ON THE HARMONIC VOLUME OF FERMAT CURVES

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ABSTRACT. We prove that B. Harris' harmonic volume of the Fermat curve of degree n is of infinite order if n has a prime divisor greater than 7. The statement is equivalent to the statement that the Griffiths' Abel-Jacobi image of the Ceresa cycle of such a curve is of infinite order for every choice of base point. In particular, these cycles are of infinite order modulo rational equivalence.

1. INTRODUCTION

Let X be a smooth projective curve over $\mathbb{C}$. The harmonic volume of X, originally defined and studied by B. Harris in [Ha83a], is an invariant of the curve that is closely related to its Ceresa cycles. Let $H^1_{\mathbb{Z}} = H^1(X,\mathbb{Z})$ and denote the kernel of the cup product

$$H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}} \longrightarrow H^2(X,\mathbb{Z})$$

by $(H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}})'$. Given $e \in X$, the pointed harmonic volume of X with base point e is a homomorphism

$$\mathcal{I}_e : H^1_{\mathbb{Z}} \otimes (H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}})' \longrightarrow \mathbb{R}/\mathbb{Z}$$

defined using Chen's iterated integrals as follows: given harmonic 1-forms η_1, η_2, η_3 on X with integral periods and such that $\eta_2 \wedge \eta_3$ is exact, set

$$\mathcal{I}_e([\eta_1] \otimes [\eta_2] \otimes [\eta_3]) = \int_{\gamma_1} \eta_2\eta_3 + \nu \quad (\mathrm{mod}\ \mathbb{Z}), \tag{1}$$

where γ_1 is a loop based at e whose homology class is Poincaré dual to $[\eta_1]$, and ν is a smooth form on X satisfying the equation $\eta_2 \wedge \eta_3 + d\nu = 0$, living in the space $\mathcal{H}^\perp$ of differential forms orthogonal to harmonic forms with respect to the inner product defined using the Hodge star operator.

Let Jac be the Jacobian of X. Let X_e be the image of X under the Albanese map $X \longrightarrow Jac$ with base point e and $X_e^- = (-1)_* X_e$. Harris showed that (after suitable identifications, see Section 4.3) the pointed harmonic volume is related to the Ceresa cycle $X_e - X_e^-$ on the Jacobian via the formula

$$2\mathcal{I}_e = AJ(X_e - X_e^-),$$

where AJ is (Griffiths') Abel-Jacobi map. Later Pulte [Pu88] further studied the pointed harmonic volume and reinterpreted the work of Harris in the framework of Hain's mixed Hodge structure on the space of quadratic iterated integrals on the curve X.

The pointed harmonic volume depends on the choice of the base point e. However, Harris showed in [Ha83a] that a certain restriction of $\mathcal{I}_e$, namely its restriction to the preimage of the primitive cohomology under the natural surjection

$$H^1_{\mathbb{Z}} \otimes (H^1_{\mathbb{Z}} \otimes H^1_{\mathbb{Z}})' \longrightarrow \bigwedge^3 H^1_{\mathbb{Z}} \cong H^3(Jac,\mathbb{Z}),$$

is indeed independent of the base point. This restriction is called the harmonic volume of X.

The connection between the harmonic volume and Ceresa cycles was exploited in [Ha83b] to prove that the Ceresa cycle of the Fermat curve of degree 4 (and hence a multiple of 4) is algebraically nontrivial. This was done by approximating of a certain value of the harmonic volume,

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showing that it was not integral. The calculation also showed that the harmonic volume of such a Fermat curve must have a large order, if finite at all. The purpose of this note is to combine results of Pulte [Pu88], Kaenders [Ka01], and Darmon, Rotger, and Sols [DRS12] on the geometry and arithmetic of the mixed Hodge structure on the space of quadratic iterated integrals on a curve together with a theorem of Gross and Rohrlich [GR78] to show the following:

Theorem 1. Let n be an integer divisible by a prime greater than 7. Then the harmonic volume of the Fermat curve $F(n)$ of degree n has infinite order.

The statement is equivalent to the statement that for a Fermat curve as above, for every choice of base point, the image of the Ceresa cycle under the Abel-Jacobi map is of infinite order.

A few words on some other prior related work are in order. A detailed account of Harris' proof of algebraic nontriviality of the Ceresa cycle of $F(4)$ can be found in [Ha04] (among other things). Shortly after Harris' original proof, Bloch used the étale Abel-Jacobi map in [Bl84] to show that the Ceresa cycle of $F(4)$ is of infinite order modulo algebraic equivalence. More recently, using Harris' method, Otsubo [Ot12] and Tadokoro (see [Ta16] and references therein) have obtained some other results on nontriviality of the Ceresa cycles of Fermat curves and their quotients modulo algebraic equivalence. Roughly speaking, in these works, one finds a sufficient condition for nontriviality with respect to algebraic equivalence in terms of non-integrality of certain values of the harmonic volume. These values can be expressed in term of special values of generalized hypergeometric functions, and their non-integrality can be easily verified numerically for an explicit given degree (Otsubo verifies this for $F(n)$ with $n \leq 1000$). One also obtains sufficient conditions for the Ceresa cycle to be of infinite order modulo algebraic equivalence, but these conditions cannot be verified numerically and hence are harder to use. While Theorem 1 does not say anything about whether the Ceresa cycles of the Fermat curves in question are nontrivial modulo algebraic equivalence, it does imply that they are of infinite order modulo rational equivalence (i.e. in the Chow group). To our knowledge, this statement was not known until now for many of the curves in question. The key additional ingredient is the realization that Gross-Rohrlich's points of infinite order on the Jacobians of Fermat curves of prime degree > 7 can be obtained from Darmon-Rotger-Sols' construction of rational points on the Jacobian using the mixed Hodge structure on the space of quadratic iterated integrals on a punctured curve.

Below, we first fix some notation. Then we briefly recall some facts and make some observations about extensions of mixed Hodge structures. In Section 4 we recall some results regarding an extension of mixed Hodge structures which arises from the space of quadratic iterated integrals on a curve, and set the stage for the proof of Theorem 1. We will then complete the proof in the final section.

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2. Notation

All integral mixed Hodge structures are assumed to have free underlying $\mathbb{Z}$-modules. We denote the category of integral (resp. rational) mixed Hodge structures by $MHS(\mathbb{Z})$ (resp. $MHS(\mathbb{Q})$). With the exception of $\mathbb{Z}(n)$ and $\mathbb{Q}(n)$ (which are, respectively, the integral and rational Hodge structures of weight $-2n$ on $\mathbb{Z}$ and $\mathbb{Q}$), we use the same notation for an object of $MHS(\mathbb{Z})$ and its forgetful image in $MHS(\mathbb{Q})$. For any integral (resp. rational) mixed Hodge structure H, as usual, $H(n)$ denotes $H \otimes \mathbb{Z}(n)$ (resp. $H \otimes \mathbb{Q}(n)$) and $H^\vee$ denotes the dual of H. By $Ext_{MHS(\mathbb{Z})}$ and $Ext_{MHS(\mathbb{Q})}$ we mean the Yoneda extension (Ext^1) groups in the subscript categories. If the category is clear from the context, we may drop the subscript from the notation. Given an integral Hodge structure H, we denote the underlying $\mathbb{Z}$-module (resp. rational vector space, etc.) by $H_\mathbb{Z}$ (resp. $H_\mathbb{Q}$, etc.). Given a pure integral Hodge structure H of odd weight $2k-1$, we denote by JH

the intermediate Jacobian

$$JH := \frac{H_{\mathbb{C}}}{F^k H_{\mathbb{C}} + H_{\mathbb{Z}}}$$

(this is a compact complex torus). Given an integral or rational pure Hodge structure of weight $2k-1$, we denote

$$J_{\mathbb{Q}}H := \frac{H_{\mathbb{C}}}{F^k H_{\mathbb{C}} + H_{\mathbb{Q}}}.$$

For H integral, this is JH modded out by its torsion subgroup.

Given any smooth complex variety X, by $H^i(X)$ we mean the mixed Hodge structure on the degree i cohomology of X. When X is projective, the Chow group of i-dimensional algebraic cycles on X (modulo rational equivalence) is denoted by $CH_i(X)$. We write $CH_i^{\hom}(X)$ for the homologically trivial subgroup of $CH_i(X)$. One has the (Griffith's) Abel-Jacobi map

$$CH_i^{\hom}(X) \longrightarrow J(H^{2i+1}(X)^\vee)$$

defined by $Z \mapsto \int_C$, where C is an integral topological chain whose boundary is Z. (See [Vo02], for instance. Note that throughout the paper, Abel-Jacobi maps are all "homological", that is, without applying Poincaré duality to go from $J(H^{2i+1}(X)^\vee)$ to $JH^{2\dim(X)-2i-1}(X)$.)

3. Remarks on extensions of mixed Hodge structures

Let us recall a few facts about extensions of mixed Hodge structures and make some easy observations. Let A be a pure integral Hodge structure of weight 1.

3.1. There are canonical isomorphisms (functorial in A)

$$Ext(A, \mathbb{Z}(0)) \cong JA^\vee \cong Hom(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}). \tag{2}$$

The first isomorphism is due to Carlson ([Ca80]); given an extension class $\mathbb{E}$ given by

$$0 \longrightarrow \mathbb{Z}(0) \longrightarrow E \longrightarrow A \longrightarrow 0,$$

the element of $JA^\vee$ corresponding to $\mathbb{E}$ is the class of $r \circ s$, where s is a section of the surjection $E_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ compatible with the Hodge filtrations, and r is a retraction of the injection $\mathbb{C} = \mathbb{Z}(0)_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ defined over $\mathbb{Z}$ (i.e. sending $E_{\mathbb{Z}}$ to $\mathbb{Z}$). As for the second isomorphism in (2), it is easy to see that any element of $JA^\vee$ has a representative defined over $\mathbb{R}$ (i.e. that restricts to a map in $A_{\mathbb{R}}{}^\vee$). One simply sends $[f]$ for a functional $f \in A_{\mathbb{C}}{}^\vee$ defined over $\mathbb{R}$, to the restriction of f to $A_{\mathbb{Z}}$ composed with the quotient map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. (See Lemma 3.5 of [Pu88].)

Similarly, there are canonical isomorphisms

$$Ext(A, \mathbb{Q}(0)) \cong J_{\mathbb{Q}}A^\vee \cong Hom(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Q}) \tag{3}$$

(the Ext group now in $MHS(\mathbb{Q})$). The constructions and arguments are similar to the ones in the integral case.

3.2. One has a commutative diagram

$$\begin{array}{ccccc} Ext(A, \mathbb{Z}(0)) & \cong & JA^\vee & \cong & Hom(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ Ext(A, \mathbb{Q}(0)) & \cong & J_{\mathbb{Q}}A^\vee & \cong & Hom(A_{\mathbb{Z}}, \mathbb{R}/\mathbb{Q}), \end{array} \tag{4}$$

where the horizontal isomorphisms are as in (2) and (3), and the vertical maps are the obvious ones: the map on the left is given by the forgetful functor $MHS(\mathbb{Z}) \longrightarrow MHS(\mathbb{Q})$, the map in the middle is the quotient map, and the one on the right is composition with the quotient map $\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Q}$. Note that the vertical maps on the right have kernels equal to the torsion subgroups of their domains. It follows that the same is true for the vertical map on the left. In particular, an extension in $Ext(A,\mathbb{Z}(0))$ is torsion if and only if its image in $Ext(A,\mathbb{Q}(0))$ is zero (i.e. it splits as an extension of rational mixed Hodge structures).

4. Algebraic cycles and Hodge theory of quadratic iterated integrals on a curve

The goal of this section is to set the stage for the proof of Theorem 1 and review certain results from [Ha83a], [Pu88], [Ka01], and [DRS12]. Since the discussion of this section is not specific to Fermat curves, we shall work with a general smooth projective curve of positive genus.

4.1. Let X be a smooth complex projective curve of genus $g>0$, and $\infty, e\in X(\mathbb{C})$ distinct points. Set $U=X-\{\infty\}$. We shall identify $H^1(X)$ and $H^1(U)$ via the map induced by the inclusion $U\subset X$, and simply write H^1 for them. Let $L_2(X,e)$ (resp. $L_2(U,e)$) be Hain's mixed Hodge structure with the underlying $\mathbb{Z}$-module

$$\left(\frac{I}{I^3}\right)^{\vee},$$

where I is the kernel of the map $\mathbb{Z}[\pi_1(X,e)]\longrightarrow\mathbb{Z}$ (resp. $\mathbb{Z}[\pi_1(U,e)]\longrightarrow\mathbb{Z}$) sending elements of the fundamental group to 1. We refer the reader to [Pu88], [Ka01], or [DRS12] for a detailed discussion of $L_2(X,e)$ and $L_2(U,e)$. The general construction of the mixed Hodge structure on the fundamental group ring of a smooth complex variety modded out by powers of the augmentation ideal (or its dual) can be found in [Ha87]. Here we shall take the mixed Hodge structures $L_2(X,e)$ and $L_2(U,e)$ as black boxes, recalling only certain facts about them that are necessary to give the proof of Theorem 1.

Let $(H^1\otimes H^1)'$ denote the kernel of the cup product map

$$H^1\otimes H^1\longrightarrow H^2(X).$$

One has a commutative diagram of mixed Hodge structures with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & L_2(X,e) & \longrightarrow & (H^1\otimes H^1)' & \longrightarrow & 0\\ & & \parallel & & \cap & & \cap & & \\ 0 & \longrightarrow & H^1 & \longrightarrow & L_2(U,e) & \longrightarrow & H^1\otimes H^1 & \longrightarrow & 0. \end{array} \tag{5}$$

(See [Pu88] and [DRS12], for instance.) The middle vertical inclusion is induced by the inclusion $U\subset X$, and the injective horizontal arrows are the duals of the natural surjections

$$\frac{I}{I^3}\longrightarrow\frac{I}{I^2}\cong(H^1_{\mathbb{Z}})^{\vee}.$$

The bottom surjective horizontal arrow is the map

$$L_2(U,e)\longrightarrow H^1\otimes H^1$$

which sends $f:I\longrightarrow\mathbb{Z}$ vanishing on I^3 (where I is the augmentation ideal for U) to the element of

$$H^1_{\mathbb{Z}}\otimes H^1_{\mathbb{Z}}\cong\big(\,H_1(U,\mathbb{Z})\otimes H_1(U,\mathbb{Z})\,\big)^{\vee}$$

given by

$$[\gamma_1]\otimes[\gamma_2]\ \mapsto\ f((\gamma_1-1)(\gamma_2-1)).$$

(Here γ_i is an element of $\pi_1(U,e)$. This map is indeed well-defined.)

We refer to the extension classes given by the sequences in the top and bottom rows of (5) respectively by $\mathbb{E}_e$ and $\mathbb{E}_e^{\infty}$. Thus $\mathbb{E}_e$ is an element of

$$Ext_{MHS(\mathbb{Z})}((H^1\otimes H^1)',H^1)\cong Ext((H^1)^{\vee}\otimes(H^1\otimes H^1)',\mathbb{Z}(0))\overset{(*)}{\cong}Ext(H^1(1)\otimes(H^1\otimes H^1)',\mathbb{Z}(0))$$

and $\mathbb{E}_e^{\infty}$ is an element of

$$Ext_{MHS(\mathbb{Z})}(H^1\otimes H^1,H^1)\cong Ext((H^1)^{\vee}\otimes H^1\otimes H^1,\mathbb{Z}(0))\overset{(*)}{\cong}Ext(H^1(1)\otimes H^1\otimes H^1,\mathbb{Z}(0)),$$

where the isomorphisms $(*)$ are induced by Poincare duality $H^1(1) \xrightarrow{\simeq} (H^1)^\vee$ defined by $[\omega] \mapsto \int_X \omega \wedge -$ (with ω a closed smooth 1-form on X). It is clear that $\mathbb{E}_e$ is the restriction of $\mathbb{E}_e^\infty$ to $(H^1 \otimes H^1)' \subset H^1 \otimes H^1$ (i.e. the pullback of $\mathbb{E}_e^\infty$ along the inclusion map $(H^1 \otimes H^1)' \longrightarrow H^1 \otimes H^1$).

4.2. The Hodge structure $(H^1)^{\otimes 3}$ can be decomposed in the category of rational Hodge structures as follows. Let ξ_Δ be the $H^1 \otimes H^1$ Künneth component of the cohomology class of the diagonal of X (in $CH_1(X \times X)$). Then

$$H^1 \otimes H^1 = \mathbb{Q}\xi_\Delta \ \oplus \ (H^1 \otimes H^1)',$$

so that

$$H^1 \otimes H^1 \otimes H^1 = H^1 \otimes \xi_\Delta \ \oplus \ H^1 \otimes (H^1 \otimes H^1)'.$$

To decompose the second summand, consider the obvious map

$$\phi : H^1 \otimes (H^1 \otimes H^1)' \longrightarrow \bigwedge^3 H^1 \tag{6}$$

(obtained by restricting the natural quotient map $(H^1)^{\otimes 3} \longrightarrow \bigwedge^3 H^1$). This is a surjective morphism of Hodge structures (see Lemma 4.7 of [Pu88] for surjectivity). Since the Mumford-Tate group of H^1 is reductive, ϕ admits a section σ in the category of rational Hodge structures, and we have

$$H^1 \otimes (H^1 \otimes H^1)' = \ker(\phi) \ \oplus \ \sigma(\bigwedge^3 H^1)$$

(the second summand being a copy of $\bigwedge^3 H^1$). Finally, identify the cohomology of the Jacobian variety Jac with the exterior algebra on H^1. Let $\overline{\xi}_\Delta$ be the image of ξ_Δ in $\bigwedge^2 H^1$. Then $\overline{\xi}_\Delta$ is an integral Kähler class of Jac, giving rise to the Lefschetz decomposition on $H^3(Jac) = \bigwedge^3 H^1$:

$$\bigwedge^3 H^1 \ = \ H^1 \wedge \overline{\xi}_\Delta \oplus \ \left(\bigwedge^3 H^1 \right)_{prim},$$

where the second summand (the primitive part) is the kernel of the map $\bigwedge^3 H^1 \longrightarrow \bigwedge^{2g-1} H^1$ given by wedging by $\overline{\xi}_\Delta^{g-2}$. To summarize, we have

$$H^1 \otimes H^1 \otimes H^1 = H^1 \otimes \xi_\Delta \ \oplus \ \underbrace{\ker(\phi) \ \oplus \ \sigma(H^1 \wedge \overline{\xi}_\Delta) \ \oplus \ \sigma(\left(\bigwedge^3 H^1 \right)_{prim})}_{H^1 \otimes (H^1 \otimes H^1)'} \tag{7}$$

in the category $MHS(\mathbb{Q})$.

4.3. We now consider the restrictions of $\mathbb{E}_e^\infty$ to each summand in (7) (twisted by $\mathbb{Q}(1)$). These restrictions have been studied in [Ha83a], [Pu88], and [Ka01]. We shall recall the results below.

4.3.1. Identify $H^1(1) \otimes \xi_\Delta$ with H^1 via the isomorphism $H^1 \longrightarrow H^1 \otimes \xi_\Delta(1)$ sending $\omega \mapsto \omega \otimes \xi_\Delta$. We have

$$Ext(H^1, \mathbb{Q}(0)) \cong J_{\mathbb{Q}}(H^1)^\vee \cong J(H^1)^\vee \ / \ \text{torsion} \ \overset{\text{Abel-Jacobi}}{\cong} \ CH_0^{\text{hom}}(X) \ / \ \text{torsion}. \tag{8}$$

By Theorem 1.2 of [Ka01] (and in view of the left square in (4)), under the identifications above, the restriction of $\mathbb{E}_e^\infty$ to $H^1(1) \otimes \xi_\Delta$ is $-2g\infty + 2e + K$ mod torsion, where K is the canonical divisor of X. (Note: The restriction of $\mathbb{E}_e^\infty$ to $H^1(1) \otimes \xi_\Delta$ is $-k_{e\infty}$ in the notation of Kaenders [Ka01]. Also note that he considers this restriction in the category $MHS(\mathbb{Z})$.)

4.3.2. The morphism ϕ of integral Hodge structures (see (6)) gives an injection

$$J\ (\ (\bigwedge^3 H^1)(1)\ )^\vee \xrightarrow{\phi^*} J(H^1\otimes(H^1\otimes H^1)'(1))^\vee \ \cong\ Ext(H^1\otimes(H^1\otimes H^1)'(1),\mathbb{Z}(0)). \tag{9}$$

Let

$$AJ : CH_1^{\text{hom}}(Jac) \longrightarrow JH^3(Jac)^\vee$$

be the Abel-Jacobi map. Denote the image of the map $X \longrightarrow Jac$ given by $x \mapsto x-e$ by X_e, and as usual, let $X_e^- := (-1)_* X_e$. Then a theorem of Harris and Pulte (Theorem 4.10 of [Pu88], also see Section 3 of [Ha83a]) asserts that

$$\phi^* \circ AJ(X_e - X_e^-) = 2\mathbb{E}_e. \tag{10}$$

(Note that for a pure integral Hodge structure A of odd weight, $JA = J(A(1))$.) The exact sequence

$$0 \longrightarrow \ker(\phi)(1) \longrightarrow H^1\otimes(H^1\otimes H^1)'(1) \xrightarrow{\phi} (\bigwedge^3 H^1)(1) \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow Ext((\bigwedge^3 H^1)(1),\mathbb{Z}(0)) \longrightarrow Ext(H^1\otimes(H^1\otimes H^1)'(1),\mathbb{Z}(0)) \longrightarrow Ext(\ker(\phi)(1),\mathbb{Z}(0)) \longrightarrow 0.$$

By the theorem of Harris and Pulte (and functoriality of Carlson's isomorphism), we see that in the category of integral mixed Hodge structures, the restriction of $2\mathbb{E}_e$ (= the restriction of $2\mathbb{E}_e^\infty$) to $\ker(\phi)(1)$ is zero. It follows that the restriction of $\mathbb{E}_e$ to $\ker(\phi)(1)$ is zero in $MHS(\mathbb{Q})$ (see Section 3.2).

4.3.3. The restriction of $\mathbb{E}_e$ (or $\mathbb{E}_e^\infty$) to $\sigma(H^1\wedge\overline{\xi}_\Delta)(1)$ in $MHS(\mathbb{Q})$ can be described as follows. Identify $\sigma(H^1\wedge\overline{\xi}_\Delta)(1)$ with H^1 via the isomorphism $H^1 \longrightarrow \sigma(H^1\wedge\overline{\xi}_\Delta)(1)$ sending $\omega\mapsto\sigma(\omega\wedge\overline{\xi}_\Delta)$. Combining Harris-Pulte's (10) and Corollary 6.7 of [Pu88], we see that via the identifications given in (8), the restriction of $\mathbb{E}_e$ to $\sigma(H^1\wedge\overline{\xi}_\Delta)(1)$ is the point

$$(2g-2)e - K\ \in\ CH_0^{\text{hom}}(X)\ /\ \text{torsion}.$$

4.3.4. The restriction of $\mathbb{E}_e$ to

$$\sigma((\ \bigwedge^3 H^1\ )_{prim})(1)$$

in the category of rational Hodge structures can be thought of as the harmonic volume of X mod torsion. More precisely, Theorem 3.9 of [Pu88] asserts that the pointed harmonic volume $\mathcal{I}_e$ with base point e (given by (1)) corresponds to the extension $\mathbb{E}_e$ under the canonical isomorphisms

$$Ext(H^1\otimes(H^1\otimes H^1)'(1),\mathbb{Z}(0))\ \cong\ J\ \big(\ H^1\otimes(H^1\otimes H^1)'(1)\big)^\vee\ \cong\ Hom(H^1_{\mathbb{Z}}\otimes(H^1\otimes H^1)'_{\mathbb{Z}},\mathbb{R}/\mathbb{Z}).$$

Thus (by functoriality of the isomorphisms in (2)) the harmonic volume $\mathcal{I}$ of X, i.e. the restriction of $\mathcal{I}_e$ to

$$\phi^{-1}\Big((\ \bigwedge^3 H^1\ )_{prim}\Big)_{\mathbb{Z}} = \phi^{-1}\Big((\ \bigwedge^3 H^1_{\mathbb{C}}\ )_{prim}\Big)\ \cap\ (H^1_{\mathbb{Z}}\otimes(H^1\otimes H^1)'_{\mathbb{Z}}),$$

corresponds to the restriction of the extension $\mathbb{E}_e$ to $\phi^{-1}\Big((\ \bigwedge^3 H^1\ )_{prim}\Big)(1)$ under the canonical isomorphisms

$$\begin{aligned} Ext(\phi^{-1}\ (\ \bigwedge^3 H^1\ )_{prim}\ (1),\mathbb{Z}(0)) \quad &\cong\quad J\ \Big(\ \phi^{-1}\Big(\bigwedge^3 H^1\Big)_{prim}(1)\ \Big)^\vee \\ &\cong\quad Hom(\phi^{-1}\Big((\ \bigwedge^3 H^1\ )_{prim}\Big)_{\mathbb{Z}},\mathbb{R}/\mathbb{Z}). \end{aligned}$$

In view of Section 3.2, under the identifications

$$\begin{aligned} Ext(\phi^{-1}\ (\ \bigwedge^3 H^1\ )_{prim}\ (1),\mathbb{Q}(0)) \quad &\cong\quad J_{\mathbb{Q}}\ \Big(\ \phi^{-1}\Big(\bigwedge^3 H^1\Big)_{prim}(1)\ \Big)^\vee \\ &\cong\quad Hom(\phi^{-1}\Big((\ \bigwedge^3 H^1\ )_{prim}\Big)_{\mathbb{Z}},\mathbb{R}/\mathbb{Q}), \end{aligned}$$

$\mathcal{I} \pmod{\mathbb{Q}}$ corresponds to the restriction of $\mathbb{E}_e$ to $\phi^{-1}\left(\left(\bigwedge^3 H^1\right)_{prim}\right)(1)$ in the category of rational Hodge structures. Since

$$\phi^{-1}\left(\left(\bigwedge^3 H^1\right)_{prim}\right)(1) = \ker(\phi)(1) \oplus \sigma\left(\left(\bigwedge^3 H^1\right)_{prim}\right)(1)$$

in $MHS(\mathbb{Q})$ and the restriction of $\mathbb{E}_e$ to $\ker(\phi)(1)$ is zero, it follows that $\mathcal{I}$ is torsion if and only if the restriction of $\mathbb{E}_e$ to the second summand above is trivial (in $MHS(\mathbb{Q})$).

Remark. (1) Harris' original definition of the harmonic volume was by an explicit formula, without reference to the extension $\mathbb{E}_e$. Pulte then observed the connection between the harmonic volume and the extension $\mathbb{E}_e$, as described above.

(2) The harmonic volume $\mathcal{I}$ is independent of the choice of the base point e (see [Ha83a]). Combining this with the results of Sections 4.3.4 and 4.3.2 above, we see that if the harmonic volume of X is of infinite order, then so are $\mathbb{E}_e$ (and hence, $\mathbb{E}_e^\infty$) and $AJ(X_e - X_e^-)$ for every choice of base point e (and every choice of ∞).

4.4. We now recall a result of Darmon, Rotger, and Sols [DRS12]. Let $Z \in CH_1(X \times X)$. Let $\Delta \in CH_1(X \times X)$ be the diagonal of X. Set $Z_{12} = Z \cdot \Delta$, $Z_1 = Z \cdot (X \times \{e\})$, and $Z_2 = Z \cdot (\{e\} \times X)$, considered as elements of $CH_0(X)$. Let

$$P_Z = Z_{12} - Z_1 - Z_2 - \deg(Z_{12})e + \deg(Z_1)e + \deg(Z_2)e.$$

The point $P_Z \in CH_0^{\mathrm{hom}}(X)$ is related to the extension $\mathbb{E}_e^\infty$ as follows. Denote by ξ_Z the $H^1 \otimes H^1$ Künneth component of the class of Z in $H^2(X \times X)$. Then pulling back extensions along the morphism $H^1 \longrightarrow (H^1)^{\otimes 3}(1)$ defined by $\omega \mapsto \omega \otimes \xi_Z$ gives a map

$$\xi_Z^{-1} : Ext((H^1)^{\otimes 3}(1), \mathbb{Z}(0)) \longrightarrow Ext(H^1, \mathbb{Z}(0)) \cong J(H^1)^\vee \overset{\text{Abel-Jacobi}}{\cong} CH_0^{\mathrm{hom}}(X).$$

By Corollary 2.6 of [DRS12] (also see Proposition 1.4 therein)[1], one has

$$\xi_Z^{-1}(\mathbb{E}_e^\infty) = \Big(\int_\Delta \xi_Z\Big)(\infty - e) - P_Z. \tag{11}$$

5. Proof of Theorem 1

We shall continue to use notation as in the previous section, specializing to $X = F(n)$ the Fermat curve of degree n, defined in projective coordinates by the equation $x^n + y^n = z^n$. We take e and ∞ to be two cusps (points satisfying $xyz = 0$). Recall that any point on the Jacobian of $F(n)$ which is supported on the set of cusps is torsion (see [Ro77]). Hence, both $K - (2g-2)e$ and $-2g\infty + 2e + K$ are torsion points on the Jacobian. By the results recalled in Sections 4.3.1 and 4.3.3, the restrictions of $\mathbb{E}_e^\infty$ in the category $MHS(\mathbb{Q})$ to the summands $H^1 \otimes \xi_\Delta(1)$ and $\sigma(H^1 \wedge \overline{\xi}_\Delta)(1)$ of $(H^1)^{\otimes 3}(1)$ are trivial. Combining with the results of Sections 4.3.2 and 4.3.4 and the decomposition (7), we see that the following statements are equivalent:

- The extension $\mathbb{E}_e^\infty$ is trivial in $MHS(\mathbb{Q})$ (or is torsion in $MHS(\mathbb{Z})$).
- The extension $\mathbb{E}_e$ is trivial in $MHS(\mathbb{Q})$.
- The Abel-Jacobi image of $F(n)_e - F(n)_e^-$ is torsion.
- The harmonic volume of $F(n)$ is torsion.

Thus to establish Theorem 1, it is enough to show that E_e^∞ is not torsion in $MHS(\mathbb{Z})$ for the triple $(F(n), e, \infty)$ if n has a prime divisor > 7 and e and ∞ are cusp points.

[1] Note that there is a typo in the definition of P_Z in Eq. (45) of [DRS12]; see the proof of Lemma 2.1 in the same reference.

Suppose $n = p$ is an odd prime number (the case of composite degree follows easily from this, see below). Let Z be the graph of the automorphism α of $F(p)$ sending

$$(x, y, z) \mapsto (-y, z, x).$$

This automorphism has two fixed points, namely

$$Q = (\eta, \eta^{-1}, 1) \qquad \text{and} \qquad \overline{Q} = (\eta^{-1}, \eta, 1),$$

where η is a primitive sixth root of unity. Thus (with notation as in Section 4.4)

$$P_Z = (Q + \overline{Q} - 2e) - (\alpha(e) + \alpha^{-1}(e) - 2e).$$

The point $\alpha(e) + \alpha^{-1}(e) - 2e$ of the Jacobian of $F(p)$ is supported on the cusps, hence is a torsion point. By a theorem of Gross and Rohrlich (Theorem 2.1 of [GR78]), for $p > 7$ the point $Q + \overline{Q} - 2e$ is of infinite order. Thus P_Z is a point of infinite order on the Jacobian of $F(p)$. Since $\infty - e$ is torsion (as both e and ∞ are cusp points), it follows from (11) that $\mathbb{E}_e^\infty$ is of infinite order in $MHS(\mathbb{Z})$, completing the proof of Theorem 1 for the case where the degree is prime.

As for the case of composite degree, suppose n is divisible by a prime number $p > 7$. By functoriality of Abel-Jacobi maps with respect to pushforwards along morphisms, the map

$$f : F(n) \longrightarrow F(p) \qquad\qquad (x, y, z) \mapsto (x^{n/p}, y^{n/p}, z^{n/p})$$

gives rise to a commutative diagram

$$\begin{array}{ccc} CH_1^{\text{hom}}(Jac(F(n))) & \xrightarrow{f_*} & CH_1^{\text{hom}}(Jac(F(p))) \\ \downarrow & & \downarrow \\ J\left(\bigwedge^3 H^1(F(n))\right)^\vee & \longrightarrow & J\left(\bigwedge^3 H^1(F(p))\right)^\vee, \end{array}$$

where the vertical arrows are Abel-Jacobi maps and the horizontal arrows are induced by the pushforward map $Jac(F(n)) \longrightarrow Jac(F(p))$. Take e to be a cusp of $F(n)$ and let $e' = f(e)$. Then $f_*(F(n)_e - F(n)_e^-) = (n/p)^2 (F(p)_{e'} - F(p)_{e'}^-)$. Since $AJ(F(p)_{e'} - F(p)_{e'}^-)$ has infinite order, the same is true for $AJ(F(n)_e - F(n)_e^-)$. Hence the harmonic volume of $F(n)$ is of infinite order.

References

[Bl84] S. Bloch, Algebraic cycles and values of L-functions, Journal fúr die reine und angewandte Mathematik 350 (1984): 94-108

[Ca80] J. A. Carlson, Extensions of mixed Hodge structures, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 107–127, Sijthoff & Noordhoff, Alphen aan den Rijn, Md., 1980.

[DRS12] H. Darmon, V. Rotger, and I. Sols, Iterated integrals, diagonal cycles, and rational points on elliptic curves, Publications Mathématiques de Besancon, vol.2, 2012, pp. 19-46.

[GR78] B. H. Gross and D. E. Rohrlich, Some results on the Mordell-Weil group of the Jacobian of the Fermat curve, Inventiones Math., 44, 201-224 (1978)

[Ha83a] B. Harris, Harmonic volumes, Acta Math. 150, 1-2 (1983), 91-123

[Ha83b] B. Harris, Homological versus algebraic equivalence in a Jacobian, Proc. Natl. Acad. Sci. USA, Vol. 80, pp. 1157-1158, February 1983

[Ha04] B. Harris, *Iterated Integrals and Cycles on Algebraic Manifolds*, Nankai Tracts in Mathematics Vol 7, 2004, World Scientific.

[Ha87] R. Hain, The geometry of the mixed Hodge structure on the fundamental group, Proceedings of Symposia in Pure Mathematics, Volume 46 (1987)

[Ka01] R. H. Kaenders, The mixed Hodge structure on the fundamental group of a punctured Riemann surface, Proceedings of the American Mathematical Society, Volume 129, 2001, Number 4, Pages 1271-1281

[Ot12] N. Otsubo, On the Abel-Jacobi maps of Fermat Jacobians, Math. Z. 270 (2012), no. 1-2, 423–444.

[Pu88] M. Pulte, The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles, Duke Math. J., 57 (1988), 721-760

[Ro77] D. Rohrlich, Points at infinity on the Fermat curves, Inventiones Math., 39, 95-127 (1977)

[Ta16] Y. Tadokoro, Harmonic Volume and its applications, in Handbook of Teichmueller Theory, Vol. VI, 167-193 IRMA Lect. Math. Theor.. Phys.,27, Eur. Math. Soc. Zurich 2016

[Vo02] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry I*, Cambridge studies in advanced mathematics 76, 2002

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