An integrable connection on the configuration space of a Riemann surface of positive genus

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ABSTRACT. Let X be a Riemann surface of positive genus. Denote by $X^{(n)}$ the configuration space of n distinct points on X. We use the Betti-de Rham comparison isomorphism on $H^1(X^{(n)})$ to define an integrable connection on the trivial vector bundle on $X^{(n)}$ with fiber the universal algebra of the Lie algebra associated to the descending central series of π_1 of $X^{(n)}$. The construction is inspired by the Knizhnik-Zamolodchikov system in genus zero and its integrability follows from Riemann period relations.

Fix $n \geq 1$. Let $\mathfrak{g}_0$ be the graded complex Lie algebra associated to the descending central series† of the classical pure braid group PB_n , i.e. the fundamental group of

$$\mathbb{C}^{(n)} := \{(z_1, \dots, z_n) : z_i \in \mathbb{C},\ z_i \neq z_j \text{ for } i \neq j\}.$$

It is generated by degree 1 elements $\{s_{ij} : 1 \leq i, j \leq n,\ i \neq j\}$, subject to the relations

$$\begin{aligned} s_{ij} &= s_{ji} \\ [s_{ij}, s_{kl}] &= 0 \qquad (i, j, k, l \text{ distinct}) \\ [s_{ij} + s_{ik}, s_{jk}] &= 0. \end{aligned} \tag{1}$$

The element $s_{ij} \in H_1(\mathbb{C}^{(n)}, \mathbb{C})$ (= degree 1 part of $\mathfrak{g}_0$) is the homology class of the j-th strand going positively around the i-th, while all other strands stay constant.

Let $\widehat{U\mathfrak{g}_0}$ be the completion of the universal algebra of $\mathfrak{g}_0$. Let $\mathcal{O}(\mathbb{C}^{(n)})$ (resp. $\Omega^{\cdot}(\mathbb{C}^{(n)})$) be the space of analytic functions (resp. complex of holomorphic differentials) on $\mathbb{C}^{(n)}$. The relations (1) assure that the Knizhnik-Zamolodchikov connection

$$\nabla_{KZ} : \widehat{U\mathfrak{g}_0} \otimes \mathcal{O}(\mathbb{C}^{(n)}) \longrightarrow \widehat{U\mathfrak{g}_0} \otimes \Omega^1(\mathbb{C}^{(n)})$$

defined by

$$\nabla_{KZ} f = df - \overbrace{\left(\sum_{i<j} \frac{1}{2\pi i} s_{ij} \otimes \frac{d(z_i - z_j)}{z_i - z_j} \right)}^{\lambda_0} f$$

is integrable. This connection and its more general variants are of great importance in conformal field theory, representation theory, and number theory.

†Let G be any group and $G_1 := G \supset \cdots \supset G_k \supset G_{k+1} := [G_k, G] \supset \cdots$ be its descending central series. By the graded complex Lie algebra associated to the descending central series of G we mean the positively graded Lie algebra with degree k component $G_k/G_{k+1} \otimes \mathbb{C}$ and Lie bracket induced by the commutator operator in G. See [4].

The connection ∇_{KZ} is related to the comparison isomorphism

$$\mathrm{comp}_{\mathbb{C}^{(n)}} : H^1(\mathbb{C}^{(n)}, \mathbb{C}) \longrightarrow H^1_{dR}(\mathbb{C}^{(n)})$$

between the singular and (say) complex-valued smooth de Rham cohomologies in the following way: λ_0 is the image of $\mathrm{comp}_{\mathbb{C}^{(n)}}$ under the map

$$H_1(\mathbb{C}^{(n)}, \mathbb{C}) \otimes H^1_{dR}(\mathbb{C}^{(n)}) \longrightarrow \widehat{U\mathfrak{g}_0} \otimes \Omega^1(\mathbb{C}^{(n)})$$

defined by

$$s_{ij} \otimes [\frac{d(z_k - z_l)}{z_k - z_l}] \ \mapsto \ s_{ij} \otimes \frac{d(z_k - z_l)}{z_k - z_l} \qquad (i < j, \ k < l).$$

(Note that the s_{ij} with $i < j$ (resp. $[\frac{d(z_k - z_l)}{z_k - z_l}]$ with $k < l$) form a basis of $H_1(\mathbb{C}^{(n)}, \mathbb{C})$ (resp. $H^1_{dR}(\mathbb{C}^{(n)})$).)

Now let $\overline{X}$ be a compact Riemann surface of genus $g > 0$, $S = \{Q_1, \ldots, Q_{|S|}\}$ a finite set of points in $\overline{X}$ (possibly empty), and $X = \overline{X} - S$. Let

$$X^{(n)} := \{(x_1, \ldots, x_n) : x_i \in X, x_i \neq x_j \text{ for } i \neq j\}.$$

Fix a base point $\underline{e} = (e_1, \ldots, e_n) \in X^{(n)}$ and let $\mathfrak{g}$ be the graded complex Lie algebra associated to the descending central series of $\pi_1(X^{(n)}, \underline{e})$. The goal of this note is to use the comparison isomorphism

$$\mathrm{comp}_{X^{(n)}} : H^1(X^{(n)}, \mathbb{C}) \longrightarrow H^1_{dR}(X^{(n)}) \tag{2}$$

to define an integrable connection ∇ on the trivial bundle $\widehat{U\mathfrak{g}} \otimes \mathcal{O}(X^{(n)})$.

1. Construction of the connection

We make three observations first:

(i) Since $g > 0$, the natural map

$$H^1_{dR}(X^n) \longrightarrow H^1_{dR}(X^{(n)}) \tag{3}$$

(induced by inclusion) is an isomorphism. Indeed, thanks to a theorem of Totaro [**7**, Theorem 1] one knows that the five-term exact sequence for the Leray spectral sequence for the constant sheaf $\mathbb{Z}$ and the inclusion $X^{(n)} \to X^n$ reads

$$0 \longrightarrow H^1(X^n, \mathbb{Z}) \xrightarrow{(3)} H^1(X^{(n)}, \mathbb{Z}) \longrightarrow \mathbb{Z}^{\{(a,b):1\leq a<b\leq n\}} \xrightarrow{(*)} H^2(X^n, \mathbb{Z}) \longrightarrow H^2(X^{(n)}, \mathbb{Z}),$$

where the map $(*)$ sends 1 in the copy of $\mathbb{Z}$ corresponding to (a, b) $(a < b)$ to the class of the pullback of the diagonal $\Delta \subset X^2$ under the projection $p_{ab} : X^n \to X^2$ (defined in the obvious way). Since $g > 0$, the class of Δ has a nonzero $H^1(X)\otimes H^1(X)$ Kunneth component (if $X = \overline{X}$ this is well-known and the noncompact case follows from the compact case in view of the functoriality of the class of the diagonal with respect to the inclusion $i : X^2 \to \overline{X}^2$ and injecivity of $i^* : H^2(\overline{X}^2) \to H^2(X^2)$ on $H^1 \otimes H^1$ components). Thus the class of $p^*_{ab}(\Delta)$ has a nonzero $p^*_{ab}(H^1(X) \otimes H^1(X))$ component. Since every other $p^*_{a'b'}(\Delta)$ has a zero $p^*_{ab}(H^1(X) \otimes H^1(X))$ component, it follows that $(*)$ is injective.

(ii) Let $\Omega^1(\overline{X} \log S)$ be the space of differentials of the third kind on $\overline{X}$ with singularities in S. Then one has a distinguished isomorphism $\Omega^1(\overline{X} \log S) \cong F^1H^1_{dR}(X)$ given by $\omega \mapsto [\omega]$ (F being the Hodge filtration). (See [**5**, (3.2.13)(ii) and (3.2.14)], for instance.)

(iii) The cohomology $H^1_{dR}(X)$ decomposes as an internal direct sum $F^1H^1_{dR}(X) \oplus H^{0,1}$ (where $H^{0,1} \subset H^1_{dR}(\overline{X}) \subset H^1_{dR}(X)$). Indeed, this is simply the Hodge decomposition in $X = \overline{X}$ case. As for the noncompact case, strictness of morphisms of mixed Hodge structures with respect to the Hodge filtration implies the two subspaces have zero intersection, and by (ii) and the Riemann-Roch theorem $F^1H^1_{dR}(X)$ has dimension $g+|S|-1$. The conclusion follows by a dimension count.

Let θ be the composition

$$H^1_{dR}(X^{(n)}) \cong H^1_{dR}(X^n) \overset{\text{Kunneth}}{\cong} H^1_{dR}(X)^{\oplus n} \xrightarrow{(\dagger)} F^1H^1_{dR}(X)^{\oplus n} \cong \Omega^1(\overline{X}\log S)^{\oplus n} \xrightarrow{(\ddagger)} \Omega^1(X^{(n)}),$$

where $(\dagger)$ is the sum of n copies of the natural projection, and $(\ddagger)$ is the sum of the pullbacks along projections $X^{(n)} \to X$. Note that the image of θ is contained in the subspace of closed forms, as it is contained in the subspace spanned by the pullbacks of holomorphic 1-forms on X along the aforementioned projections. Let ι be the composition of the inclusion $H_1(X^{(n)}, \mathbb{C}) \subset \mathfrak{g}$ and the natural map $\mathfrak{g} \to \widehat{U\mathfrak{g}}$. Denote by λ the image of the comparison isomorphism (2) under the map

$$\iota \otimes \theta : H_1(X^{(n)}, \mathbb{C}) \otimes H^1_{dR}(X^{(n)}) \longrightarrow \widehat{U\mathfrak{g}} \otimes \Omega^1(X^{(n)}).$$

Define the connection

$$\nabla : \widehat{U\mathfrak{g}} \otimes \mathcal{O}(X^{(n)}) \longrightarrow \widehat{U\mathfrak{g}} \otimes \Omega^1(X^{(n)})$$

by

$$\nabla(f) = df - \lambda f.$$

(Note that λ multiplies with an element of $\widehat{U\mathfrak{g}} \otimes \mathcal{O}(X^{(n)})$ through the multiplication in the universal algebra in the first factor and the algebra of differential forms in the second.)

2. Integrability

We prove that the connection ∇ is integrable. Since $\lambda \in \widehat{U\mathfrak{g}} \otimes \Omega^1_{\text{closed}}(X^{(n)})$, it is enough to show that

$$\lambda^2 \in \widehat{U\mathfrak{g}} \otimes \Omega^2(X^{(n)})$$

is zero. For simplicity denote $d = \dim H_1(X, \mathbb{Z})$ (thus $d = 2g$ if $X = \overline{X}$ and $d = 2g+|S|-1$ otherwise). Let $\{\alpha_i\}_{1\le i\le d}$ be a basis of $H_1(X, \mathbb{Z})$ such that for $i \le g$, α_i and α_{i+g} are (classes of) transversal loops around the i-th handle with $\alpha_i \cdot \alpha_{i+g} = 1$ in $H_1(\overline{X}, \mathbb{Z})$, and for $1 \le i \le |S|-1$, α_{2g+i} is a simple loop going positively around the puncture Q_i , contractible in $X \cup \{Q_i\}$. Let $\{\omega_i\}_{1\le i\le d}$ be 1-forms such that $\{\omega_i\}_{i\le g}$ form a basis for holomorphic differentials on $\overline{X}$, $\omega_{g+i} = \overline{\omega_i}$ for $i \le g$, and ω_{2g+i} ($1 \le i \le |S|-1$) is a differential of the third kind with residual divisor $\frac{1}{2\pi i}(Q_i - Q_{|S|})$. With abuse of notation we denote a differential form (resp. a loop) and its cohomology (resp. homology) class by the same symbol. Write the comparison isomorphism $\mathrm{comp}_X \in H_1(X, \mathbb{C}) \otimes H^1_{dR}(X)$ as $\sum_{i,j} \pi_{ij}\alpha_i \otimes \omega_j$. (Here and in all the sums in the sequel, unless otherwise indicated the indices run over all their possible values.) The matrix $(\pi_{ij})_{ij}$ (with ij-entry π_{ij}) is the inverse of the matrix whose ij-entry is $\int_{\alpha_j} \omega_i$, and is of the form

$$\begin{pmatrix} P^{-1} & 0 \\ & I_{|S|-1} \end{pmatrix},$$

where P is the matrix of periods of $\overline{X}$ with respect to the ω_i and α_j, and I denotes the identity matrix.

Let $\{\alpha_i^{(k)}\}_{\substack{1\leq k\leq n\\ 1\leq i\leq d}}$ be pure braids in X with n strands based at $\underline{e}$ (= loops in $X^{(n)}$ based at $\underline{e}$) such that the following hold:

(i) The only nonconstant strand in $\alpha_i^{(k)}$ is the one based at e_k.

(ii) For $i \leq g$, the strands of $\alpha_i^{(k)}$ and $\alpha_{i+g}^{(k)}$ based at e_k are transversal loops around the i-th handle.

(iii) For $1 \leq i \leq |S|-1$, the strand of $\alpha_{2g+i}^{(k)}$ based at e_k is a simple loop going around Q_i.

(iv) The k-th projection $X^{(n)} \to X$ sends $\alpha_i^{(k)}$ to α_i in homology.

Let $\omega_i^{(k)}$ be the pullback of ω_i under the k-th projection $X^{(n)} \to X$. Then $\{\alpha_i^{(k)}\}$ and $\{\omega_i^{(k)}\}$ are bases of $H_1(X^{(n)}, \mathbb{C})$ and $H^1_{dR}(X^{(n)})$, and

$$\mathtt{comp}_{X^{(n)}} = \sum_{i,j,k} \pi_{ij}\alpha_i^{(k)} \otimes \omega_j^{(k)}.$$

Let $\mathcal{F} = \{1, \ldots, d\} - \{g+1, \ldots, 2g\}$. Then

$$\lambda = \sum_{\substack{j\in\mathcal{F}\\ i,k}} \pi_{ij}\alpha_i^{(k)} \otimes \omega_j^{(k)}.$$

We have

$$\begin{aligned}\lambda^2 &= \sum_{\substack{j,j'\in\mathcal{F};\ i,i'\\ k,k'}} \pi_{ij}\pi_{i'j'}\alpha_i^{(k)}\alpha_{i'}^{(k')} \otimes \omega_j^{(k)} \wedge \omega_{j'}^{(k')}\\ &= \sum_{\substack{j,j'\in\mathcal{F};\ i,i'\\ k<k'}} \pi_{ij}\pi_{i'j'}[\alpha_i^{(k)}, \alpha_{i'}^{(k')}] \otimes \omega_j^{(k)} \wedge \omega_{j'}^{(k')}.\end{aligned}$$

Simple calculations using Bellingeri's description of $\pi_1(X^{(n)})$ given in [**1**, Theorems 5.1 and 5.2] (also see [**2**] for a misprint corrected) show that in $\mathfrak{g}$, for arbitrary distinct k, k', $[\alpha_i^{(k)}, \alpha_{i'}^{(k')}] = 0$ unless $i, i' \leq 2g$ and $|i-i'| = g$ (i.e. unless $\alpha_i^{(k)}, \alpha_{i'}^{(k')}$ correspond to transversal loops going around the same handle), and moreover that

$$[\alpha_i^{(k)}, \alpha_{i+g}^{(k')}] \qquad (i \leq g) \tag{4}$$

only depends on the *set* $\{k, k'\}$. (Note that one can take $\alpha_i^{(k)} \in \pi_1(X^{(n)})$ to be Bellingeri's $A_{2i-1,d+k}$, $A_{2(i-g),d+k}$, or $A_{i,d+k}$ depending on whether $i \leq g$, $g < i \leq 2g$, or $2g < i \leq d$ respectively.) Denoting (4) by $s_{kk'}$ (=$s_{k'k}$), we thus have

$$\lambda^2 = \sum_{\substack{j,j'\leq g\\ k<k'}} \left(\sum_{i\leq g} \pi_{ij}\pi_{i+g,j'} - \pi_{i+g,j}\pi_{ij'} \right) s_{kk'} \otimes \omega_j^{(k)} \wedge \omega_{j'}^{(k')},$$

which is zero by Riemann period relations.

REMARKS. (1) In the case $X = \overline{X}$, one can replace $\mathfrak{g}$ by the Lie algebra $\mathfrak{l}$ of the nilpotent completion of $\pi_1(X^{(n)})$. Thanks to a theorem of Bezrukavnikov [**3**] one knows similar relations to the ones in $\mathfrak{g}$ used above to prove integrability also hold in $\mathfrak{l}$.

(2) It would be interesting to relate the connection defined here with the one defined by Enriquez in [**6**] on configuration spaces of compact Riemann surfaces.

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References

[1] P. Bellingeri, On presentations of surface braid groups, *Journal of Algebra* 274 (2004) 543-563
[2] P. Bellingeri, S. Gervais and J. Guaschi, Lower central series for Artin Tits and surface braid groups, *Journal of Algebra* 319 (2008), 1409-1427
[3] R. Bezrukavnikov, Koszul DG-algebras arising from configuration spaces, *Geometric and Functional Analysis*, Vol. 4, No. 2 (1994)
[4] N. Bourbaki, *Lie groups and Lie algebras: Chapters 1-3*, Springer 1988
[5] P. Deligne, Theorie de Hodge, II, *Publications Mathématiques de l'I.H.É.S.*, tome 40 (1971), p. 5-57
[6] B. Enriquez, Flat connections on configuration spaces and braid groups of surfaces, *Advances in Mathematics* 252 (2014) 204226
[7] B. Totaro, Configuration spaces of algebraic varieties, *Topology*, Vol. 35, No. 4 pp 1057-1067 (1996)

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