$

Moreover, M has a max. U^{mot} and a period matrix

$$\begin{pmatrix} (2\pi i)^{-(n+1)} & (2\pi i)^{-(n+1)}\zeta(n) & ? \\ 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log(a) \\ 0 & 0 & 1 \end{pmatrix}.$$

$$U^{mot}(M) \text{ max.} \quad \Rightarrow \quad G^{mot}(M) = \{ \begin{pmatrix} a^{n+1} & * & * \\ 0 & a & * \\ 0 & 0 & 1 \end{pmatrix} \} \subset GL_3.$$

$$\text{GPC for } M \quad \Longleftrightarrow \quad \{\pi, \log(a), \zeta(n), ?\} \text{ alg. indep.}$$

Problem (open)

Find the missing period. (Rem: M is a mixed Tate motive over $\mathbb{Z}[1/a]$.)

Thank you!

Deligne's characterization of $U^{mot}(M)$

- $\mathfrak{u}^{mot}(M) =$ Lie algebra of $U^{mot}(M)$; by Tan. formalism a submotive of $W_{-1}\underline{\mathrm{End}}(M)$.
- Let $\mathcal{E}_n(M)$ be the extension

$$0 \longrightarrow W_n M \longrightarrow M \longrightarrow M/W_n M \longrightarrow 0.$$

- Deligne's result in slogans: "$\sum_n \mathcal{E}_n(M) \rightsquigarrow \mathfrak{u}^{mot}(M)$".
-

$$\begin{aligned} \mathcal{E}_n(M) &\in Ext^1(M/W_n M, W_n M) \\ &\cong Ext^1(\mathbb{1}, \underline{\mathrm{Hom}}(M/W_n M, W_n M)) \\ &\hookrightarrow Ext^1(\mathbb{1}, W_{-1}\underline{\mathrm{End}}(M)) \end{aligned}$$

Theorem (Deligne, 2014)

$\mathfrak{u}^{mot}(M)$ is the smallest submotive of $W_{-1}\underline{\mathrm{End}}(M)$ such that $(\sum_n \mathcal{E}_n(M))/\mathfrak{u}^{mot}(M)$ splits.

A refinement of Deligne's result

Let

$$J_1^n := \{(i,j) \in \mathbb{Z}^2 : i \leq n < j\}$$
$$J_2^n := \{(i,j) \in \mathbb{Z}^2 : i < j\} \setminus J_1^n .$$

Theorem (E. - K. Murty)

Fix n. Suppose $gr^W M$ is semisimple and that M satisfies the following "independence axiom": the two objects

$$\bigoplus_{(i,j)\in J_1^n} \underline{Hom}(gr_j^W M, gr_i^W M) \quad \text{and} \quad \bigoplus_{(i,j)\in J_2^n} \underline{Hom}(gr_j^W M, gr_i^W M)$$

have no nonzero isomorphic subobjects.
Then

$$\mathcal{E}_n(M)/\mathfrak{u}^{mot}(M)$$

splits.

Theorem (E. - K. Murty)

Let $n < 0$ and M be a motive such that

$$M/W_nM \simeq \mathbb{1}, \quad Gr_n^W M \neq 0$$

(so that in particular, 0 and n are the highest two weights of M). Suppose moreover that:

(i) $U^{mot}(W_nM)$ *is maximal,*

(ii) $U^{mot}(M/W_{n-1}M)$ *is maximal, and*

(iii) M *satisfies the same independence axiom as the previous theorem.*

Then $U^{mot}(M)$ is large.

Theorem (E. - K. Murty)

Let $-2k < p < 0$ and $p \neq -k$. Let A be a nonzero simple motive of weight p. Suppose moreover that $Ext^1(\mathbb{1}, \mathbb{Q}(k)) = 0$. Then there is a bijection

$$\left\{\begin{array}{c}\text{the collection of objects } M\\ \text{with } gr^W M \simeq \mathbb{Q}(k) \oplus A \oplus \mathbb{1}\\ \text{and max. } U^{mot}(M)\text{, up to}\\ \text{isomorphism}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{the collection of compatible}\\ \text{pairs of nonsplit extentions in}\\ Ext^1(A, \mathbb{Q}(k)) \times Ext^1(\mathbb{1}, A),\\ \text{up to equivalence,}\end{array}\right\}$$

which assigns to the isomorphism class of an object M the equivalence class of the compatible pairs to which M is attached. If we omit the condition $Ext^1(\mathbb{1}, \mathbb{Q}(k)) = 0$, this map is well-defined and surjective.