ON BLENDED EXTENSIONS IN FILTERED TANNAKIAN CATEGORIES AND MOTIVES WITH MAXIMAL UNIPOTENT RADICALS

PAYMAN ESKANDARI

Abstract. Grothendieck's theory of blended extensions (*extensions panachées*) gives a natural framework to study 3-step filtrations in abelian categories. We give a generalization of this theory that is suitable for filtrations with an arbitrary finite number of steps. We use this generalization to study two natural classification problems for objects with a fixed associated graded in a filtered tannakian category over a field of characteristic zero. We then give an application to the study of mixed motives with a given associated graded and maximal unipotent radicals of motivic Galois groups. We prove a homological classification result for such motives when the given associated graded is "graded-independent", a condition defined in the paper. The special case of this result for motives with 3 weights was proved with K. Murty in [22] under some extra hypotheses. The paper also includes some discussions on periods.

Contents

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1. Introduction

1.1. About this paper. Let $\mathbf{T}$ be a tannakian category over a field of characteristic zero, equipped with a weight filtration $W_\bullet$ similar to the weight filtration on the category of rational mixed Hodge structures or any reasonable tannakian category of mixed motives over a field of characteristic zero. That is, $W_\bullet$ is indexed by $\mathbb{Z}$, functorial, exact, increasing, finite on every object, and compatible with the tensor structure. Consider a graded object

$$A = \bigoplus_{r=1}^{k} A_r,$$

where the A_r are pure and in an increasing order of weights. One may consider the following two classification problems:

(1) Classify the equivalence classes of all pairs (X, ϕ) of an object X of $\mathbf{T}$ whose associated graded
$$Gr^W X = \bigoplus_n W_n X / W_{n-1} X$$
is isomorphic to A, and an isomorphism $\phi : Gr^W X \to A$. Here, the equivalence relation for such pairs is defined as follows: two pairs (X, ϕ) and (X', ϕ') are considered equivalent if there exists an isomorphism $f : X \to X'$ such that $\phi' \circ Gr^W f = \phi$.

(2) Classify, up to isomorphism in $\mathbf{T}$, all objects X whose associated graded $Gr^W X$ is isomorphic to A. Note that the data of a choice of isomorphism $Gr^W X \to A$ is not recorded here at all.

In general, the group
$$Aut(A) = \prod_r Aut(A_r)$$
acts on the collection of equivalence classes of pairs (X, ϕ) as in (1), and the orbits of this action will be in bijection with the collection of isomorphism classes of X as in (2).

When $k = 2$ and $A = A_1 \oplus A_2$, the homological concept that classifies the pairs (X, ϕ) up to equivalence of such pairs is the Ext group $Ext^1(A_2, A_1)$. As for the isomorphism classes of X, the answer is given by the quotient of $Ext^1(A_2, A_1)$ by the group
$$Aut(A_1) \times Aut(A_2),$$
where the actions of automorphisms of A_1 and A_2 on extension classes is by pushforward and pullback.

When $k = 3$, the homological concept related to problems (1) and (2) is the concept of a blended extension[1], introduced by Grothendieck in SGA 7.I [27, §9.3] to study 3-step filtrations. A nice detailed discussion of the relation between problem (1) in the case $k = 3$ and blended extensions can be found in the appendix of the work [36] of Ramis, Sauloy and Zhang.

The first main objective of the present paper is to give a new approach towards problems (1) and (2) for an arbitrary k. We introduce a new concept, which we call a *generalized extension*, that is a generalization of the notion of a blended extension and provides a natural homological framework for the study of problems (1) and (2). This concept naturally leads to an inductive approach (namely, induction on the *level*) towards the problems that seems more interesting and powerful than the more obvious approach of induction on the number of weights.

The other main objective of the paper, which was our original motivation for the work, is to consider the problem of classifying the isomorphism classes of motives X with associated graded

[1] The original French term for the concept is *extension panachée*. The English translation to the term *blended extension*, which we first found in [9], is attributed by Bertrand to L. Breen.

isomorphic to a given semisimple motive A and with maximal unipotent radicals of motivic Galois groups. By maximality of the unipotent radical here we mean that the Lie algebra of the unipotent radical of the motivic Galois group of X is equal to $W_{-1}\underline{Hom}(X,X)$, where $\underline{Hom}$ is the internal Hom (that is, the unipotent radical is as large as it can be under the constraints imposed on it by the weight filtration). The interest in motives with maximal unipotent radicals is inspired by Grothendieck's period conjecture, which predicts that the transcendence degree of the field generated by the periods of any motive over $\overline{\mathbb{Q}}$ should be equal to the dimension of the motivic Galois group of the motive. See André's letter to Bertolin published at the end of [8] for more about this deep conjecture, including some very interesting remarks on the history of it. It would follow from this conjecture that among motives over $\overline{\mathbb{Q}}$ with the same associated graded, the field generated by the periods of a motive with a maximal unipotent radical should have the largest transcendence degree.

When A satisfies a certain property (what we call *graded-independence*), we give a particularly nice homological answer to the classification problem of motives with associated graded isomorphic to A and maximal unipotent radicals. The special case of this result for when $A = A_1 \oplus A_2 \oplus A_3$ with the A_r pure and in an increasing order of weights, $A_3 = \mathbb{1}$ and $Ext^1(\mathbb{1}, A_1) = 0$ was proved with K. Murty in [22].

As an example that illustrates the result beyond the case with 3 weights, in the final section of the paper we give a classification of graded-independent 4-dimensional mixed Tate motives over $\mathbb{Q}$ with 4 weights and maximal unipotent radicals. This builds on the discussion of the 3-dimensional case in [22] and raises some interesting questions about periods.

1.2. A more detailed overview of the paper and summary of the main results.

1.2.1. *Contents of §3: Classification of objects in a filtered tannakian category with a given associated graded.* Let $\mathbf{T}$ be a filtered tannakian category over a field of characteristic zero, i.e. a neutral tannakian[2] category $\mathbf{T}$ over a field of characteristic zero, equipped with a filtration $W_\bullet$ satisfying similar properties to the weight filtration on mixed Hodge structures, i.e. $W_\bullet$ is indexed by $\mathbb{Z}$, functorial, exact, increasing, finite on every object, and it respects the tensor structure. This means that for every integer n we have an exact linear functor $W_n : \mathbf{T} \to \mathbf{T}$ such that for every object X of $\mathbf{T}$ we have

$$\begin{aligned} W_{n-1}X &\subset W_nX \qquad (\forall n) \\ W_nX &= 0 \qquad (\forall n \ll 0) \\ W_nX &= X \qquad (\forall n \gg 0), \end{aligned}$$

and such that the inclusions $W_nX \subset X$ for various X give a morphism of functors from W_n to the identity. Compatibility with the tensor structure means that for every objects X and Y and every n,

$$W_n(X \otimes Y) = \sum_{p+q=n} W_p(X) \otimes W_q(Y).$$

We will refer to $W_\bullet$ as the weight filtration. Adopting the terminology of mixed Hodge structures (or mixed motives), an object X will be called pure if there is an integer n such that $W_{n-1}X = 0$ and $W_nX = X$. If X is nonzero and pure, there is a unique such an integer, which is called the weight of X.

Fix nonzero pure objects $A_1, \ldots, A_k$ of increasing weights $p_1 < \cdots < p_k$, respectively. Set A to be the direct sum of the A_r over $1 \le r \le k$. We are interested in objects X of $\mathbf{T}$ such that $Gr^W X$ is isomorphic to A. We denote the set of isomorphism classes of such X by $S(A)$, and

[2] We will freely use the language of of tannakian categories. The reader can refer to [17] for the basic theory of tannakian categories.

the set of equivalence classes of pairs (X,ϕ) as described in problem (1) earlier by $S'(A)$. As a minor detail, we note that since $\mathbf{T}$ is tannakian, $S(A)$ and $S'(A)$ are indeed sets (rather than proper classes).

The classification problem for $S'(A)$ has been considered previously in the setting of difference modules over difference rings by Ramis, Salouy and Zhang [36] and in the setting of real mixed Hodge structures in Ferrario's PhD thesis[3] [24]. In the former setting, Ramis et al show that $S'(A)$ is actually a scheme. The study of $S'(A)$ in the latter setting leads Ferrario to obtains some interesting results on a complex analogue of Grothendieck's section conjecture. In both works, the authors study $S'(A)$ by an inductive approach on the number of weights. We will say a few more words about this approach later in this Introduction.

The goal of §3 of the present paper is to study $S'(A)$ and $S(A)$ in the general setting of an arbitrary $\mathbf{T}$ by a different approach, which has better naturalness properties than the approach of induction on the number of weights. We should point out that we find the classification problem for $S(A)$ more interesting, even though one sometimes prefers to work with $S'(A)$ because of its potential for better moduli properties (as the work [36] in the setting of difference modules illustrates). Note that in passing from $S'(A)$ to $S(A)$ we do not content ourselves with a cursory description of $S(A)$ as $S'(A)$ modded out by the obvious action of $Aut(A)$ (see §3.1 for more details on this action). In fact, one of the advantages of the approach proposed in this paper is that it allows us to trace this action more explicitly, resulting in a particularly simplified outcome in the important totally nonsplit case (see below).

Let us first discuss the case $k=3$ in more details. What we will say here is a special case of what will be proved in the case of arbitrary $k\geq 3$. Given an object X of $\mathbf{T}$ with

$$Gr^W X \simeq A = A_1\oplus A_2\oplus A_3$$

where A_r is nonzero pure of weight p_r and $p_1<p_2<p_3$, set $X_r := W_{p_r}X$ so that the X_r form a 3-step filtration. Choosing an isomorphism $\phi : Gr^W X \to A$ we obtain a blended extension

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & X_2 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & X \arrow[r] \arrow[d] & X/X_1 \arrow[r] \arrow[d] & 0 \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{1}$$

with obvious arrows. It is easy to see that by sending the equivalence class of (X,ϕ) to the classes of the extensions of the top row and the right column we obtain a (well-defined) map

$$S'(A)\to Ext^1(A_2,A_1)\times Ext^1(A_3,A_2). \tag{2}$$

Given $\mathscr{N}\in Ext^1(A_3,A_2)$ and $\mathscr{L}\in Ext^1(A_2,A_1)$, after choosing representative extensions for $\mathscr{N}$ and $\mathscr{L}$ denote the set of equivalence classes of blended extensions of $\mathscr{N}$ by $\mathscr{L}$ with respect to the

[3] I thank Peter Jossen for bringing this to my attention. Note that Ferrario calls a pair $(X, Gr^W X \xrightarrow{\phi,\simeq} A)$ an amalgam of A, see Definition 3.2.4 of [24]. Our $S'(A)$ is the same as $\mathrm{Am}(A)$ in his notation (= the set of isomorphism classes of amalgams of A).

standard equivalence (morphisms between the middle objects that are identity on the top row and the right column) by $Extpan(\mathscr{N},\mathscr{L})$. In view of the fact that

$$Hom(A_2,A_1)\cong Hom(A_3,A_2)\cong 0,$$

one easily sees that the fiber of the map (2) above $(\mathscr{L},\mathscr{N})$ is in a canonical bijection with $Extpan(\mathscr{N},\mathscr{L})$. This bijection sends the equivalence class of a pair (X,ϕ) above $(\mathscr{L},\mathscr{N})$ to the class of the blended extension (1), with the top row and right column replaced by the chosen representatives of $\mathscr{L}$ and $\mathscr{N}$ via the canonical isomorphisms. (Note that since $Hom(A_2,A_1)$ and $Hom(A_3,A_2)$ vanish, there is a canonical isomorphism between any two extensions representing $\mathscr{L}$ or $\mathscr{N}$.) Thanks to the general theory of blended extensions (see §2.1 for a brief review), each fiber of (2) is thus either empty or a torsor over $Ext^1(A_3,A_1)$. Moreover, one can see that the torsor structure on the fiber of (2) above $(\mathscr{L},\mathscr{N})$ (when nonempty) is canonical, in the sense that it does not depend on the choice of representative extensions for $\mathscr{N}$ and $\mathscr{L}$. Finally, it follows also from the theory of blended extensions that the image of (2) is the kernel of the Yoneda composition

$$Ext^1(A_2,A_1)\times Ext^1(A_3,A_2)\to Ext^2(A_3,A_1).$$

Let us turn our attention to $S(A)$. The claims that will be made in this discussion will be proved as we prove the general result for an arbitrary k. The map (2) descends to a map

$$S(A)\to\big(\,Ext^1(A_2,A_1)\times Ext^1(A_3,A_2)\,\big)\,/Aut(A), \tag{3}$$

where the action of $Aut(A)$ is by pushforwards and pullbacks of extensions: an element

$$(\sigma_1,\sigma_2,\sigma_3)\in Aut(A_1)\times Aut(A_2)\times Aut(A_3)=Aut(A)$$

sends $(\mathscr{L},\mathscr{N})$ to $((\sigma_1)_*(\sigma_2^{-1})^*\mathscr{L},(\sigma_2)_*(\sigma_3^{-1})^*\mathscr{N})$. The fiber of (3) above the $Aut(A)$-orbit of $(\mathscr{L},\mathscr{N})$ is the image in $S(A)$ of the fiber of (2) above $(\mathscr{L},\mathscr{N})$. One can show that there is a group $\Gamma(\mathscr{L},\mathscr{N})$ with a natural action on $Extpan(\mathscr{N},\mathscr{L})$ such that the fiber of (3) above the orbit of $(\mathscr{L},\mathscr{N})$ can be identified with

$$Extpan(\mathscr{N},\mathscr{L})/\Gamma(\mathscr{L},\mathscr{N}).$$

Thus the fiber of (3) above the orbit of $(\mathscr{L},\mathscr{N})$ is either empty or the quotient of an $Ext^1(A_3,A_1)$-torsor by the group $\Gamma(\mathscr{N},\mathscr{L})$.

We briefly include the description of $\Gamma(\mathscr{L},\mathscr{N})$ as it will give some intuition for the picture for an arbitrary k. Fixing representatives

$$0\longrightarrow A_1\longrightarrow L\longrightarrow A_2\longrightarrow 0$$

and

$$0\longrightarrow A_2\longrightarrow N\longrightarrow A_3\longrightarrow 0$$

for $\mathscr{L}$ and $\mathscr{N}$, the group $\Gamma(\mathscr{L},\mathscr{N})$ is the subgroup of $Aut(L)\times Aut(N)$ consisting of pairs (σ_L,σ_N) such that the automorphisms of A_2 induced by σ_L and σ_N coincide. Then the action of $\Gamma(\mathscr{L},\mathscr{N})$ on $Extpan(\mathscr{N},\mathscr{L})$ is by twisting the arrows: the class of the blended extension of the left below is sent to the class of the one on the right (here, σ_{A_r} is the automorphism of A_r induced by σ_L

or σ_N):

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\iota"] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "j"] & X \arrow[r, "\pi"] \arrow[d, "\omega"] & N \arrow[r] \arrow[d] & 0 \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\iota\sigma_L^{-1}"] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "j\sigma_{A_1}^{-1}"] & X \arrow[r, "\sigma_N\pi"] \arrow[d, "\sigma_{A_3}\omega"] & N \arrow[r] \arrow[d] & 0. \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{4}
$$

Note that the arrows on the top row and the right column remain unchanged. The stabilizer of the class of the blended extension on the left under this action is the image of the natural injection

$$Aut(X) \hookrightarrow Aut(L) \times Aut(N).$$

With applications to motives with maximal unipotent radicals in mind, of particular interest is the case in which the extensions $\mathscr{N}$ and $\mathscr{L}$ are totally nonsplit (see Definition 2.2.1 to recall what this means). One can see that in this case, the subgroup $\Gamma(\mathscr{L},\mathscr{N})$ of $Aut(L) \times Aut(N)$ is just the diagonal copy of the group of nonzero scalar maps, so that $Aut(X) \cong \Gamma(\mathscr{L},\mathscr{N})$ for every X as in the diagram. Thus the action of the group $\Gamma(\mathscr{L},\mathscr{N})$ on $Extpan(\mathscr{N},\mathscr{L})$ is trivial, so that in passing from (2) to (3) the fibers above totally nonsplit extension pairs do not change.

The goal of §3 is to establish a generalization of the above picture for the case of an arbitrary number of steps in the weight filtration. The main result is as follows:

Theorem 1.2.1. *Let $k \geq 2$. There exist sets $S'_\ell(A)$ for $1 \leq \ell \leq k-1$ and maps*

$$S'_{k-1}(A) \xrightarrow{\Theta_{k-1}} S'_{k-2}(A) \xrightarrow{\Theta_{k-2}} S'_{k-3}(A) \to \cdots \to S'_2(A) \xrightarrow{\Theta_2} S'_1(A)$$

with the following properties:

(a) *There are canonical bijections $S'_{k-1}(A) \cong S'(A)$ and $S'_1(A) \cong \prod_r Ext^1(A_{r+1}, A_r)$.*

(b) *Let $2 \leq \ell \leq k-1$. Every nonempty fiber of the map $\Theta_\ell : S'_\ell(A) \to S'_{\ell-1}(A)$ is canonically a torsor for*

$$\prod_r Ext^1(A_{r+\ell}, A_r).$$

(c) *Let $2 \leq \ell \leq k-1$. If the Ext^2 groups*

$$Ext^2(A_{r+\ell}, A_r) \qquad (1 \leq r \leq k-\ell)$$

vanish, then the map Θ_ℓ is surjective.

(d) *There is a natural action of $Aut(A)$ on each $S'_\ell(A)$ $(1 \leq \ell \leq k-1)$ such that setting $S_\ell(A) = S'_\ell(A)/Aut(A)$, the maps of (a) and (b) descend to maps*

$$S_{k-1}(A) \to S_{k-2}(A) \to S_{k-3}(A) \to \cdots \to S_2(A) \to S_1(A)$$

and

$$S_{k-1}(A) \cong S(A) \quad \text{and} \quad S_1(A) \cong \Big(\prod_r Ext^1(A_{r+1}, A_r)\Big) / \ Aut(A)$$

(the action of $Aut(A) = \prod_{1\leq j\leq k} Aut(A_j)$ *on* $\prod_r Ext^1(A_{r+1}, A_r)$ *being by pushforwards and pullbacks; see §3.5 for more details).*

(e) *Let* $2 \leq \ell \leq k-1$. *Denote the induced map* $S_\ell(A) \to S_{\ell-1}(A)$ *also by* Θ_ℓ. *For every* $\epsilon \in S_{\ell-1}(A)$ *and every* $\epsilon' \in S'_{\ell-1}(A)$ *above* ϵ, *the fiber* $\Theta_\ell^{-1}(\epsilon)$ *is the image of the fiber* $\Theta_\ell^{-1}(\epsilon')$ *under the quotient map* $S'_\ell(A) \twoheadrightarrow S_\ell(A)$. *Moreover, there exists a group* $\Gamma(\epsilon')$ *acting on the fiber* $\Theta_\ell^{-1}(\epsilon')$ *such that the map* $\Theta_\ell^{-1}(\epsilon') \twoheadrightarrow \Theta_\ell^{-1}(\epsilon)$ *induces a bijection*

$$\Theta_\ell^{-1}(\epsilon')/\Gamma(\epsilon') \cong \Theta_\ell^{-1}(\epsilon).$$

(f) *With* ℓ, ϵ *and* ϵ' *as in (e), if* ϵ *is weakly totally nonsplit (see the comments below), then the action of* $\Gamma(\epsilon')$ *is trivial and the map* $S'_\ell(A) \to S_\ell(A)$ *restricts to a bijection*

$$\Theta_\ell^{-1}(\epsilon') \cong \Theta_\ell^{-1}(\epsilon).$$

In particular, when ϵ *is weakly totally nonsplit, then the choice of* ϵ' *above* ϵ *makes* $\Theta_\ell^{-1}(\epsilon)$ *a torsor for* $\prod_r Ext^1(A_{r+\ell}, A_r)$.

We now discuss the idea of the construction of the sets $S'_\ell(A)$. Let X be an object of $\mathbf{T}$ whose associated graded is isomorphic to A. Fix an isomorphism $Gr^W X \to A$ to identify the two. For any integers m, n with $0 \leq m < n \leq k$, set

$$X_{m,n} := W_{p_n}X/W_{p_m}X,$$

where we have set $p_0 = p_1 - 1$ (so that $W_{p_0}X = 0$). It is convenient to introduce the following notation: for m, n with $0 \leq m < n \leq k$ and $n - m \geq 2$, let $\mathfrak{X}^h_{m,n}$ and $\mathfrak{X}^v_{m,n}$ be the following two extensions with $X_{m,n}$ in the middle:

$$\mathfrak{X}^h_{m,n}: \qquad 0 \longrightarrow A_{m+1} \longrightarrow X_{m,n} \longrightarrow X_{m+1,n} \longrightarrow 0$$

and

$$\mathfrak{X}^v_{m,n}: \qquad 0 \longrightarrow X_{m,n-1} \longrightarrow X_{m,n} \longrightarrow A_n \longrightarrow 0$$

where we have used our fixed isomorphism $Gr^W X \to A$ to identify each $X_{r-1,r}$ with A_r. Here, the superscripts h and v stand for horizontal and vertical, respectively; the reason for the choice of notation is that these will be considered respectively as horizontal and vertical extensions in diagrams of blended extensions.

The $X_{m,n}$ fit into the commutative diagram

$$\begin{tikzcd}
A_1 \arrow[d, hook] \\
X_{0,2} \arrow[r, two heads] \arrow[d, hook] & A_2 \arrow[d, hook] \\
X_{0,3} \arrow[r, two heads] \arrow[d, hook] & X_{1,3} \arrow[r, two heads] \arrow[d, hook] & A_3 \arrow[d, hook] \\
X_{0,4} \arrow[r, two heads] \arrow[d, hook] & X_{1,4} \arrow[r, two heads] \arrow[d, hook] & X_{2,4} \arrow[r, two heads] \arrow[d, hook] & A_4 \arrow[d, hook] \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
X_{0,k-1} \arrow[r, two heads] \arrow[d, hook] & X_{1,k-1} \arrow[r, two heads] \arrow[d, hook] & \cdots \arrow[rr, two heads] & & X_{k-3,k-1} \arrow[r, two heads] \arrow[d, hook] & A_{k-1} \arrow[d, hook] \\
X_{0,k} \arrow[r, two heads] & X_{1,k} \arrow[r, two heads] & \cdots \arrow[rr, two heads] & & X_{k-3,k} \arrow[r, two heads] & X_{k-2,k} \arrow[r, two heads] & A_k.
\end{tikzcd} \tag{5}$$

Every $X_{m,n}$ appears in the diagram exactly once. Each horizontal arrow is surjective and is given by modding out by the first step in the weight filtration on the domain. The vertical arrows are all injective and are the inclusions $X_{m,n-1} \hookrightarrow X_{m,n}$ given by the weight filtration.

Roughly speaking, our goal is to obtain all X or $(X, Gr^W X \to A)$ up to the appropriate equivalence relation. Our approach is to do this step by step as follows: First consider possibilities for $(X_{r-1,r+1})_r$ (i.e. the first diagonal below the A_r); the object $X_{r-1,r+1}$ is an extension of A_{r+1} by A_r. So we must look at

$$\prod_r EXT(A_{r+1}, A_r)$$

up to some equivalence. Here, $EXT(A_{r+1}, A_r)$ means the collection of extensions of A_{r+1} by A_r before modding out by any equivalence relation. Fixing $(X_{r-1,r+1})_r$, we now consider the possibilities for $(X_{r-1,r+2})_r$ (i.e. the second diagonal below the A_r. The object $X_{r-1,r+2}$ will be the middle object of the blended extension

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_r \arrow[r] \arrow[d, equal] & X_{r-1,r+1} \arrow[r] \arrow[d] & X_{r,r+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_r \arrow[r] & X_{r-1,r+2} \arrow[r] \arrow[d] & X_{r,r+2} \arrow[r] \arrow[d] & 0 \\
 & & A_{r+2} \arrow[r, equal] \arrow[d] & A_{r+2} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

of $\mathfrak{X}^v_{r,r+2}$ by $\mathfrak{X}^h_{r-1,r+1}$. We must thus look at

$$\prod_r EXTPAN(\mathfrak{X}^v_{r,r+2}, \mathfrak{X}^h_{r-1,r+1})$$

up to some equivalence, where following [27] the notation $EXTPAN(\mathfrak{X}^v_{r,r+2}, \mathfrak{X}^h_{r-1,r+1})$ means the collection of all blended extensions of $\mathfrak{X}^v_{r,r+2}$ by $\mathfrak{X}^h_{r-1,r+1}$ (before taking any equivalence relations into account). We continue in the same fashion until we get to the possibilities for $X_{0,k} = X$. Of course, one also has to keep track of the appropriate equivalence relations in each step.

To make this approach precise, we introduce the notion of a *generalized extension of A of a given level ℓ* $(1 \leq \ell \leq k-1)$ (see Definitions 3.2.1 and 3.2.2). A generalized extension of level ℓ of A is the abstract data of a diagram as in (5), but only with the first ℓ diagonals below the A_r included. Thus a generalized extension of level $k-1$ is an abstract version of the full diagram (5). When $k = 2$, a generalized extension of level 1 of A is simply an extension of A_2 by A_1. When $k = 3$, a generalized extension of level 2 of A is simply the data of a blended extension as in the left diagram of (4), with varying L and N, but A_1, A_2, A_3 fixed. For any k, the data of a generalized extension of level 1 of A consists of an extension of A_{r+1} by A_r for each $1 \leq r \leq k-1$. We highlight that our notion of a generalized extension becomes interesting when the level is less than $k-1$ (as in level $k-1$, everything is determined by the bottom left object).

The sets $S'_\ell(A)$ and $S_\ell(A)$ in Theorem 1.2.1 are the quotients of the collection of all generalized extensions of level ℓ of A by suitable equivalence relations.[4] A pair (X, ϕ) gives rise to a generalized extension of level $k-1$, inducing the identification $S'(A) \cong S_{k-1}(A)$. The maps $S'_\ell(A) \to S'_{\ell-1}(A)$ and $S_\ell(A) \to S_{\ell-1}(A)$ are simply induced by truncation.

The weakly[5] totally nonsplit case of Theorem 1.2.1 is crucial in §4 (reviewed below shortly), where we give an application to the classification problem of motives with maximal unipotent radicals and a fixed associated graded. A weakly totally nonsplit generalized extension of any level is one in which certain extensions arising from the objects on the lowest diagonal are totally nonsplit (see Definition 3.9.1). The notion descends to elements of $S_\ell(A)$. The weakly totally nonsplit elements of

$$S_1(A) \cong \Big(\prod_r Ext^1(A_{r+1}, A_r) \Big) \big/ \ Aut(A)$$

are simply orbits of tuples of extension classes $(\mathscr{E}_r)$ in which all the $\mathscr{E}_r$ are totally nonsplit.

Of course, as mentioned earlier, there is a somewhat more straightforward inductive approach towards the study of $S'(A)$ and $S(A)$, namely to induct on the number of weights of A. Setting

$$A_{\le \ell} := \bigoplus_{r \le \ell} A_r,$$

the weight filtration gives rise to maps

$$S'(A) = S'(A_{\le k}) \to S'(A_{\le k-1}) \to \cdots \to S'(A_{\le 3}) \to S'(A_{\le 2})$$

and

$$S(A) = S(A_{\le k}) \to S(A_{\le k-1}) \to \cdots \to S(A_{\le 3}) \to S(A_{\le 2}).$$

The fiber of $S'(A_{\le \ell}) \to S'(A_{\le \ell-1})$ above the equivalence class of (X, ϕ) is in a canonical bijection with $Ext^1(A_\ell, X)$ (this is written in detail in [24], see Proposition 3.2.9 and Remark 3.2.10 therein).

The inductive approach using the level proposed in this paper is not merely more elegant. The application to mixed motives with maximal unipotent radicals given in §4 illustrates the usefulness of the approach proposed in this paper. We also draw the reader's attention to the better naturalness properties of the approach of induction on the level compared to induction on the number of weights: Every nonempty fiber of $S'_\ell(A) \to S'_{\ell-1}(A)$ is canonically a torsor for the same group $\prod_r Ext^1(A_{r+\ell}, A_r)$ (compare with the structure of the fibers of $S'(A_{\le \ell}) \to S'(A_{\le \ell-1})$). Also, the fiber of $S(A_{\le \ell}) \to S(A_{\le \ell-1})$ above the isomorphism class of X is in bijection with

$$Ext^1(A_\ell, X) \ \big/ \ Aut(X) \times Aut(A_\ell),$$

where the actions of $Aut(X)$ and $Aut(A_\ell)$ are by pushforward and pullback of extensions. Unless $Ext^1(A_\ell, X)$ is trivial, the action of $Aut(X) \times Aut(A_\ell)$ on $Ext^1(A_\ell, X)$ is never trivial. Compare this situation with part (f) of Theorem 1.2.1. It would be very interesting to see if the improved properties of the approach of inducting on the level can lead to new moduli results.

We end our review of §3 by noting that while we have written the article working in the setting of filtered tannakian categories, under some standard hypotheses, one should be able to adapt, with some minor adjustments, much of the constructions and results above to the study of k-step filtrations in tannakian (in fact, aside from the results about the totally nonsplit case, abelian) categories, even if the filtrations do not come from a functorial filtration. The standard

[4] This characterization of $S_\ell(A)$ will be equivalent to taking $S_\ell(A)$ to be the quotient of $S'_\ell(A)$ by $Aut(A)$.

[5] In the later parts of the paper we will also introduce a stronger notion of total nonsplitting for elements of $S_\ell(A)$.

hypotheses that one needs are that $Hom(A_j, A_i) = 0$ whenever $i < j$. (These hypotheses also appeared in [11] and [36].)

Map of the proof of Theorem 1.2.1: In §3.1 we briefly recall the definitions of $S'(A)$ and $S(A)$ and describe the action of $Aut(A)$ on $S'(A)$. The machinery of generalized extensions of A of a given level is developed in §3.2 - §3.3. This is used in §3.4 to define the sets $S'_\ell(A)$ and $S_\ell(A)$ and the maps Θ_ℓ. The characterization of the sets $S'_\ell(A)$ and $S'_\ell(A)$ in levels $\ell = 1$ and $\ell = k-1$ asserted in parts (a) and (d) of the theorem is established in §3.5. The remaining assertions are addressed in the remainder of §3. Parts (b) and (c) are the subject of §3.6-3.7. Part (e) is the subject of §3.8. Finally, in §3.9 we define the notion of weakly totally nonsplit elements of $S_\ell(A)$ and establish part (f) of the theorem.

1.2.2. *Contents of §4: Application to mixed motives with maximal unipotent radicals.* We now assume that $\mathbf{T}$ is a filtered tannakian category over a field K of characteristic 0 such that the pure objects of $\mathbf{T}$ are semisimple. The prototype examples are the category of graded-polarizable mixed Hodge structures over $\mathbb{Q}$ and any reasonable tannakian category of mixed motives over subfields of $\mathbb{C}$, e.g. those of Ayoub [3] and Nori [31], or those of Deligne [18] and Jannsen [32] defined earlier using realizations, or Voevodsky's category of mixed Tate motives over a number field. In fact, inspired by these prototype examples, we refer to the objects of $\mathbf{T}$ as motives, even though aside from §4.5 the discussion is valid in the generality of $\mathbf{T}$ described above. We note that when $\mathbf{T}$ is the category of graded-polarizable rational mixed Hodge structures or a reasonable category of mixed motives over a number field, it is either known or expected that the Ext^2 groups all vanish[6], so that the truncation maps in Theorem 1.2.1 are or should be all surjective.

Let X be a motive. One has a canonical subobject $\underline{\mathfrak{u}}(X)$ of $W_{-1}\underline{End}(X)$ (where $\underline{End}(X)$ is the internal Hom $\underline{Hom}(X, X)$) associated with the Lie algebra of the unipotent radical of the tannakian group of X (see §4.1 for a brief review). The object $\underline{\mathfrak{u}}(X)$ has been studied in various contexts by many, including Deligne ([18] and [34, Appendix]), André [1], Bertrand [10], Bertolin ([6] and [7]), Hardouin ([29] and [30]), Jossen [34], and the author and Murty ([21] and [22]).

Let us say that $\underline{\mathfrak{u}}(X)$ is maximal[7], or that X has a maximal unipotent radical, if $\underline{\mathfrak{u}}(X) = W_{-1}\underline{End}(X)$. It is easy to see that if X has a maximal unipotent radical, then so does each of its subquotients. In [22, Theorem 6.3.1], with Murty we gave sufficient conditions, which we called *independence axioms*, under which (at least in some situations) if W_pX and $X/W_{p-1}X$ have maximal unipotent radicals, then so does X.[8] This allowed us in §6.4-6.7 of the same article to give a homological classification of the isomorphism classes of motives X with maximal unipotent radicals and an associated graded isomorphic to

$$A_1 \oplus A_2 \oplus \mathbb{1},$$

where A_1 and A_2 are pure of negative weights $p_1 < p_2$, $Ext^1(\mathbb{1}, A_1) = 0$, and $A_1 \oplus A_2 \oplus \mathbb{1}$ satisfies the following independence axiom: A_2 and $\underline{Hom}(A_2, A_1)$ have no nonzero isomorphic subobjects.

Broadly speaking, an independence axiom in this context is a property that guarantees that $Gr^W\underline{\mathfrak{u}}(X)$ decomposes according to the decomposition of

$$Gr^W W_{-1}\underline{End}(X) = \bigoplus_{i<j} \underline{Hom}(Gr^W_j X, Gr^W_i X)$$

[6] This is known for the categories of mixed Hodge structures [5] and mixed Tate motives over a number field [19]. It is expected for the category of mixed motives over a number field. See for instance, [35] or [33].

[7] In [22] instead of "maximal" we used the word "large" for this.

[8] The statement becomes false if we remove the independence axioms from the hypotheses; see §6.3 of [22].

given by a suitable partition of the set $\{(i,j)\in\mathbb{Z}^2 : i<j\}$. In [22], our independence axioms simplified the relationship between the extensions of the form

$$0 \longrightarrow W_pX \longrightarrow X \longrightarrow X/W_pX \longrightarrow 0 \tag{6}$$

and $\underline{\mathfrak{u}}(X)$, thereby refining a result of Deligne [34, Proposition A.3] regarding this relationship. This refinement, stated as Corollaries 5.3.2 and 5.3.3 of [22], was one of the main ingredients of the maximality criterion of [22, Theorem 6.3.1] and hence the classification result mentioned earlier.

The aim of §4 of the present article is to generalize the classification result of [22, §6] regarding motives with maximal unipotent radicals from the case of 3 weights to an arbitrary number of weights. We also no longer need to assume that the graded piece with the highest weight is $\mathbb{1}$, or assume anything about the Ext groups between the different graded pieces (compare with the hypotheses of the classification result in the case of 3 weights in [22]). After reviewing some background and basic observations, in §4.3 we define the notion of graded-independence (Definition 4.3.1), which is an independence axiom in the above sense; in fact, in the case where the associated graded is $A_1\oplus A_2\oplus\mathbb{1}$ it becomes exactly the independence axiom mentioned earlier. We then give a simple criterion for maximality of the unipotent radical of a graded-independent motive (Theorem 4.3.2): it turns out that a graded-independent motive X with weights $p_1<\cdots<p_k$ has a maximal unipotent radical if and only if each of the extensions

$$0 \longrightarrow Gr^W_{p_{j-1}}X \longrightarrow W_{p_j}X/W_{p_{j-2}}X \longrightarrow Gr^W_{p_j}X \longrightarrow 0$$

is totally nonsplit. For context, note that without the graded-independence hypothesis, even total nonsplitting of all of the extensions

$$0\to W_mX/W_\ell X\to W_nX/W_\ell X\to W_nX/W_mX\to 0 \qquad (\ell<m<n)$$

is not enough to guarantee maximality of $\underline{\mathfrak{u}}(X)$ (see Lemma 4.2.1(c)).

Assume as before that $A=\bigoplus_{1\le r\le k}A_r$ with the A_r nonzero pure and in an increasing order of weights. Let $S^*(A)$ be the subset of $S(A)$ consisting of the isomorphism classes of motives with maximal unipotent radicals. Let $S^*_1(A)$ be the subset of

$$S_1(A)\cong\Big(\prod_r Ext^1(A_{r+1},A_r)\Big)/\,Aut(A)$$

(see Theorem 1.2.1) consisting of the $Aut(A)$-orbits of tuples in which every entry is totally nonsplit. Combining the maximality criterion of Theorem 4.3.2 with the totally nonsplit case of Theorem 1.2.1, we obtain that when A is graded-independent, there exist naturally defined sets $S^*_\ell(A)$ for $2\le\ell\le k-1$ (defined as suitable subsets of the $S_\ell(A)$) and maps

$$S^*(A)\cong S^*_{k-1}(A)\to S^*_{k-2}(A)\to S^*_{k-3}(A)\to\cdots\to S^*_2(A)\to S^*_1(A) \tag{7}$$

(restrictions of the maps of Theorem 1.2.1(d)) such that every nonempty fiber of $S^*_\ell(A)\to S^*_{\ell-1}(A)$ is a torsor over

$$\prod_r Ext^1(A_{r+\ell},A_r).$$

If the Ext^2 groups of Theorem 1.2.1(c) vanish in $\mathbf{T}$, then the map $S^*_\ell(A)\to S^*_{\ell-1}(A)$ above is surjective. These are recorded as Theorem 4.4.4 in the text. The special case of this result when $k=3$, $A_3=\mathbb{1}$ and $Ext^1(\mathbb{1},A_1)=0$ was proved in §6 (see in particular, §6.7) of [22]. (An interesting feature of the construction is that the fiber of $S_\ell(A)\to S_{\ell-1}(A)$ above an element of $S^*_{\ell-1}(A)$ is entirely in $S^*_\ell(A)$, so that the fibers of (7) have the said algebraic structure.)

To illustrate our results, in [22, §6.8] with Murty we gave a classification of isomorphism classes of 3-dimensional graded-independent mixed Tate motives over $\mathbb{Q}$ with 3 weights and maximal unipotent radicals. This classification led to some questions about periods. In §4.5 of the present paper, as an example to illustrate Theorem 4.4.4 in a case with more than 3 weights we consider the analogous problem for graded-independent 4-dimensional mixed Tate motives over $\mathbb{Q}$. More explicitly, we give a classification up to isomorphism of all mixed Tate motives over $\mathbb{Q}$ with maximal unipotent radicals and associated graded isomorphic to

$$\mathbb{Q}(a+b+c)\oplus\mathbb{Q}(a+b)\oplus\mathbb{Q}(a)\oplus\mathbb{1},$$

where a, b, c are distinct positive integers, $c \neq a+b$ and $a \neq b+c$ (the latter two conditions together with the distinctness of a, b, c being the graded-independence conditions in this situation). This leads to more questions about periods.

The questions about the periods become particularly interesting when $1 \in \{a, b, c\}$, in which case our motives will have a Kummer motive as a subquotient. Let r be a squarefree integer > 1 and L_r the Kummer motive for $\log(r)$, i.e. an extension of $\mathbb{1}$ by $\mathbb{Q}(1)$ with $(2\pi i)^{-1}\log(r)$ as a period (geometrically, L_r can be thought of as the relative homology $H_1(\mathbb{G}_m, \{1, r\})$). For each odd integer $n \geq 3$ let Z_n be the motive that sits as the middle object of nonsplit extensions of $\mathbb{1}$ by $\mathbb{Q}(n)$ in the category of mixed Tate motives over $\mathbb{Q}$. Then Z_n is unique up to isomorphism and has $(2\pi i)^{-n}\zeta(n)$ as a period. In view of the known description of Ext groups in the category of mixed Tate motives over $\mathbb{Q}$ (thanks to Voevodsky and Borel), it follows from the results of this paper that given any two distinct odd integers $b, c \geq 3$, the set of isomorphism classes of mixed Tate motives X over $\mathbb{Q}$ with

$$Gr^W X \simeq \mathbb{Q}(1+b+c)\oplus\mathbb{Q}(1+b)\oplus\mathbb{Q}(1)\oplus\mathbb{1} \tag{8}$$

$$X/W_{-3}X \simeq L_r, \qquad W_{-1}X/W_{-2-2b-1}X \simeq Z_b(1), \qquad W_{-2-2b}X \simeq Z_c(1+b)$$

is a torsor over

$$Ext^1(\mathbb{1}, \mathbb{Q}(1+b+c)) \simeq \mathbb{Q}.$$

Moreover, it follows that these motives all have maximal unipotent radicals. Any such X is unramified outside r, i.e. belongs to the category of mixed Tate motives over $\mathbb{Z}[1/r]$, and with respect to suitable bases of Betti and de Rham realizations has a period matrix of the form

$$\begin{pmatrix} (2\pi i)^{-1-b-c} & (2\pi i)^{-1-b-c}\zeta(c) & (2\pi i)^{-1-b-c}z_{c,b} & (2\pi i)^{-1-b-c}\lambda_{c,b,r}(X) \\ & (2\pi i)^{-1-b} & (2\pi i)^{-1-b}\zeta(b) & (2\pi i)^{-1-b}p'_{b,r} \\ & & (2\pi i)^{-1} & (2\pi i)^{-1}\log(r) \\ & & & 1 \end{pmatrix},$$

where the (1,3) and (2,4) entries can be made to be fixed (i.e. independent of X). By the maximality of its unipotent radical, the motivic Galois group of X has dimension 7. Thus Grothendieck's period conjecture predicts that the numbers π, $\zeta(c)$, $\zeta(b)$, $\log(r)$, $z_{c,b}$, $p'_{b,r}$, and $\lambda_{c,b,r}(X)$ (for a fixed X) are algebraically independent.

Since $W_{-2}X$ is a mixed Tate motive over $\mathbb{Z}$, by Brown's work [12] $z_{c,b}$ is in the algebra generated by multiple zeta values and $1/(2\pi i)$. Unless $r \in \{2, 3, 6\}$, we do not know the nature of $p'_{b,r}$ (= the "new period" of the motive $M_{1+b,r}$ in the notation of [22, §6.8]) and $\lambda_{c,b,r}(X)$. For $r \in \{2, 3, 6\}$, one has Deligne's work [20] for the full subcategory of the category of mixed Tate motives over $\mathbb{Q}(\mu_r)$ consisting of the motives unramified outside r. It follows that for these values of r, the unknown periods are generated by the periods of the fundamental group of $\mathbb{G}_m - \mu_r$ (i.e. generated by $1/(2\pi i)$ and cyclotomic multiple zeta values, see [19]). For more general r, as far as the author knows, little is understood about the periods of the category of mixed Tate motives over $\mathbb{Z}[1/r]$. We refer the reader to [15, §3] and the references therein (in particular, [25]) for a nice discussion of this and some conjectures.

The period $\lambda_{c,b,r}(X)$ offers an extra layer of mystery, thanks to its dependence on X. Since

$$Ext^1(\mathbb{1}, \mathbb{Q}(1+b+c))$$

is generated by the motive of $\zeta(1+b+c)$, it will be very interesting to understand if $\zeta(1+b+c)$ plays a role in how $\lambda_{c,b,r}(X)$ changes as X varies in a set of representatives of the isomorphism classes. Is there a relation between these periods that reflects the role that Z_{1+b+c} plays for the set of isomorphism classes of X via the torsor structure?

In an appendix to the paper, we prove another result about graded-independent motives with maximal unipotent radicals that is tangential to the rest of the paper, but adds to the discussion of periods. Going back to the generality of filtered tannakian categories with semisimple pure objects and assuming $A = \bigoplus_{1\leq r\leq k} A_r$ (the notation as in earlier) is graded-independent, we show in Theorem A.1 that for any object X for which the unipotent radical is maximal and $Gr^W X$ is isomorphic to A, the Ext^1 groups

$$Ext^1_{\langle X\rangle}(A_j, A_i) \qquad (j-i\geq 2)$$

for the tannakian subcategory $\langle X\rangle$ generated by X all vanish.

Going back to the example of the isomorphism classes of mixed Tate motives X over $\mathbb{Q}$ satisfying (8), we thus see that in fact, for any such X one has

$$Ext^1_{\langle X\rangle}(\mathbb{1}, \mathbb{Q}(1+b+c)) = 0.$$

From this one easily concludes (see §4.5) that assuming Grothendieck's period conjecture, $\zeta(1+b+c)$ is algebraically independent from the 7 numbers π, $\zeta(c)$, $\ldots$, $\lambda_{c,b,r}(X)$ listed above.

1.3. Conventions. Here we make some comments about the conventions and notations of the paper. A tannakian category will always be assumed to be a neutral one, i.e. one for which a fiber functor with values in the base field exists. By the term "filtered tannakian category" we mean a tannakian category equipped with a filtration $W_\bullet$ (called the weight filtration) as described in the beginning of §1.2.1. By a weight of an object X of a filtered tannakian category we mean an integer n such that $W_{n-1}X \neq W_n X$.

For the purposes of this paper it is important to carefully distinguish between some terms that are sometimes abused in the literature. By an extension we mean a 1-extension, i.e. a short exact sequence. We will have to distinguish between an extension and the object that sits in its middle. We will often use upper case English letters in script font (e.g. $\mathcal{N}$, $\mathcal{X}$) for an extension or its class in the Ext^1 group (which of the two is the intended interpretation will be clear from the context or explicitly mentioned), and use upper case English letters in ordinary font (e.g. N, X) for objects of $\mathbf{T}$. The notation $EXT^i(X,Y)$ (or simply $EXT(X,Y)$ when $i=1$) will be used for the collection of i-extensions (or simply, extensions when $i=1$) of X by Y. As usual, $Ext^i(X,Y)$ denotes the group (or vector space, since our categories are tannakian) of the equivalence classes of i-extensions with respect to the standard equivalence given by isomorphism of i-extensions, i.e. commuting morphisms between the middle objects that induce identity on X and Y. Note that throughout, our notations for Ext and Hom groups as well as for the proper classes EXT do not include a mention of the category under consideration (simply denoting these by Ext^i, Hom, etc). In a few occasions where the category is not clear from the context, it will be included in the notation as a subscript (e.g. as in $Ext^1_{\langle X\rangle}$).

Finally, internal Homs are denoted by $\underline{Hom}$ and our group actions are always designed to be left actions.

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2. Background

2.1. Recollections on blended extensions. In this section we recall some of the basics of the theory of blended extensions. The original reference is §9.3 of Grothendieck's [27]. Another excellent reference is Bertrand's [11].

Let $\mathbf{T}$ be an abelian category. Let A_1, A_2, and A_3 be objects of T. Fix two extensions

$$\mathscr{L}: \qquad 0 \longrightarrow A_1 \longrightarrow L \longrightarrow A_2 \longrightarrow 0$$

$$\mathscr{N}: \qquad 0 \longrightarrow A_2 \longrightarrow N \longrightarrow A_3 \longrightarrow 0$$

in $\mathbf{T}$. A blended extension of $\mathscr{N}$ by $\mathscr{L}$ by definition is a diagram of the form

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\ & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{9}$$

where the rows and columns are exact. The collection of all blended extensions of $\mathscr{N}$ by $\mathscr{L}$ is denoted by $EXTPAN(\mathscr{N},\mathscr{L})$ (for *extension panachées*). We will refer to the object X in the diagram as the middle object.

The standard notion of a morphism of blended extensions of $\mathscr{N}$ by $\mathscr{L}$ is a morphism in $\mathbf{T}$ between the middle objects which induces identity on L and N (and hence on A_1, A_2, and A_3). Via this notion of morphisms, $EXTPAN(\mathscr{N},\mathscr{L})$ is a category in which every morphism is an isomorphism (i.e. is a groupoid category). The collection of isomorphism classes of blended extensions of $\mathscr{N}$ by $\mathscr{L}$ is denoted by $Extpan(\mathscr{N},\mathscr{L})$.

We recall three basic results about blended extensions here, which together form the contents of Proposition 9.3.8 of [27]. The first is that when $Extpan(\mathscr{N},\mathscr{L})$ is nonempty, it has a natural structure of a torsor over $Ext^1(A_3,A_1)$.[9] The action of $Ext^1(A_3,A_1)$ on $Extpan(\mathscr{N},\mathscr{L})$ can be described as follows. Denote the map $N \twoheadrightarrow A_3$ in $\mathscr{N}$ by ω. Let $\mathfrak{X} \in EXTPAN(\mathscr{N},\mathscr{L})$ be the blended extension (9). Let $\mathfrak{X}^h \in EXT(N,A_1)$ be its second row. Given an element $\mathscr{E} \in EXT(A_3,A_1)$, consider the Baer sum

$$\mathfrak{X}^h + \omega^*\mathscr{E} \in EXT(N,A_1).$$

There is a canonical map from L to the middle object of $\mathfrak{X}^h+\omega^*\mathscr{E}$ and a canonical map from this middle object to A_3, and these make $\mathfrak{X}^h+\omega^*\mathscr{E}$ the second row of an element of $EXTPAN(\mathscr{N},\mathscr{L})$.

[9] In fact, one can do this at the level of the categories and make $EXTPAN(\mathscr{N},\mathscr{L})$ a torsor over $EXT(A_3,A_1)$. See page 105 of [27].

Denote this element by $\mathscr{E} * \mathscr{X}$ (we may call it the translation of $\mathscr{X}$ by $\mathscr{E}$). One can check that the map

$$EXT(A_3, A_1) \times EXTPAN(\mathscr{N}, \mathscr{L}) \to EXTPAN(\mathscr{N}, \mathscr{L}) \qquad (\mathscr{E}, \mathscr{X}) \mapsto \mathscr{E} * \mathscr{X}$$

descends to a map

$$Ext^1(A_3, A_1) \times Extpan(\mathscr{N}, \mathscr{L}) \to Extpan(\mathscr{N}, \mathscr{L}). \tag{10}$$

When $Extpan(\mathscr{N}, \mathscr{L})$ is nonempty, the map above makes it a torsor over $Ext^1(A_3, A_1)$. We also use the symbol $*$ for the descended action.

Of course, there is an alternative way to try to define the translation of $\mathscr{X}$ by $\mathscr{E}$, namely by pushing an extension of A_3 by A_1 forward along the injection $A_1 \hookrightarrow L$ in $\mathscr{L}$, and then adding it in $EXT(A_3, L)$ to the first vertical extension in $\mathscr{X}$ (in fact, this is the original construction given in [27]). Bertrand [11, Appendix] has checked that the two constructions coincide after passing to the level of equivalence classes (10).

For referencing purposes we record the other two basic results as a lemma below. Before stating the lemma, recall the Yoneda product

$$EXT(A_2, A_1) \times EXT(A_3, A_2) \to EXT^2(A_3, A_1)$$

given by splicing. With $\mathscr{L}$ and $\mathscr{N}$ as above, the Yoneda product of $\mathscr{L}$ and $\mathscr{N}$ is given by

$$0 \longrightarrow A_1 \longrightarrow L \longrightarrow N \longrightarrow A_3 \longrightarrow 0$$

where the maps $A_1 \to L$ and $N \to A_3$ come from $\mathscr{L}$ and $\mathscr{N}$, and $L \to N$ is the composition of the map $L \to A_2$ of $\mathscr{L}$ with the map $A_2 \to N$ of $\mathscr{N}$. (See [37, §3].)

Lemma 2.1.1. *(a) Given $\mathscr{N} \in EXT(A_3, A_2)$ and $\mathscr{L} \in EXT(A_2, A_1)$, there exists a blended extension of $\mathscr{N}$ by $\mathscr{L}$ if and only if the Yoneda product of $\mathscr{L}$ and $\mathscr{N}$ vanishes in $Ext^2(A_3, A_1)$.*
(b) The automorphism group of a blended extension of $\mathscr{N}$ by $\mathscr{L}$ is in a canonical bijection with $Hom(A_3, A_1)$.

For proofs, see Proposition 9.3.8(a,c) of [27] (or Lemma 6.4.1 of [22] for part (a)). Note that throughout the paper, we shall only deal with blended extensions for which

$$Hom(A_3, A_1) \cong 0,$$

so that they always have a trivial automorphism group.

We end this subsection with a remark about various equivalence relations for blended extensions and an observation. Throughout the paper, by the standard equivalence relation on blended extensions of $\mathscr{N}$ by $\mathscr{L}$ we mean the one considered above (where two blended extensions are considered equivalent if there exists a morphism between their middle objects that induces identity on the first rows and second columns), and the notation $Extpan(\mathscr{N}, \mathscr{L})$ is always used in reference to this relation. There are however two coarser equivalence relations that one may alternatively consider. In the first alternative relation, two blended extensions of $\mathscr{N}$ by $\mathscr{L}$ are considered equivalent if there is a morphism between the middle objects that induces morphisms on the analogous objects of the two diagrams such that the induced morphisms on A_1, A_2 and A_3 (but not necessarily on L and N) are identity. In the second alternative relation, which is the coarsest of all three equivalence relations, one declares two blended extensions to be equivalent if there exists a morphism between the middle objects that induces isomorphisms between the analogous objects of the two diagrams.

We should note that in the paper, our blended extensions will also always satisfy

$$Hom(A_2, A_1) \cong Hom(A_3, A_2) \cong 0. \tag{11}$$

In this case, the only automorphism of L (resp. N) that induces identity and A_2 and A_1 (resp. A_3) is the identity, so that the standard notion of equivalence on $EXTPAN(\mathscr{N}, \mathscr{L})$ coincides with the one only requiring morphisms to be identity on the A_i. Later in the paper, when we introduce the notion of generalized extensions, the equivalence that requires inducing identity on the A_i is denoted by $\sim'$, whereas the one that allows for arbitrary automorphisms of the A_i is denoted by $\sim$.[10]

Finally, we make a further observation about blended extensions in the case where (11) holds. In general, without assuming (11), thanks to the fact that isomorphisms in $EXTPAN(\mathscr{N}, \mathscr{L})$ are identity on $\mathscr{N}$ and $\mathscr{L}$, by sending the isomorphism class of a blended extension to the class of the second row of a representative we have a well-defined map

$$-^h : Extpan(\mathscr{N}, \mathscr{L}) \to Ext^1(N, A_1).$$

As before, let ω be the map $N \twoheadrightarrow A_3$ in $\mathscr{N}$. Let ι be the map $A_2 \hookrightarrow N$. Given $\mathfrak{X} \in Extpan(\mathscr{N}, \mathscr{L})$ and $\mathscr{E} \in Ext^1(A_3, A_1)$, by definition of the torsor structure on $Extpan(\mathscr{N}, \mathscr{L})$ we have

$$(\mathscr{E} * \mathfrak{X})^h = \omega^* \mathscr{E} + \mathfrak{X}^h.$$

Thus $(\mathscr{E} * \mathfrak{X})^h = \mathfrak{X}^h$ in $Ext^1(N, A_1)$ if and only if

$$\mathscr{E} \in \ker\Big(Ext^1(A_3, A_1) \xrightarrow{\omega^*} Ext^1(N, A_1) \Big) = \ Im \Big(Hom(A_2, A_1) \to Ext^1(A_3, A_1) \Big),$$

where the latter map is the connecting homomorphism in the long exact sequence obtained by applying $Hom(-, A_1)$ to $\mathscr{N}$. Since the action of $Ext^1(A_3, A_1)$ on $Extpan(\mathscr{N}, \mathscr{L})$ is transitive, we obtain the following statement:

Lemma 2.1.2. *Suppose that $Hom(A_2, A_1) \cong 0$. Then the map*

$$Extpan(\mathscr{N}, \mathscr{L}) \to Ext^1(N, A_1)$$

which sends the class of a blended extension to the class of its second row is injective.

2.2. Totally nonsplit extensions. In this subsection we recall the notion of a totally nonsplit extension. This notion, which as far as the author knows is due to Bertrand [10], will play a crucial role in §3.9 and §4.

Assume that $\mathbf{T}$ is a tannakian category over a field of characteristic zero.

Definition 2.2.1. Let X and Y be objects of $\mathbf{T}$.
(a) An extension or an extension class $\mathscr{E}$ of $\mathbb{1}$ by X is called totally nonsplit if for every proper subobject X' of X the pushforward of $\mathscr{E}$ along the quotient map $X \to X/X'$ is nonsplit.
(b) An extension or an extension class $\mathscr{E}$ of Y by X is called totally nonsplit if the extension class of $\mathbb{1}$ by $\underline{Hom}(Y, X)$ corresponding to $\mathscr{E}$ under the canonical isomorphism

$$Ext^1(Y, X) \cong Ext^1(\mathbb{1}, \underline{Hom}(Y, X)) \tag{12}$$

is totally nonsplit.

We first make a remark about the special case of the definition about extensions of $\mathbb{1}$ by X. In view of the long exact sequence obtained by applying the functor $Hom(\mathbb{1}, -)$ to the sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X/X' \longrightarrow 0,$$

[10] To avoid confusion, the reader is warned that in our definition of morphisms of generalized extensions (to be given in §3.2) we will only require commutativity of diagrams. The morphisms on the A_i will be allowed to be arbitrary (even zero).

an extension (or extension class) $\mathscr{E}$ of $\mathbb{1}$ by X is totally nonsplit if and only if for every proper subobject X' of X, the extension class of $\mathscr{E}$ is not in the image of the pushforward map

$$Ext^1(\mathbb{1}, X') \to Ext^1(\mathbb{1}, X).$$

We also make a cautionary remark about the general case of the definition, when $\mathscr{E}$ is an extension of Y by X. The reader should keep in mind that in this definition, for $\mathscr{E}$ to be totally nonsplit we need to first consider $\mathscr{E}$ as an extension of $\mathbb{1}$. The notion would remain the same if we considered the extension as an extension by $\mathbb{1}$ (with the statements being dualized). However, the more naively defined notion in which given an extension $\mathscr{E}$ of Y by X one only considers quotients by subobjects of X and pullbacks to subobjects of Y is not as well behaved. In any case, we do not work with that weaker notion.

We end this subsection by recalling an important property of totally nonsplit extensions, which follows immediately from [23, Theorem 1.2].

Lemma 2.2.2. *Suppose that* $\mathbf{T}$ *is a filtered tannakian category over a field of characteristic zero. Let* X *and* Y *be two nonzero objects of* $\mathbf{T}$*. Let* $\mathscr{E}$ *be a totally nonsplit extension of* Y *by* X*, with its middle object denoted by* E*. Suppose that every weight of* X *is less than every weight of* Y*. Then the only endomorphisms of* E *are the scalar maps.*

3. Objects with a prescribed associated graded

3.1. Two classification problems. From this point on, $\mathbf{T}$ is a filtered tannakian category over a field K of characteristic zero. Let $A_1, \dots, A_k$ be nonzero pure objects of $\mathbf{T}$, respectively of weights $p_1, \dots, p_k$ with $p_1 < \cdots < p_k$. Set

$$A := \bigoplus_{1 \le r \le k} A_r.$$

In this section we study the sets $S(A)$ and $S'(A)$ introduced in §1.2.1. Let us recall the definitions.

Definition 3.1.1. (a) We denote by $S(A)$ the set of isomorphism classes of objects of $\mathbf{T}$ whose associated graded (with respect to the weight filtration) is isomorphic to A:

$$S(A) := \ \{X \in \mathrm{Obj}(\mathbf{T}) : Gr^W X \text{ is isomorphic to } A\} \big/ \text{ isomorphism in } \mathbf{T}.$$

Note that here, we do not keep track of the data of the isomorphisms between the associated gradeds and A.
(b) We denote by $S'(A)$ the set of equivalence classes of pairs

$$(X,\, Gr^W X \xrightarrow{\phi, \simeq} A)$$

of an object X of $\mathbf{T}$ whose associated graded is isomorphic to A together with a choice of an isomorphism $\phi : Gr^W X \to A$. Here, two pairs (X, ϕ) and (X', ϕ') are declared to be equivalent if there exists an isomorphism $f : X \to X'$ for which we have $\phi' \, Gr^W f = \phi$, where $Gr^W f : Gr^W X \to Gr^W X'$ is the isomorphism induced by f.

The group $Aut(A)$ acts on $S'(A)$ by twisting the isomorphism between the associated graded and A. More precisely, given a pair (X, ϕ) as in (b) and $\sigma \in Aut(A)$, we set

$$\sigma \cdot (X, \phi) = (X, \sigma\phi).$$

This defines an action of $Aut(A)$ on the collection of pairs (X, ϕ) as in (b) which is easily seen to descend to an action on $S'(A)$.

There is an obvious surjection

$$S'(A) \twoheadrightarrow S(A)$$

induced by forgetting the data of ϕ. The reader can easily see that two elements of $S'(A)$ are mapped to the same element of $S(A)$ if and only if they belong to the same orbit of $Aut(A)$. For future referencing, we record the conclusion:

Lemma 3.1.2. *The natural surjection $S'(A) \twoheadrightarrow S(A)$ induced by $(X,\phi) \mapsto X$ descends to a bijection*

$$S'(A)/Aut(A) \cong S(A).$$

Our goal in this section is to prove Theorem 1.2.1. As it was explained in §1.2.1, the constructions rely on the concept of generalized extensions of A of various levels. This concept is the subject of the next two subsections. The remaining subsections contain the proof of Theorem 1.2.1. We refer the reader to the end of §1.2.1 for a map of the argument.

3.2. Generalized extensions - Definitions. In this subsection we define our notion of generalized extensions of a given level. This notion will be the key to our approach to the classification problems of interest in the paper.

From this point on in this section, we fix

$$A = \bigoplus_{1\leq r\leq k} A_r$$

as in §3.1. That is, for each $1 \leq r \leq k$, the object A_r is nonzero pure of weight p_r and $p_1 < \cdots < p_k$. For now, we assume $k \geq 2$, but the real case of interest is when $k \geq 3$.

As mentioned in §1.2.1, given any pair (X,ϕ) of an object X of $\mathbf{T}$ and an isomorphism $\phi : Gr^W X \to A$, setting $X_{m,n} = W_{p_n}X/W_{p_m}X$ for $0 \leq m < n \leq k$ with $p_0 := p_1 - 1$, the natural inclusions and quotient maps among the $X_{m,n}$ give rise to the diagram (5). The following definition, modelled based on this diagram, formalizes the situation.

Definition 3.2.1 (Generalized extensions of level $k-1$ of A)**.**
(a) By a generalized extension of level $k-1$ of A we mean the data of a collection of objects

$$(X_{m,n})_{0\leq m<n\leq k}$$

of $\mathbf{T}$ with $X_{r-1,r} = A_r$ for all $1 \leq r \leq k$, together with a surjective morphism $X_{m,n} \to X_{m+1,n}$ and an injective morphism $X_{m,n-1} \to X_{m,n}$ for every m, n in the eligible[11] ranges, such that the following axioms hold:

(i) Every diagram of the form

$$\begin{tikzcd} X_{m,n-1} \arrow[r, two heads] \arrow[d, hook] & X_{m+1,n-1} \arrow[d, hook] \\ X_{m,n} \arrow[r, two heads] & X_{m+1,n} \end{tikzcd} \tag{13}$$

(with the maps as in the given data) commutes.

(ii) The diagram

$$0 \longrightarrow X_{m,n-1} \longrightarrow X_{m,n} \longrightarrow A_n \longrightarrow 0 \tag{14}$$

is an exact sequence for every m, n in the eligible range. Here, the morphism $X_{m,n} \to A_n$ is the composition

$$X_{m,n} \twoheadrightarrow X_{m+1,n} \twoheadrightarrow X_{m+2,n} \twoheadrightarrow \cdots \twoheadrightarrow X_{n-1,n} = A_n.$$

[11] Here and elsewhere throughout, by the adjective *eligible* in the context of indices we mean the range in which the indices in question make sense. So here, for instance, we have an injective map $X_{m,n-1} \to X_{m,n}$ for every pair of integers (m,n) with $0 \leq m < n \leq k$ and $m < n-1$.

(b) The collection of all generalized extensions of level $k-1$ of A is denoted by $D_{k-1}(A)$.

The reason for including axiom (ii) in the definition is to make sure that the $X_{m,n}$ cannot be larger than what we like. With (ii) included as a requirement, one is guaranteed to also get exact sequences

$$0 \longrightarrow A_{m+1} \longrightarrow X_{m,n} \longrightarrow X_{m+1,n} \longrightarrow 0$$

(see Lemma 3.3.1).

Given an object X of $\mathbf{T}$ whose associated graded is isomorphic to A, choosing an isomorphism $\phi : Gr^W X \to A$ to identify the two, the subquotients $(X_n/X_m)_{0\leq m<n\leq k}$ with $X_r = W_{p_r}X$ together with the natural successive inclusion and projection maps between them form a generalized extension of level $k-1$ of A. We call this the generalized extension of A associated with (X, ϕ) and denote it by $ext(X, \phi)$.

In general, a generalized extension of level $k-1$ of A can be visualized by a diagram as in (5). We will simply speak of a generalized extension $(X_{m,n})_{0\leq m<n\leq k}$, or often merely $(X_{m,n})$ or $(X_{\bullet,\bullet})$ without including the arrows or range of indices in the notation. For simplicity and to save space we might sometimes drop the arrows even from our diagrams.

Note that while the definition of a generalized extension $(X_{\bullet,\bullet})$ only includes maps between objects in adjacent positions in the diagram, for every pairs (m, n) and (m', n') with $m' \geq m$ and $n' \geq n$ by composing the morphisms along any path from $X_{m,n}$ to $X_{m',n'}$ we get a map

$$X_{m,n} \to X_{m',n'}.$$

Commutativity of the diagram for $(X_{\bullet,\bullet})$ guarantees that the outcome does not depend on the choice of the path.

Example. Let $k = 3$. Then the data of a generalized extension of level 2 is the same as the data of a blended extension whose top row is an extension of A_2 by A_1 and whose right column is an extension of A_3 by A_2. Indeed, given a generalized extension $(X_{m,n})$ (with $0 \leq m < n \leq 3$), we have a blended extension

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 = X_{0,1} \arrow[r] \arrow[d, equal] & X_{0,2} \arrow[r] \arrow[d] & X_{1,2} = A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & X_{0,3} \arrow[r] \arrow[d] & X_{1,3} \arrow[r] \arrow[d] & 0 \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 = X_{2,3}, \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the maps are all compositions of the structure maps. The passage from blended extensions to generalized extensions is also clear from this.

Back to working with an arbitrary k, we crucially also need truncated versions of the notion, which only include the data of the top several (top left to bottom right) diagonals of (5).

Definition 3.2.2 (Generalized extensions of various levels of A)**.**
(a) Let $1 \leq \ell \leq k-1$. By a generalized extension of level ℓ of A we mean the data of an object $X_{m,n}$ of $\mathbf{T}$ for each pair (m, n) of integers with $0 \leq m < n \leq k$ and $n - m \leq \ell + 1$, with

$X_{r-1,r} = A_r$ for all $1 \leq r \leq k$, together with the data of a surjective morphism $X_{m,n} \to X_{m+1,n}$ and an injective morphism $X_{m,n-1} \to X_{m,n}$ for every m and n in the eligible range such that axioms (i) and (ii) of Definition 3.2.1(a) hold.
(b) The collection of all generalized extensions of level ℓ of A is denoted by $D_\ell(A)$. For convenience, we set $D_{k'}(A) = D_{k-1}(A)$ for $k' \geq k$.

Note that this definition agrees with the previous one in level $k-1$. We visualize a generalized extension of level ℓ (with $1 \leq \ell \leq k-1$) by a truncated version of (5), with ℓ diagonals below the diagonal consisting of the A_r.

Example. A generalized extension of level 1 of A can be visualized as a diagram of the form

$$\begin{matrix} A_1 & & & & & \\ X_{0,2} & A_2 & & & & \\ & X_{1,3} & A_3 & & & \\ & & \ddots & \ddots & & \\ & & & X_{k-3,k-1} & A_{k-1} & \\ & & & & X_{k-2,k} & A_k \end{matrix}$$

where the arrows, dropped from the writing for convenience, satisfy axiom (ii) of the definition. This is simply the data of $k-1$ extensions

$$0 \longrightarrow A_r \longrightarrow X_{r-1,r+1} \longrightarrow A_{r+1} \longrightarrow 0 \qquad (1 \leq r \leq k-1).$$

Note that these are short exact sequences, rather than elements of Ext^1, since we have not yet introduced any equivalence relations on $D_1(A)$.

We make each $D_\ell(A)$ $(1 \leq \ell \leq k-1)$ the collection of objects of a category by defining the notion of morphisms of generalized extensions as follows: Let $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ be generalized extensions of A of the same level. A morphism of generalized extensions $(X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is a collection of morphisms $f_{m,n} : X_{m,n} \to X'_{m,n}$ (one for each pair (m,n) in the eligible range) that commute with the structure morphisms of $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$; that is, such that each diagram below commutes for all eligible (m,n):

$$\begin{tikzcd} X_{m,n} \arrow[r, two heads] \arrow[d, "f_{m,n}"] & X_{m+1,n} \arrow[d, "f_{m+1,n}"] \\ X'_{m,n} \arrow[r, two heads] & X'_{m+1,n} \end{tikzcd} \qquad \begin{tikzcd} X_{m,n-1} \arrow[r, hook] \arrow[d, "f_{m,n-1}"] & X_{m,n} \arrow[d, "f_{m,n}"] \\ X'_{m,n-1} \arrow[r, hook] & X'_{m,n} \end{tikzcd} \tag{15}$$

Note that the morphisms

$$f_{r-1,r} : A_r \to A_r$$

here are *not* necessarily isomorphisms. With abuse of notation, the category of generalized extensions of level ℓ of A is also denoted by $D_\ell(A)$. A morphism $(f_{\bullet,\bullet})$ in $D_\ell(A)$ is an isomorphism if and only if all the $f_{m,n}$ are isomorphisms. (It will follow from Lemma 3.3.1 below that $(f_{\bullet,\bullet})$ is an isomorphism if and only if every $f_{r-1,r} : A_r \to A_r$ is an isomorphism.)

<u>Truncation and cropping functors</u>: One can naturally define two types of forgetful functors between categories of generalized extensions. The first are the functors

$$\Theta_\ell : D_\ell(A) \to D_{\ell-1}(A) \qquad (2 \leq \ell \leq k-1)$$

defined when $k \geq 3$ as follows: For each ℓ, the functor Θ_ℓ merely erases the lowest diagonal of each generalized extension of level ℓ (together with the arrows going to and coming from the lowest diagonal). Its action on morphisms of generalized extensions is by restricting the data to the part of the diagrams that survive. We refer to the functors Θ_ℓ as *truncation* functors.

The second are the functors that crop diagrams horizontally and vertically to include only the part between two particular graded pieces of A. Given integers i and j with $0 \leq i, j \leq k$ and $i+1 < j$, we have a functor

$$D_\ell(A) \to D_\ell\Big(\bigoplus_{i<r\leq j} A_r\Big) \tag{16}$$

which only keeps the part of diagrams that lie in the intersection of the columns between A_{i+1} and A_j inclusively, and the rows between A_{i+1} and A_j inclusively. The action on morphisms is again by restricting the data. We call the second type of functors *cropping* functors.

3.3. Basic properties of generalized extensions. In this section we gather some basic results about generalized extensions which will be used in the remainder of the paper. Throughout, unless otherwise indicated, a generalized extension means a generalized extension of A (with A as introduced earlier in §3.2). We visualize a generalized extension by a diagram of the form (5) (with general objects $X_{\bullet,\bullet}$ and arrows that form the data of a generalized extension) or truncated versions of it if the level is $< k-1$. The references in the text to "the lowest diagonal", "the above and right of an object", "entry (m,n)", etc. all refer to this visualization (with the object at entry (m,n) of $(X_{\bullet,\bullet})$ being $X_{m,n}$).

For any generalized extension $(X_{\bullet,\bullet})$ of any level, for convenience we set $X_{r,r} = 0$ for $0 \leq r \leq k$. To simplify the writing, as before, if there is no ambiguity we will often simply refer to "indices in the eligible range" or "eligible indices"; this simply means that the indices are in the range for which the objects in the equations are available.

Lemma 3.3.1. *Let $(X_{\bullet,\bullet})$ be a generalized extension of any level ℓ.*
(a) For every $m \leq r \leq n$ in the eligible range (depending on the level), we have

$$Im(X_{m,r} \hookrightarrow X_{m,n}) = W_{p_r} X_{m,n}.$$

That is, the weight filtration for each object of the diagram is given by the objects directly above it. (Note that in particular, the statement asserts that $W_{p_n} X_{m,n} = X_{m,n}$.)
(b) The isomorphisms

$$Gr^W X_{m,n} \cong \bigoplus_{m<r\leq n} A_r \tag{17}$$

given by

$$Gr^W X_{m,n} \overset{(a)}{\cong} \bigoplus_{m<r\leq n} \frac{X_{m,r}}{X_{m,r-1}} \overset{(14)}{\cong} \bigoplus_{m<r\leq n} A_r$$

for every (m,n) in the eligible range are compatible with the natural injective and surjective maps. That is, we have commutative diagrams

$$\begin{array}{ccc} Gr^W X_{m,n-1} & \hookrightarrow & Gr^W X_{m,n} \\ \| & & \| \\ \bigoplus_{m<r\leq n-1} A_r & \hookrightarrow & \bigoplus_{m<r\leq n} A_r \end{array} \quad \text{and} \quad \begin{array}{ccc} Gr^W X_{m,n} & \twoheadrightarrow & Gr^W X_{m+1,n} \\ \| & & \| \\ \bigoplus_{m<r\leq n} A_r & \twoheadrightarrow & \bigoplus_{m+1<r\leq n} A_r \end{array} \tag{18}$$

in which the top arrows are Gr^W applied to the structure arrows of $(X_{\bullet,\bullet})$ and the bottom arrows are the natural embedding and projection maps. The identifications shown as equality in the diagrams are given by the isomorphisms (17)*.*
(c) For each m, n in the eligible range, we have an exact sequence

$$0 \longrightarrow A_{m+1} \longrightarrow X_{m,n} \longrightarrow X_{m+1,n} \longrightarrow 0, \tag{19}$$

where the morphism $A_{m+1} \to X_{m,n}$ is the composition

$$A_{m+1} = X_{m,m+1} \hookrightarrow X_{m,m+2} \hookrightarrow X_{m,m+3} \hookrightarrow \cdots \hookrightarrow X_{m,n}.$$

(d) Let $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ be a morphism in $D_\ell(A)$. For each eligible (m, n), the canonical isomorphisms (17) *for $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ fit into a commutative diagram*

$$\begin{tikzcd}
Gr^W X_{m,n} \arrow[r, equal] \arrow[d, "Gr^W f_{m,n}"] & \bigoplus_{m<r\leq n} A_r \arrow[d, "(f_{r-1,r})"] \\
Gr^W X'_{m,n} \arrow[r, equal] & \bigoplus_{m<r\leq n} A_r.
\end{tikzcd}$$

Proof. (a) Fixing m, this is seen by induction on n in view of the extension (14). In the induction step, we first apply the exact functor $W_{p_{n-1}}$ to (14). Since $W_{p_{n-1}} A_n = 0$ we get the assertion for $r \leq n-1$. As for when $r = n$, this follows from exactness of W_{p_n} and the fact that $W_{p_n} A_n = A_n$.
(b) This follows from the construction of the canonical isomorphisms and the commutativity of the diagram of a generalized extension. We leave the details to the reader.
(c) The exactness of (19) is clear at the first and third object. As for at the middle, from part (a) we know that $W_{p_{m+1}} X_{m,n} = A_{m+1}$ and $W_{p_{m+1}} X_{m+1,n} = 0$. Applying $W_{p_{m+1}}$ to $X_{m,n} \twoheadrightarrow X_{m+1,n}$ we see that A_{m+1} is in the kernel of $X_{m,n} \twoheadrightarrow X_{m+1,n}$. On the other hand, by part (b) the dimension of the kernel of $X_{m,n} \twoheadrightarrow X_{m+1,n}$ is equal to the dimension of A_{m+1}.
(d) Let $m < r \leq n$. By definition of the canonical isomorphism (17) we have a commutative diagram

$$\begin{tikzcd}
 & A_r \arrow[dr, equal] & \\
X_{m,r} \arrow[ur, two heads] \arrow[rr, two heads] & & Gr^W_{p_r} X_{m,n}
\end{tikzcd}$$

where the horizontal surjective arrow is given by $X_{m,r} \hookrightarrow X_{m,n}$ (mapping $X_{m,r}$ isomorphically to $W_{p_r} X_{m,n}$) and then passing to $Gr^W_{p_r} X_{m,n}$, and the slanted surjective arrow is the composition of the surjective arrows $X_{m',r} \twoheadrightarrow X_{m'+1,r}$ for $m \leq m' < r-1$. The side donated by equality is the identification of (17). There is an analogous triangle for $(X'_{\bullet,\bullet})$. The two triangle can be put into a (to be seen to be commutative) diagram:

$$\begin{tikzcd}
 & A_r \arrow[dr, equal] \arrow[dd] & \\
X_{m,r} \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, "f_{m,r}"] & & Gr^W_{p_r} X_{m,n} \arrow[dd, "Gr^W_{p_r} f_{m,n}"] \\
 & A_r \arrow[dr, equal] & \\
X'_{m,r} \arrow[ur, two heads] \arrow[rr, two heads] & & Gr^W_{p_r} X'_{m,n}
\end{tikzcd} \tag{20}$$

where the map $A_r \to A_r$ on the top is $f_{r-1,r}$. The front and back (triangular) faces are commutative. The rectangular faces on the top left and the bottom are both commutative, the former (resp. latter) by compatibility of morphisms of generalized extensions with the surjective (resp. injective) structure arrows. It follows that the top right face is also commutative. □

Note that in particular, the previous lemma asserts that for every generalized extension $(X_{\bullet,\bullet})$ of level $k-1$ of A we have

$$Gr^W X_{0,n} \cong \bigoplus_{1\leq r\leq n} A_r$$

for all $1 \leq n \leq k$.

Part (d) of the previous lemma has the following consequence:

Lemma 3.3.2. *Let $(X_{\bullet,\bullet})$ be a generalized extension of level ℓ of A. The forgetful map*

$$Aut((X_{\bullet,\bullet})) \to Aut(A) = \prod_{1\leq r\leq k} Aut(A_r)$$

given by $(\sigma_{\bullet,\bullet}) \mapsto (\sigma_{r-1,r})$ is injective. (That is, every automorphism of $(X_{\bullet,\bullet})$ is uniquely determined by its action on A. Of course, the forgetful map above need not be surjective.)

Proof. Suppose $(\sigma_{\bullet,\bullet})$ is an automorphism of $(X_{\bullet,\bullet})$ which is identity on A. Applying Lemma 3.3.1(d) with $(X'_{\bullet,\bullet}) = (X_{\bullet,\bullet})$ and $(f_{\bullet,\bullet}) = (\sigma_{\bullet,\bullet})$ we obtain that each $Gr^W\sigma_{m,n}$ and hence $\sigma_{m,n}$ is identity. □

Before we proceed any further, let us introduce a notation:

Notation 3.3.3. Given a generalized extension $(X_{\bullet,\bullet})$ of A of any level, we denote the two extensions (14) and (19) respectively by $\mathfrak{X}^v_{m,n}$ and $\mathfrak{X}^h_{m,n}$.

The following two lemmas will be useful in constructing morphisms between generalized extensions. The first lemma asserts that given two generalized extensions $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ of the same level, every morphism from an object of $(X_{\bullet,\bullet})$ to an object of $(X'_{\bullet,\bullet})$ at the same entry extends (or spreads) uniquely to the part of the diagrams to the above and right of that entry.

Lemma 3.3.4. *Let $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ be generalized extensions of level ℓ of A.*
(a) Given any i,j in the eligible range with $j-i>1$ (so that $X_{i,j}$ is below the diagonal of the A_r) and a morphism $f : X_{i,j} \to X'_{i,j}$, there exists a unique collection of morphisms

$$f_{m,n} : X_{m,n} \to X'_{m,n} \qquad (i \leq m < n \leq j)$$

such that $f_{i,j} = f$ and the $f_{m,n}$ commute with the morphisms in $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$. (That is, such that the $f_{m,n}$ give a morphism between the parts of $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ between A_{i+1} and A_j. See (16).)
(b) The extension of f to $(f_{m,n})$ as above behaves well with respect to compositions: if $(X''_{\bullet,\bullet})$ is also a generalized extension of level ℓ of A and $f' : X'_{i,j} \to X''_{i,j}$ is a morphism, then

$$(f' \circ f)_{m,n} = f'_{m,n} \circ f_{m,n}$$

for all eligible m,n.

Proof. We may assume that the level is $k-1$ and $(i,j)=(0,k)$, as the seemingly more general statements follow from applying this case to the cropped generalized extensions (cropped in the sense of (16)).

By Lemma 3.3.1, for any $n \leq k$ we have $W_{p_n}X_{0,k} \cong X_{0,n}$ and $W_{p_n}X'_{0,k} \cong X'_{0,n}$ (identifications via the injective structure arrows of the generalized extensions). By functoriality of the weight filtration, the morphism f restricts to morphisms $f_{0,n} : X_{0,n} \to X'_{0,n}$ compatible with each other. So far, we have extended f to the first column of the two generalized extensions.

Assume f has been extended in the desired way to the $f_{m,n}$ for all $m < m'$. For each eligible n we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{m'} \arrow[r] \arrow[d, "f_{m'-1,m'}"] & X_{m'-1,n} \arrow[r] \arrow[d, "f_{m'-1,n}"] & X_{m',n} \arrow[r] & 0 \\
0 \arrow[r] & A_{m'} \arrow[r] & X'_{m'-1,n} \arrow[r] & X'_{m',n} \arrow[r] & 0
\end{tikzcd}$$

with rows $\mathfrak{X}^h_{m'-1,n}$ and $\mathfrak{X}'^h_{m'-1,n}$. We thus get a morphism $f_{m',n} : X_{m',n} \to X'_{m',n}$ making this diagram commute.

We have extended f to the column of entries (m', n) (with m' fixed), and the commutativity with the surjective arrows from the column for $m'-1$ to the one for m' is known by construction. We have a diagram

$$\begin{tikzcd}
& X_{m'-1,n-1} \arrow[rr, two heads] \arrow[dl, hook'] \arrow[dd] & & X_{m',n-1} \arrow[dl, hook'] \arrow[dd, "f"] \\
X_{m'-1,n} \arrow[rr, two heads, crossing over] \arrow[dd, "f"] & & X_{m',n} & \\
& X'_{m'-1,n-1} \arrow[rr, two heads] \arrow[dl, hook'] & & X'_{m',n-1} \arrow[dl, hook'] \\
X'_{m'-1,n} \arrow[rr, two heads] & & X'_{m',n} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the top (resp. bottom) face is a part of $(X_{\bullet,\bullet})$ (resp. $(X'_{\bullet,\bullet})$), and the downward maps are the maps induced by f. We know the top and bottom faces as well as the left, back and front faces are commutative. In view of the surjectivity of $X_{m'-1,n-1} \twoheadrightarrow X_{m',n-1}$ we get the commutativity of the right face. This completes the proof of the fact that f extends to a morphism of generalized extensions.

As for uniqueness, the map $f_{0,k}$ determines every map $f_{m,n}$ with $0 \leq m < n \leq k$ because of the commutativity requirements. Indeed, $f_{0,k}$ determines each $f_{0,n}$ and in turn, $f_{0,n}$ determines each $f_{m,n}$ by the commutativity of the following two diagrams:

$$\begin{tikzcd}
X_{0,n} \arrow[r, hook] \arrow[d, "f_{0,n}"] & X_{0,k} \arrow[d, "f_{0,k}"] & X_{0,n} \arrow[r, two heads] \arrow[d, "f_{0,n}"] & X_{m,n} \arrow[d, "f_{m,n}"] \\
X'_{0,n} \arrow[r, hook] & X'_{0,k} & X'_{0,n} \arrow[r, two heads] & X'_{m,n}
\end{tikzcd}$$

Part (b) is easily seen from the construction of the $f_{m,n}$ given above. □

The next lemma says that morphisms between the lowest diagonals of two generalized extensions of the same level glue together to give a morphism of generalized extensions if and only if they agree on the diagonal just above the lowest.

Lemma 3.3.5. *Suppose $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ are generalized extensions of level ℓ of A. Suppose that for each eligible pair (i, j) with $j-i = \ell+1$ (i.e. on the lowest diagonal) we have a morphism $f_{i,j} : X_{i,j} \to X'_{i,j}$. For each such (i, j), let $f^v_{i,j-1}$ and $f^h_{i+1,j}$ be the unique morphisms*[12] *fitting in*

[12]That these exist is by Lemma 3.3.1 and functoriality of the weight filtration.

the commutative diagrams

$$\begin{tikzcd} X_{i,j-1} \arrow[r, hook] \arrow[d, "f^v_{i,j-1}"] & X_{i,j} \arrow[d, "f_{i,j}"] \\ X'_{i,j-1} \arrow[r, hook] & X'_{i,j} \end{tikzcd} \qquad \begin{tikzcd} X_{i,j} \arrow[r, two heads] \arrow[d, "f_{i,j}"] & X_{i+1,j} \arrow[d, "f^h_{i+1,j}"] \\ X'_{i,j} \arrow[r, two heads] & X'_{i+1,j}. \end{tikzcd}$$

Then there exists a morphism of generalized extensions $(X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ extending the given $f_{i,j}$ from the lowest diagonals to the full diagrams if and only if for every eligible (m,n) with $n-m=\ell$ (i.e. on the diagonal just above the lowest), we have $f^v_{m,n} = f^h_{m,n}$. Moreover, when such an extension exists, it is unique. (In other words, if two morphisms of generalized extensions agree on the lowest diagonal, then the two morphisms are the same.)

Proof. The uniqueness is immediate from the uniqueness statement in Lemma 3.3.4. That the given condition is necessary follows from the same result and its proof. The new assertion here is that the compatibility condition on the diagonal just above the lowest is sufficient for the morphisms between the lowest diagonals to glue to make a morphism of generalized extensions.

By Lemma 3.3.4 the morphism $f_{0,\ell+1}$ extends to the right and top of entry $(0,\ell+1)$. Assume that the morphisms on the lowest diagonal glue all the way up to and including the entry (i,j) on the lowest diagonal, so that for all eligible pairs (m,n) of indices with $n \le j$ we already have morphisms $f_{m,n} : X_{m,n} \to X'_{m,n}$ commuting with the structure injections and surjections. We will argue that if $j \ne k$, we can also glue $f_{i+1,j+1}$ to the current data.

Consider the map $f_{i+1,j} : X_{i+1,j} \to X'_{i+1,j}$ (which is already available). Then $f_{i+1,j}$ is induced by $f_{i,j}$ and hence is $f^h_{i+1,j}$. By Lemma 3.3.4, $f_{i+1,j+1}$ extends uniquely to a morphism of generalized extensions between the parts of the diagrams above and to the right of entry $(i+1,j+1)$, in particular, inducing the map $f^v_{i+1,j} : X_{i+1,j} \to X'_{i+1,j}$. By the compatibility condition, $f^h_{i+1,j} = f^v_{i+1,j}$. Applying the uniqueness statement of Lemma 3.3.4 at entry $(i+1,j)$ it follows that the map induced by $f_{i+1,j+1}$ at every entry (m,n) with $m \ge i+1$ and $n \le j$ coincides with the map $f_{m,n}$ already there. Together with the new maps $f_{m,j+1}$ for $m \ge i+1$ induced by $f_{i+1,j+1}$ we have extended the maps between the diagrams one row further down.

There is nothing new to check for the commutativity with the structure injections and surjections: every square that needs to commute is already known to commute. □

3.4. Equivalence relations on generalized extensions. In this subsection we define the sets $S'_\ell(A)$ and $S_\ell(A)$ as well as the maps from level ℓ to $\ell-1$ in Theorem 1.2.1. Recall that for each integer ℓ with $1 \le \ell \le k-1$ the collection (as well as the category) of generalized extensions of level ℓ of A is denoted by $D_\ell(A)$. (As before, we may drop the phrase "of A" from the writing.)

There are two natural equivalence relations on each $D_\ell(A)$. The first is simply given by isomorphisms in the category $D_\ell(A)$, and the second is the finer equivalence given by isomorphisms that are identity on A.

Notation 3.4.1. Let $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ be generalized extensions of the same level.
(a) We write $(X_{\bullet,\bullet}) \sim (X'_{\bullet,\bullet})$ if there exists an isomorphism of generalized extensions $(X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$. That is, if there an isomorphism $f_{m,n} : X_{m,n} \to X'_{m,n}$ for each pair (m,n) in the eligible range such that the diagrams (15) commute for all (m,n).
(b) We write $(X_{\bullet,\bullet}) \sim' (X'_{\bullet,\bullet})$ if there exists an isomorphism $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ that is identity on A, i.e. such that for every r the isomorphism $f_{r-1,r} : A_r \to A_r$ is the identity map.
(c) For each $1 \le \ell \le k-1$, denote the set of equivalence classes of objects of $D_\ell(A)$ with respect to $\sim$ (resp. $\sim'$) by $S_\ell(A)$ (resp. $S'_\ell(A)$).

There is a natural surjection $S'_\ell(A) \to S_\ell(A)$ induced by the identity map on $D_\ell(A)$. Note that by Lemma 3.3.2, if $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ are generalized extensions of level ℓ such that $(X_{\bullet,\bullet}) \sim' (X'_{\bullet,\bullet})$, then there is only one isomorphism $(X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ that is identity on A.

Recall that for each $2 \le \ell \le k-1$ we have a truncation functor $\Theta_\ell : D_\ell(A) \to D_{\ell-1}(A)$, simply erasing the lowest diagonal of a generalized extension. The truncation functors clearly preserve both $\sim$ and $\sim'$, inducing maps $S'_\ell(A) \to S'_{\ell-1}(A)$ and $S_\ell(A) \to S_{\ell-1}(A)$ both of which we shall also refer to as truncation maps and (with abuse of notation) denote by Θ_ℓ. For each ℓ, we have a commutative diagram

(21)
$$\begin{tikzcd}
D_\ell(A) \arrow[r, "\Theta_\ell"] \arrow[d, two heads] & D_{\ell-1}(A) \arrow[d, two heads] \\
S'_\ell(A) \arrow[r, "\Theta_\ell"] \arrow[d, two heads] & S'_{\ell-1}(A) \arrow[d, two heads] \\
S_\ell(A) \arrow[r, "\Theta_\ell"] & S_{\ell-1}(A)
\end{tikzcd}$$

where the vertical arrows are the natural maps: modding out by $\sim'$ first and then further by $\sim$.

The sets $S_\ell(A)$ and the maps $\Theta_\ell : S_\ell(A) \to S_{\ell-1}(A)$ were characterized differently in the statement of Theorem 1.2.1 in §1.2.1. We now describe the action of $Aut(A)$ on $S'_\ell(A)$ and discuss the equivalence of definitions given here and the ones in the statement of Theorem 1.2.1.

The group $Aut(A)$ acts on $D_\ell(A)$ by twisting the arrows to and from the A_r: making the action a left action as usual, $\sigma = (\sigma_r)$ in $Aut(A) = \prod_r Aut(A_r)$ sends the diagram below on the left to the one on the right. Note that the rest of the arrows in the diagram remain unchanged.

(22)
$$\begin{tikzcd}
A_1 \arrow[d, hook, "j_1"'] \\
X_{0,2} \arrow[r, two heads, "\omega_2"] & A_2 \arrow[d, hook, "j_2"'] \\
\vdots & X_{1,3} \arrow[r, two heads, "\omega_3"] & A_3 \arrow[d, hook, "j_3"'] \\
& & \ddots & \ddots \\
& & X_{k-3,k-1} \arrow[r, two heads, "\omega_{k-1}"] & A_{k-1} \arrow[d, hook, "j_{k-1}"'] \\
& & \cdots & X_{k-2,k} \arrow[r, two heads, "\omega_k"] & A_k
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[d, hook, "j_1\sigma_1^{-1}"'] \\
X_{0,2} \arrow[r, two heads, "\sigma_2\omega_2"] & A_2 \arrow[d, hook, "j_2\sigma_2^{-1}"'] \\
\vdots & X_{1,3} \arrow[r, two heads, "\sigma_3\omega_3"] & A_3 \arrow[d, hook, "j_3\sigma_3^{-1}"'] \\
& & \ddots & \ddots \\
& & X_{k-3,k-1} \arrow[r, two heads, "\sigma_{k-1}\omega_{k-1}"] & A_{k-1} \arrow[d, hook, "j_{k-1}\sigma_{k-1}^{-1}"'] \\
& & \cdots & X_{k-2,k} \arrow[r, two heads, "\sigma_k\omega_k"] & A_k
\end{tikzcd}$$

We use the notation $\sigma \cdot (X_{\bullet,\bullet})$ for the image of $(X_{\bullet,\bullet})$ under $\sigma \in Aut(A)$.

If $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is an isomorphism that is identity on A, then $(f_{\bullet,\bullet})$ is also such an isomorphism $\sigma \cdot (X_{\bullet,\bullet}) \to \sigma \cdot (X'_{\bullet,\bullet})$. Indeed, if either of the two diagrams

$$\begin{tikzcd}
A_r \arrow[r, hook, "j_r"] \arrow[d, equal] & X_{r-1,r+1} \arrow[d, "f_{r-1,r+1}"] \\
A_r \arrow[r, hook, "j'_r"] & X'_{r-1,r+1}
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
X_{r-2,r} \arrow[r, two heads, "\omega_r"] \arrow[d, "f_{r-2,r}"] & A_r \arrow[d, equal] \\
X'_{r-2,r} \arrow[r, two heads, "\omega'_r"] & A_r
\end{tikzcd}$$

commutes, then so it does after twisting the horizontal arrows by σ_r or σ_r^{-1} in $Aut(A_r)$. Thus the action of $Aut(A)$ on $D_\ell(A)$ descends to an action on $S'_\ell(A)$. Moreover, it is clear from the definitions that the top horizontal map in (21) is $Aut(A)$-equivariant; thus so is the truncation map $S'_\ell(A) \to S'_{\ell-1}(A)$.

The equivalence of the two definitions of $S_\ell(A)$ and $\Theta_\ell : S_\ell(A) \to S_{\ell-1}(A)$ given in this subsection and the ones given in the statement of Theorem 1.2.1 in §1.2.1 can now be seen from the following lemma:

Lemma 3.4.2. *Let $1 \le \ell \le k-1$. The natural surjection*

$$S'_\ell(A) \twoheadrightarrow S_\ell(A)$$

descending from the identity map on $D_\ell(A)$ descends further to a bijection

$$S'_\ell(A)/Aut(A) \cong S_\ell(A).$$

Proof. We will show that two generalized extensions $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ are $\sim$-equivalent (i.e. isomorphic) if and only if there exists $\sigma \in Aut(A)$ such that $\sigma \cdot (X_{\bullet,\bullet}) \sim' (X'_{\bullet,\bullet})$.

Suppose $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is an isomorphism. Then $\sigma_r := f_{r-1,r}$ is an automorphism of A_r. For each eligible pair (m,n), let $g_{m,n} = f_{m,n}$ if $n - m > 1$. Let $g_{r-1,r}$ be the identity on A_r. Set $\sigma = (\sigma_r) \in Aut(A)$. Then $(g_{\bullet,\bullet}) : \sigma \cdot (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is an isomorphism of generalized extensions that is identity on A. Thus $\sigma \cdot (X_{\bullet,\bullet}) \sim' (X'_{\bullet,\bullet})$.

Conversely, assume $\sigma \cdot (X_{\bullet,\bullet}) \sim' (X'_{\bullet,\bullet})$. Let $(g_{\bullet,\bullet}) : \sigma \cdot (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ be an isomorphism that is identity on A. Then $f_{\bullet,\bullet} : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ defined by $f_{m,n} = g_{m,n}$ if $n - m > 1$ and $f_{r-1,r} = \sigma_r$ is an isomorphism. □

In what follows, we adopt the definitions given in this subsection for the sets $S_\ell(A)$ and the maps between them.

3.5. Generalized extensions in levels 1 and $k-1$. We now study the sets $S_\ell(A)$ and $S'_\ell(A)$ when ℓ is 1 or $k-1$. The goal is to establish the characterizations given in parts (a) and (d) of Theorem 1.2.1.

We start by recalling the action of

$$Aut(A) = \prod_r Aut(A_r)$$

on

$$\prod_r Ext^1(A_{r+1}, A_r)$$

that appeared in the statement of Theorem 1.2.1(d). Given $\sigma = (\sigma_r) \in Aut(A)$ and

$$(\mathscr{E}_r) \in \prod_r Ext^1(A_{r+1}, A_r),$$

the element $\sigma \cdot (\mathscr{E}_r)$ is the element whose r-entry is $(\sigma_r)_*(\sigma_{r+1}^{-1})^*\mathscr{E}_r$. Note that after taking a representative for $\mathscr{E}_r$, i.e. lifting it an extension

$$0 \longrightarrow A_r \xrightarrow{j_r} E_r \xrightarrow{\omega_{r+1}} A_{r+1} \longrightarrow 0,$$

$(\sigma_r)_*(\sigma_{r+1}^{-1})^*\mathscr{E}_r$ is the class of the extension obtained by replacing ω_{r+1} (resp. j_r) by $\sigma_{r+1}\omega_{r+1}$ (resp. $j_r\sigma_r^{-1}$).

As we already observed, by definition, the data of a generalized extension of level 1 of A is equivalent to the data of a collection of objects $X_{0,2}, X_{1,3}, \ldots, X_{k-2,k}$ and short exact sequences

$$(23) \qquad 0 \longrightarrow A_r \longrightarrow X_{r-1,r+1} \longrightarrow A_{r+1} \longrightarrow 0 \qquad (1 \le r \le k-1).$$

Referring to the notation earlier introduced (see Notation 3.3.3), the extension above is both $\mathfrak{X}^h_{r-1,r+1}$ and $\mathfrak{X}^v_{r-1,r+1}$. A morphism $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ of generalized extensions of level

1 is the data of morphisms $f_{r-1,r+1} : X_{r-1,r+1} \to X'_{r-1,r+1}$ and $f_{r-1,r} : A_r \to A_r$ (for each r) making the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_r & \longrightarrow & X_{r-1,r+1} & \longrightarrow & A_{r+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{r-1,r}} & & \downarrow{\scriptstyle f_{r-1,r+1}} & & \downarrow{\scriptstyle f_{r,r+1}} & & \\ 0 & \longrightarrow & A_r & \longrightarrow & X_{r-1,r+1} & \longrightarrow & A_{r+1} & \longrightarrow & 0 \end{array}$$

commute. By definition, two generalized extensions $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ of level 1 are $\sim'$-equivalent if and only if there are morphisms $f_{r-1,r+1}$ that together with the identity maps on the A_r make the diagrams commute. In other words, $(X_{\bullet,\bullet}) \sim' (X'_{\bullet,\bullet})$ if and only if for every r the extensions $\mathfrak{X}^h_{r-1,r+1}$ and $\mathfrak{X}'^h_{r-1,r+1}$ represent the same element in the corresponding Ext^1 group. This is summarized in part (a) below. Part (b) of the statement follows from the fact that the actions of $Aut(A)$ on both $S'(A)$ and $\prod_r Ext^1(A_{r+1}, A_r)$ are given by twisting the same arrows in the same way, and the fact (just observed) that $\sim'$ translates to the usual equivalence of 1-extensions.

Lemma 3.5.1. *(a) Two generalized extensions $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ of level 1 are $\sim'$-equivalent if and only if for each r the extensions $\mathfrak{X}^v_{r-1,r+1}$ and $\mathfrak{X}'^v_{r-1,r+1}$ (i.e.* (23) *and its counterpart for $(X'_{\bullet,\bullet})$) coincide in $Ext^1(A_{r+1}, A_r)$. The association $(X_{\bullet,\bullet}) \mapsto (\mathfrak{X}^h_{r-1,r+1})_r$ induces a bijection*

$$S'_1(A) \xrightarrow{\simeq} \prod_r Ext^1(A_{r+1}, A_r).$$

(b) Considering the previously defined actions of $Aut(A)$ on $S'_1(A)$ and $\prod\limits_r Ext^1(A_{r+1}, A_r)$, the bijection of part (a) is $Aut(A)$-equivariant and descends to a bijection

$$S_1(A) \xrightarrow{\simeq} \left(\prod_r Ext^1(A_{r+1}, A_r) \right) / \, Aut(A).$$

We now turn our attention to the case $\ell = k-1$. Recall from §3.2 that for every pair (X, ϕ) of an object X of $\mathbf{T}$ whose associated graded is isomorphic to A and an isomorphism $\phi : Gr^W X \to A$, we have an associated generalized extension $ext(X, \phi)$ of level $k-1$. The object at (m, n) entry of $ext(X, \phi)$ is $W_{p_n}X/W_{p_m}X$, with the graded component $W_{p_r}X/W_{p_{r-1}}X$ identified with A_r via ϕ. The structure morphisms in $ext(X, \phi)$ are the natural injections and projections. The following statement is easily seen from the definitions:

Lemma 3.5.2. *For any pair (X, ϕ) as above, the canonical isomorphism* (17) *for $ext(X, \phi)$ with $(m, n) = (0, k)$ is ϕ.*

It is easily seen from the constructions that for every $\sigma \in Aut(A)$,

$$ext(\sigma \cdot (X, \phi)) = ext(X, \sigma\phi) = \sigma \cdot ext(X, \phi). \tag{24}$$

Note that here $\sigma \cdot (X, \phi)$ refers to the action of $Aut(A)$ on the collection of pairs (X, ϕ). This action was defined by $\sigma \cdot (X, \phi) = (X, \sigma\phi)$ (see §3.1).

Recall from §3.1 that two pairs $(X, Gr^W X \xrightarrow{\phi, \simeq} A)$ and $(X', Gr^W X' \xrightarrow{\phi', \simeq} A)$ are said to be equivalent if there exists an isomorphism $f : X \to X'$ such that $\phi' \circ Gr^W f = \phi$. Also recall that we denoted the set of equivalence classes of such pairs by $S'(A)$, and that the action of $Aut(A)$ on the collection of pairs (X, ϕ) descends to an action on $S'(A)$.

Lemma 3.5.3. *(a) Two pairs (X, ϕ) and (X', ϕ') as above are equivalent if and only if the generalized extensions $ext(X, \phi)$ and $ext(X', \phi')$ are $\sim'$-equivalent.*

(b) Let $(X_{\bullet,\bullet})$ be a generalized extension of level $k-1$. Let $\phi : Gr^W X_{0,k} \to A$ be the canonical isomorphism of (17) for $(X_{\bullet,\bullet})$ and $(m,n) = (0,k)$. Then the identity map on $X_{0,k}$ extends to an isomorphism $(X_{\bullet,\bullet}) \to ext(X_{0,k}, \phi)$ that is identity on A.
(c) The association $(X,\phi) \mapsto ext(X,\phi)$ descends to a bijection

$$S'(A) \xrightarrow{\simeq} S'_{k-1}(A).$$

(d) Considering the previously defined actions of $Aut(A)$ on $S'(A)$ and $S'_{k-1}(A)$, the bijection of part (b) is $Aut(A)$-equivariant and it descends to a bijection

$$S'(A)/Aut(A) \xrightarrow{\simeq} S'_{k-1}(A)/Aut(A).$$

(e) Using the bijections of Lemmas 3.1.2 and 3.4.2 to translate the bijection of part (d) to a map

$$S(A) \xrightarrow{\simeq} S_{k-1}(A),$$

this bijection is described as follows: It sends the isomorphism class of X (an object whose associated graded is isomorphic to A) to the isomorphism class (i.e. image in $S_{k-1}(A)$) of the generalized extension $ext(X,\phi)$ for any choice of isomorphism $\phi : Gr^W X \to A$.

Proof. (a) Suppose (X,ϕ) and (X',ϕ') are equivalent, with $f : X \to X'$ an isomorphism for which $\phi' \circ Gr^W f = \phi$. By Lemma 3.3.4, f extends uniquely to an isomorphism $(f_{\bullet,\bullet}) : ext(X,\phi) \to ext(X',\phi')$. In view of Lemmas 3.5.2 and 3.3.1(d) (the latter applied with $(m,n) = (0,k)$), the fact that $\phi' \circ Gr^W f = \phi$ implies that $(f_{\bullet,\bullet})$ is identity on A.

Conversely, suppose $(f_{\bullet,\bullet}) : ext(X,\phi) \to ext(X',\phi')$ is an isomorphism that is identity on A. Then $f_{0,k} : X \to X'$ is an isomorphism that satisfies $\phi' Gr^W f_{0,k} = \phi$. This again follows from Lemmas 3.5.2 and 3.3.1(d).
(b) By Lemma 3.3.4, the identity map on $X_{0,k}$ extends uniquely to an isomorphism $(f_{\bullet,\bullet})$: $(X_{\bullet,\bullet}) \to ext(X_{0,k},\phi)$. Now apply Lemma 3.3.1(d) with $(m,n) = (0,k)$, $(X'_{\bullet,\bullet}) = ext(X_{0,k},\phi)$ and $(f_{\bullet,\bullet})$ as in here. The left arrow of the diagram is $Gr^W f_{0,k} = Gr^W Id$ which is just the identity. The top and bottom canonical isomorphisms are both ϕ. Hence the arrow on the right is the identity map.
(c) By part (a), $(X,\phi) \mapsto ext(X,\phi)$ descends to an injection $S'(A) \to S'_{k-1}(A)$, which is also surjective by part (b).
(d) We have a commutative diagram

$$\begin{tikzcd} \{\ (X,\phi)\ \} \arrow[r, "ext(-)"] \arrow[d, two heads] & D_{k-1}(A) \arrow[d, two heads] \\ S'(A) \arrow[r, "\text{part (b)}", "\simeq"'] & S'_{k-1}(A), \end{tikzcd}$$

where $\{(X,\phi)\}$ means the collection of all pairs $(X, Gr^W X \xrightarrow{\phi,\simeq} A)$. By (24), the top arrow is $Aut(A)$-equivariant. By definition of the $Aut(A)$-actions on $S'(A)$ and $S'_{k-1}(A)$, so are the two side arrows. It follows that the map of part (b) is $Aut(A)$-equivariant and it descends to a map

$$S'(A)/Aut(A) \to S'_{k-1}(A)/Aut(A),$$

which is surjective thanks to part (b). It remains to show that it is also injective.

Consider two pairs (X,ϕ) and (X',ϕ') such that the classes of $ext(X,\phi)$ and $ext(X',\phi')$ in $S'_{k-1}(A)$ are in the same $Aut(A)$-orbit. In view of (24) this means that there exists $\sigma \in Aut(A)$ such that $ext(X,\sigma\phi)$ is $\sim'$-equivalent to $ext(X',\phi')$. That is, there exists an isomorphism $(f_{\bullet,\bullet})$ of generalized extensions $ext(X,\sigma\phi) \to ext(X',\phi')$ such that for each r, the morphism $f_{r-1,r}$ is the identity map on A_r.

Consider $f_{0,k} : X \to X'$. We claim that $\phi' \circ Gr^W f_{0,k} = \sigma\phi$; this would show that pairs $(X, \sigma\phi) = \sigma \cdot (X, \phi)$ and (X', ϕ') are equivalent, so that the classes of (X, ϕ) and (X', ϕ') in $S'(A)$ are in the same $Aut(A)$-orbit.

To see the claim, apply Lemma 3.3.1(d) to $(f_{\bullet,\bullet})$ for $(m, n) = (0, k)$. Since the canonical isomorphism (17) for $ext(X, \sigma\phi)$ (resp. $ext(X', \phi')$) is simply $\sigma\phi$ (resp. ϕ'), we get that the diagram

$$\begin{tikzcd} Gr^W X \arrow[r, "\sigma\phi", "\simeq"'] \arrow[d, "Gr^W f_{0,k}"'] & A \arrow[d, "(f_{r-1,r}) = Id"] \\ Gr^W X' \arrow[r, "\phi'", "\simeq"'] & A \end{tikzcd}$$

commutes.

(e) The given description is clear from the definitions of the other three arrows of the commutative diagram

$$\begin{tikzcd} S'(A)/Aut(A) \arrow[r, "\text{part (d)}", "\simeq"'] \arrow[d, "\simeq"'] & S'_{k-1}(A)/Aut(A) \arrow[d, "\simeq"] \\ S(A) \arrow[r, "\simeq"'] & S_{k-1}(A). \end{tikzcd}$$

□

The following diagram summarizes our picture so far.

(25)

$$\begin{tikzcd} \{(X,\phi)\} \arrow[r, "ext(-)"] \arrow[d, two heads] & D_{k-1}(A) \arrow[r, "\Theta_{k-1}"] \arrow[d, two heads] & D_{k-2}(A) \arrow[r, "\Theta_{k-2}"] \arrow[d, two heads] & \cdots \arrow[r, "\Theta_2"] & D_1(A) \cong \prod_r EXT(A_{r+1}, A_r) \arrow[d, two heads] \\ S'(A) \arrow[r, "\cong"] \arrow[d, two heads] & S'_{k-1}(A) \arrow[r, "\Theta_{k-1}"] \arrow[d, two heads] & S'_{k-2}(A) \arrow[r, "\Theta_{k-2}"] \arrow[d, two heads] & \cdots \arrow[r, "\Theta_2"] & S'_1(A) \cong \prod_r Ext^1(A_{r+1}, A_r) \arrow[d, two heads] \\ S(A) \arrow[r, "\cong"] & S_{k-1}(A) \arrow[r, "\Theta_{k-1}"] & S_{k-2}(A) \arrow[r, "\Theta_{k-2}"] & \cdots \arrow[r, "\Theta_2"] & S_1(A) \cong \dfrac{\prod_r Ext(A_{r+1}, A_r)}{Aut(A)} \end{tikzcd}$$

Here, $\{(X, \phi)\}$ means the collection of all pairs (X, ϕ) of consisting of an object X of $\mathbf{T}$ and an isomorphism $\phi : Gr^W X \to A$. The map $\{(X, \phi)\} \twoheadrightarrow S'(A)$ sends a pair to its $\sim'$-equivalence class. The map $S'(A) \twoheadrightarrow S(A)$ is induced by $(X, \phi) \mapsto X$. The maps $D_\ell(A) \twoheadrightarrow S'_\ell(A)$ and $S'_\ell(A) \twoheadrightarrow S_\ell(A)$, respectively, are given by modding out by $\sim'$ and (further) by $\sim$. All the maps between the middle and bottom rows can also be thought of as modding out by the action of $Aut(A)$. What remains of Theorem 1.2.1 to be established is the assertions about the structure of the fibers. This will be the subject of the rest of the section.

3.6. Fibers of truncation maps I: Torsor structures. Assume $2 \le \ell \le k - 1$. In this subsection we fix a generalized extension $(X_{m,n})_{n-m \le \ell}$ of level $\ell - 1$ and first describe the fiber of the truncation functor $\Theta_\ell : D_\ell(A) \to D_{\ell-1}(A)$ above it, i.e., the collection of all generalized extensions of level ℓ that become $(X_{m,n})_{n-m \le \ell}$ once their lowest diagonal is erased. We then consider the equivalence relation $\sim'$ on this fiber.

Recall that given objects X, Y, Z of $\mathbf{T}$, $\mathscr{N} \in EXT(Z, Y)$ and $\mathscr{L} \in EXT(Y, X)$, the notation $EXTPAN(\mathscr{N}, \mathscr{L})$ means the collection of all blended extensions of $\mathscr{N}$ by $\mathscr{L}$ (with no identification made). Recall also the notations $\mathfrak{X}^v_{m,n}$ and $\mathfrak{X}^h_{m,n}$ for extensions respectively coming from the arrows $X_{m,n-1} \hookrightarrow X_{m,n}$ and $X_{m,n} \twoheadrightarrow X_{m+1,n}$ of a generalized extension $(X_{\bullet,\bullet})$ (i.e. extensions given by (14) and (19), respectively, see Notation 3.3.3).

Construction 3.6.1. There is a natural (to be seen to be bijective) map

$$\Theta_\ell^{-1}((X_{m,n})_{n-m\leq\ell}) = \left\{ \begin{array}{l} \text{fiber of} \\ D_\ell(A) \xrightarrow{\Theta_\ell} D_{\ell-1}(A) \\ \text{above } (X_{m,n})_{n-m\leq\ell} \end{array} \right\} \longrightarrow \prod_r EXTPAN(\mathfrak{X}^v_{r,r+\ell}, \mathfrak{X}^h_{r-1,r+\ell-1}) \tag{26}$$

(where the index r on the right runs through the integers $1 \leq r \leq k-\ell$) described as follows. Consider an element $(X_{m,n})_{n-m\leq\ell+1}$ of $D_\ell(A)$ in the fiber above $(X_{m,n})_{n-m\leq\ell}$. For each $X_{r-1,r+\ell}$ on its lowest diagonal, the morphisms

$$\begin{tikzcd} X_{r-1,r+\ell-1} \arrow[r, two heads] \arrow[d, hook] & X_{r,r+\ell-1} \arrow[d, hook] \\ X_{r-1,r+\ell} \arrow[r, two heads] & X_{r,r+\ell} \end{tikzcd}$$

lead to a blended extension

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_r \arrow[r] \arrow[d, equal] & X_{r-1,r+\ell-1} \arrow[r] \arrow[d] & X_{r,r+\ell-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_r \arrow[r] & X_{r-1,r+\ell} \arrow[r] \arrow[d] & X_{r,r+\ell} \arrow[r] \arrow[d] & 0, \\ & & A_{r+\ell} \arrow[r, equal] \arrow[d] & A_{r+\ell} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{27}$$

in which every map is a composition (uniquely determined by the indices) of the structure arrows. The extensions on the top and right are respectively $\mathfrak{X}^h_{r-1,r+\ell-1}$ and $\mathfrak{X}^v_{r,r+\ell}$. The map (26) sends $(X_{m,n})_{n-m\leq\ell+1}$ to the tuple with this blended extension in its r-entry.

Lemma 3.6.2. *The map* (26) *is bijective.*

Proof. We construct the inverse of (26). Note that for each r the two extensions $\mathfrak{X}^v_{r,r+\ell}$ and $\mathfrak{X}^h_{r-1,r+\ell-1}$ come from the data of the generalized extension $(X_{m,n})_{n-m\leq\ell}$ of level $\ell-1$. For each r, consider a blended extension of $\mathfrak{X}^v_{r,r+\ell}$ by $\mathfrak{X}^h_{r-1,r+\ell-1}$. It is given by a diagram of the form (27), with the top and right extensions being $\mathfrak{X}^h_{r-1,r+\ell-1}$ and $\mathfrak{X}^v_{r,r+\ell}$, respectively. Now enlarge $(X_{m,n})_{n-m\leq\ell}$ to a generalized extension of level ℓ by adding to its data, for each r, the object $X_{r-1,r+\ell}$ in entry $(r-1,r+\ell)$ and the morphisms $X_{r-1,r+\ell-1} \hookrightarrow X_{r-1,r+\ell}$ and $X_{r-1,r+\ell} \twoheadrightarrow X_{r,r+\ell}$ of the corresponding blended extension. The augmented data $(X_{m,n})_{n-m\leq\ell+1}$ is a generalized extension of level ℓ. Indeed, the only new squares formed by the structure arrows are the ones in the top rights of our blended extensions. So axiom (i) of the definition of a generalized extension holds. As for axiom (ii) (the exactness of the sequences (14)), the new sequences we must consider are exactly the sequences in the middle columns of our blended extensions.

By sending the tuple of blended extensions we started with to the generalized extension $(X_{m,n})_{n-m\leq \ell+1}$ we obtain a map

$$\prod_r EXTPAN(\mathcal{X}^v_{r,r+\ell},\mathcal{X}^h_{r-1,r+\ell-1}) \longrightarrow \Theta_\ell^{-1}((X_{m,n})_{n-m\leq \ell}). \tag{28}$$

The reader easily sees that this maps is the inverse to (26). □

It is convenient to have a notation for blended extensions of the form (27):

Notation 3.6.3. For a generalized extension $(X_{\bullet,\bullet})$ of level $\ell\geq 2$, for any r we denote the blended extension (27) with middle object $X_{r-1,r+\ell}$ by $\mathcal{X}_{r-1,r+\ell}$ (without a superscript h or v).

Recall from §2.1 that given blended extensions $\mathcal{X}$ and $\mathcal{X}'$ of an extension $\mathcal{N}$ by an extension $\mathcal{L}$, a morphism (automatically an isomorphism) of blended extensions from $\mathcal{X}$ to $\mathcal{X}'$ is a morphism from the middle object X of $\mathcal{X}$ to the middle object X' of $\mathcal{X}'$ that induces identity maps on objects in $\mathcal{L}$ and $\mathcal{N}$; that is, a morphism $X\to X'$ that together with the identity maps on the objects in $\mathcal{L}$ and $\mathcal{N}$ commute with the arrows in $\mathcal{X}$ and $\mathcal{X}'$ (in the obvious sense). The collection of isomorphism classes of blended extensions of $\mathcal{N}$ by $\mathcal{L}$ is denoted by $Extpan(\mathcal{N},\mathcal{L})$.

Lemma 3.6.4. *(a) Two elements of the fiber of $\Theta_\ell : D_\ell(A)\to D_{\ell-1}(A)$ above $(X_{\bullet,\bullet})$ are $\sim'$-equivalent if and only if their images under the map* (26) *coincide in*

$$\prod_r Extpan(\mathcal{X}^v_{r,r+\ell},\mathcal{X}^h_{r-1,r+\ell-1}).$$

(b) The map (26) *descends to a bijection*

$$\big(\ \Theta_\ell^{-1}((X_{\bullet,\bullet}))\ \big)/\sim' \xrightarrow{\ \simeq\ } \prod_r Extpan(\mathcal{X}^v_{r,r+\ell},\mathcal{X}^h_{r-1,r+\ell-1}).$$

In particular,

$$\big(\ \Theta_\ell^{-1}((X_{\bullet,\bullet}))\ \big)/\sim'$$

is either empty or a torsor over

$$\prod_r Ext^1(A_{r+\ell},A_r).$$

Moreover, it is nonempty if and only if for each r, the image of the Yoneda product of $\mathcal{X}^v_{r,r+\ell}$ and $\mathcal{X}^h_{r-1,r+\ell-1}$ in $Ext^2(A_{r+\ell},A_r)$ vanishes.

Proof. We first note that part (b) follows immediately from part (a), Lemma 3.6.2, and the general theory of blended extensions (see §2.1, in particular, Lemma 2.1.1(a)). So we will focus on part (a).

Suppose $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ are in the fiber above $(X_{\bullet,\bullet})$, so that dropping the lowest diagonals, $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ are just $(X_{\bullet,\bullet})$. The blended extensions (27) for $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ (respectively, denoted by $\mathcal{Y}_{r-1,r+\ell}$ and $\mathcal{Y}'_{r-1,r+\ell}$) have the same top rows and the same right columns, coming from $(X_{\bullet,\bullet})$. Since every arrow in $\mathcal{Y}_{r-1,r+\ell}$ (resp. $\mathcal{Y}'_{r-1,r+\ell}$) is the appropriate composition of the structure arrows of $(Y_{\bullet,\bullet})$ (resp. $(Y'_{\bullet,\bullet})$), every morphism $(f_{\bullet,\bullet}) : (Y_{\bullet,\bullet})\to(Y'_{\bullet,\bullet})$ of generalized extensions includes the data of a collection of maps between the corresponding objects of $\mathcal{Y}_{r-1,r+\ell}$ and $\mathcal{Y}'_{r-1,r+\ell}$ that commute with the arrows in the two blended extensions.

Let $(f_{\bullet,\bullet}) : (Y_{\bullet,\bullet})\to(Y'_{\bullet,\bullet})$ be a morphism that is identity on A. Both $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ truncate to $(X_{\bullet,\bullet})$, so that $(f_{\bullet,\bullet})$ must be identity on $(X_{\bullet,\bullet})$ (see Lemma 3.3.2). Combining with the earlier comments, it follows that for each r, the isomorphism $f_{r-1,r+\ell}$ from $Y_{r-1,r+\ell}$ to $Y'_{r-1,r+\ell}$ gives an isomorphism of blended extensions from $\mathcal{Y}_{r-1,r+\ell}$ to $\mathcal{Y}'_{r-1,r+\ell}$.

Conversely, suppose that for each r, the classes of blended extensions $\mathcal{Y}_{r-1,r+\ell}$ and $\mathcal{Y}'_{r-1,r+\ell}$ coincide in $Extpan(\mathfrak{X}^v_{r,r+\ell}, \mathfrak{X}^h_{r-1,r+\ell-1})$. Let $f_{r-1,r+\ell}$ be the morphism $Y_{r-1,r+\ell} \to Y'_{r-1,r+\ell}$ that gives an isomorphism of blended extensions $\mathcal{Y}_{r-1,r+\ell} \to \mathcal{Y}'_{r-1,r+\ell}$. Then $f_{r-1,r+\ell}$ induces identity on $X_{r-1,r+\ell-1}$ and $X_{r,r+\ell}$. By Lemma 3.3.5 the collection of morphisms $f_{r-1,r+\ell}$ glues together to give a morphism $(f_{\bullet,\bullet}) : (Y_{\bullet,\bullet}) \to (Y'_{\bullet,\bullet})$. This morphism is identity on the diagonal just above the lowest, and hence is identity on all of $(X_{\bullet,\bullet})$. □

3.7. Fibers of truncation maps II: Canonicity of the torsor structures. We continue to assume $2 \leq \ell \leq k-1$. Let $(X_{\bullet,\bullet})$ be a generalized extension of level $\ell-1$. Denote the class of $(X_{\bullet,\bullet})$ in $S'_{\ell-1}(A)$ by $[(X_{\bullet,\bullet})]_{\sim'}$. In the previous subsection we studied $\Theta_\ell^{-1}((X_{\bullet,\bullet}))/\sim'$ and saw that it has a torsor structure. To finish the proof of Theorem 1.2.1(b,c) we need to show that

$$\Theta_\ell^{-1}([(X_{\bullet,\bullet})]_{\sim'}) \cong \Theta_\ell^{-1}((X_{\bullet,\bullet}))/\sim',$$

and that moreover the torsor structure obtained this way on the fiber of $S'_\ell(A) \to S'_{\ell-1}(A)$ above $\epsilon := [(X_{\bullet,\bullet})]_{\sim'}$ is independent of the choice of representative $(X_{\bullet,\bullet})$ for ϵ. By the end of this subsection these will be established and the proof of Theorem 1.2.1(b,c) will be completed.

The following simple definition is convenient.

Definition 3.7.1. Let $(X_{\bullet,\bullet})$ be a generalized extension of any level. Let (i,j) be an eligible pair, and $f : X_{i,j} \to X'$ an isomorphism.
(a) The transport of $(X_{\bullet,\bullet})$ along f, denoted by $tr((X_{\bullet,\bullet}), f)$, is the generalized extension obtained from $(X_{\bullet,\bullet})$ by replacing the object $X_{i,j}$ at entry (i,j) by X' via f. That is, by replacing $X_{i,j}$ by X', and the arrows to (resp. from) $X_{i,j}$ by their composition with f (resp. f^{-1}).
(b) The collection of morphisms $(f_{\bullet,\bullet})$ given by $f_{m,n} = Id_{X_{m,n}}$ for every eligible pair $(m,n) \neq (i,j)$ and $f_{i,j} = f$ is called the isomorphism given by the transport datum f.

One easily sees that the transport defined above is indeed a generalized extension[13], and that the collection of isomorphisms $(f_{\bullet,\bullet})$ of (b) commutes with the structure arrows of $(X_{\bullet,\bullet})$ and its transport $tr((X_{\bullet,\bullet}), f)$. One also easily sees that the transport construction behaves well with respect to compositions: given $(X_{\bullet,\bullet})$ and isomorphisms $X_{i,j} \xrightarrow{f} Y \xrightarrow{g} Z$, we have

$$tr(tr((X_{\bullet,\bullet}), f), g) = tr((X_{\bullet,\bullet}), gf) \tag{29}$$

and the composition of the isomorphisms given by the transport datum f first and then g

$$(X_{\bullet,\bullet}) \to tr((X_{\bullet,\bullet}), f) \to tr(tr((X_{\bullet,\bullet}), f), g)$$

is just the isomorphism given by the transport datum gf.

More generally, given a generalized extension $(X_{\bullet,\bullet})$ of any level, a set I of eligible pairs of indices and for each $(m,n) \in I$ an isomorphism $f_{m,n} : X_{m,n} \to X'_{m,n}$, we may talk about the transport $tr((X_{\bullet,\bullet}), (f_{m,n})_{(m,n)\in I})$ of $(X_{\bullet,\bullet})$ along $(f_{m,n})_{(m,n)\in I}$. Making the transport all at once is the same as making it step by step for one $(m,n) \in I$ at a time (note that the transport operations along morphisms from $X_{m,n}$ for different (m,n) commute with one another). The collection of morphisms $(g_{\bullet,\bullet})$ where $g_{m,n} = Id_{X_{m,n}}$ for every eligible $(m,n) \notin I$ and $g_{m,n} = f_{m,n}$ if $(m,n) \in I$ commutes with the structure arrows of $(X_{\bullet,\bullet})$ and $tr((X_{\bullet,\bullet}), (f_{m,n})_{(m,n)\in I})$; in line with Definition 3.7.1(b), we call $(g_{\bullet,\bullet})$ the isomorphism given by the transport data $(f_{m,n})_{(m,n)\in I}$.

Now let $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ be generalized extensions of level $\ell-1$ of A, and $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ an isomorphism of generalized extensions. Given $(Y_{\bullet,\bullet}) \in \Theta_\ell^{-1}((X_{\bullet,\bullet}))$, it follows from

[13]We note that in the generality of Definition 3.7.1, the transport may not be a generalized extension *of* A. However, in all applications of this construction in the paper, whenever $i = j-1$ (so that $X_{i,j} = A_j$) we will also take X' to be A_j, so that the transport will always be indeed a generalized extension of A as well.

the commutativity of the diagrams of (15) that the transport of $(Y_{\bullet,\bullet})$ along $(f_{\bullet,\bullet})$ is in the fiber of Θ_ℓ above $(X'_{\bullet,\bullet})$. The map

$$\Theta_\ell^{-1}((X_{\bullet,\bullet})) \to \Theta_\ell^{-1}((X'_{\bullet,\bullet})) \qquad (Y_{\bullet,\bullet}) \mapsto tr((Y_{\bullet,\bullet}),(f_{\bullet,\bullet})) \tag{30}$$

is a bijection, with its inverse given by transport along $(f^{-1}_{\bullet,\bullet})$. For every $(Y_{\bullet,\bullet})$, the isomorphism given by the transport data $(f_{\bullet,\bullet})$ is an isomorphism of generalized extensions $(Y_{\bullet,\bullet}) \to tr((Y_{\bullet,\bullet}),(f_{\bullet,\bullet}))$. Thus (30) descends to a bijection between the $\sim$-equivalence classes.

If $(Y^{(1)}_{\bullet,\bullet})$ and $(Y^{(2)}_{\bullet,\bullet})$ in $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$ are $\sim'$-equivalent with $(g_{\bullet,\bullet}) : (Y^{(1)}_{\bullet,\bullet}) \to (Y^{(2)}_{\bullet,\bullet})$ an isomorphism that is identity on A, then the composition

$$tr((Y^{(1)}_{\bullet,\bullet}),(f_{\bullet,\bullet})) \xrightarrow[\text{tr. data } (f^{-1}_{\bullet,\bullet})]{\text{iso. given by}} (Y^{(1)}_{\bullet,\bullet}) \xrightarrow{(g_{\bullet,\bullet})} (Y^{(2)}_{\bullet,\bullet}) \xrightarrow[\text{tr. data } (f_{\bullet,\bullet})]{\text{iso. given by}} tr((Y^{(2)}_{\bullet,\bullet}),(f_{\bullet,\bullet}))$$

is identity on A as well (even if $(f_{\bullet,\bullet})$ is not identity on A). Hence (30) also descends to a bijection between the $\sim'$-equivalence classes, with its inverse induced by transport along $(f^{-1}_{\bullet,\bullet})$.

If the morphism $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is identity on A, then every $(Y_{\bullet,\bullet})$ above $(X_{\bullet,\bullet})$ is $\sim'$-equivalent to its transport along $(f_{\bullet,\bullet})$, with the isomorphism given by the transport data giving the $\sim'$-equivalence.

In particular, we note from the above that if $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ in $D_{\ell-1}(A)$ are $\sim$-equivalent (resp. $\sim'$-equivalent), then for every $(Y_{\bullet,\bullet}) \in \Theta^{-1}((X_{\bullet,\bullet}))$ there exists $(Y'_{\bullet,\bullet}) \in \Theta^{-1}((X'_{\bullet,\bullet}))$ that is $\sim$-equivalent (resp. $\sim'$-equivalent) to $(Y_{\bullet,\bullet})$.

We obtain the following lemma regarding the fibers of truncation maps $S'_\ell(A) \to S'_{\ell-1}(A)$ and $S_\ell(A) \to S_{\ell-1}(A)$ (recall that we refer to both of these also by Θ_ℓ).

Lemma 3.7.2. *Let $(X_{\bullet,\bullet}) \in D_{\ell-1}(A)$. Denote the classes of $(X_{\bullet,\bullet})$ in $S_\ell(A)$ and $S'_\ell(A)$ respectively by $[(X_{\bullet,\bullet})]_\sim$ and $[(X_{\bullet,\bullet})]_{\sim'}$.*
(a) The natural injection

$$\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\!\sim\; \to\; \Theta_\ell^{-1}([(X_{\bullet,\bullet})]_\sim)$$

is bijective. If $(X'_{\bullet,\bullet}) \in D_{\ell-1}(A)$ and $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is an isomorphism, then we have a commutative diagram

$$\begin{tikzcd}
\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\!\sim \arrow[rr, "tr.\ along\ (f_{\bullet,\bullet})", "\simeq"'] \arrow[dr, "\simeq"'] & & \big(\,\Theta_\ell^{-1}((X'_{\bullet,\bullet}))\,\big)/\!\sim \arrow[dl, "\simeq"] \\
& \Theta_\ell^{-1}([(X_{\bullet,\bullet})]_\sim) &
\end{tikzcd}$$

where the horizontal arrow is given by transport along $(f_{\bullet,\bullet})$ (descending from (30)*) and the other two arrows are the natural maps.*
(b) The natural injection

$$\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\!\sim'\; \to\; \Theta_\ell^{-1}([(X_{\bullet,\bullet})]_{\sim'}) \tag{31}$$

is bijective. If $(X'_{\bullet,\bullet}) \in D_{\ell-1}(A)$ and $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is an isomorphism that is identity on A, then we have a commutative diagram

$$\begin{tikzcd}
\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\!\sim' \arrow[rr, "tr.\ along\ (f_{\bullet,\bullet})", "\simeq"'] \arrow[dr, "\simeq"'] & & \big(\,\Theta_\ell^{-1}((X'_{\bullet,\bullet}))\,\big)/\!\sim' \arrow[dl, "\simeq"] \\
& \Theta_\ell^{-1}([(X_{\bullet,\bullet})]_{\sim'}) &
\end{tikzcd} \tag{32}$$

where the horizontal arrow is given by transport along $(f_{\bullet,\bullet})$ (descending from (30)) and the other two arrows are the natural maps. If $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ is an isomorphism that is not necessarily identity on A, we still have a bijection

$$\left(\Theta_\ell^{-1}((X_{\bullet,\bullet})) \right)/\sim' \xrightarrow{\text{tr. along } (f_{\bullet,\bullet})} \left(\Theta_\ell^{-1}((X'_{\bullet,\bullet})) \right)/\sim' \tag{33}$$

which forms a commutative diagram if we pass along on both sides to $\Theta_\ell^{-1}([(X_{\bullet,\bullet})]_\sim)$.

Combining part (b) with Lemma 3.6.4(b) we obtain torsor structures on the nonempty fibers of $S'_\ell(A) \to S'_{\ell-1}(A)$. At the moment however, the torsor structure on the fiber of $S'_\ell(A) \to S'_{\ell-1}(A)$ above the class of $(X_{\bullet,\bullet})$ appears to depend on the choice of representative $(X_{\bullet,\bullet})$. Our next task is to rule out this dependence.

The group $Aut(A)$ acts on

$$\prod_r Ext^1(A_{r+\ell}, A_r) \tag{34}$$

similarly to the action we already considered when $\ell = 1$, i.e. by pushforwards and pullbacks. An element $\sigma = (\sigma_r)$ (with $\sigma_r \in Aut(A_r)$) sends a tuple of extension classes $\mathscr{E} = (\mathscr{E}_r)$ to the tuple that has $(\sigma_r)_*(\sigma_{r+\ell}^{-1})^*\mathscr{E}_r$ in its r-entry. Denoting the image of $\mathscr{E}$ under the action by σ by $\sigma \cdot \mathscr{E}$, thus the r-entry of $\sigma \cdot \mathscr{E}$ is obtained, after taking a representative for $\mathscr{E}_r$ in $EXT(A_{r+\ell}, A_r)$, by composing the arrow coming out of A_r by σ_r^{-1} and the arrow going to $A_{r+\ell}$ by $\sigma_{r+\ell}$.

Lemma 3.7.3. *Let $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ be in $D_{\ell-1}(A)$ and $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ an isomorphism. Suppose that the fiber of $\Theta_\ell : D_\ell(A) \to D_{\ell-1}(A)$ above $(X_{\bullet,\bullet})$ (and hence $(X'_{\bullet,\bullet})$) is nonempty. Consider*

$$\left(\Theta_\ell^{-1}((X_{\bullet,\bullet})) \right)/\sim' \quad \text{and} \quad \left(\Theta_\ell^{-1}((X'_{\bullet,\bullet})) \right)/\sim'$$

as torsors for (34) via the canonical bijection of Lemma 3.6.4(b) (for $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$, respectively). Then the bijection (33) satisfies the following identity: denoting the action of the group (34) on the torsors above by $*$, the map (33) (as well as (30)) by $tr(-, (f_{\bullet,\bullet}))$, and the restriction of $(f_{\bullet,\bullet})$ to A by f_A, then for every $(Y_{\bullet,\bullet}) \in \Theta_\ell^{-1}((X_{\bullet,\bullet}))$ and for every tuple of extension classes $\mathscr{E} = (\mathscr{E}_r)$ in (34) we have

$$tr(\mathscr{E} * [(Y_{\bullet,\bullet})]_{\sim'}, (f_{\bullet,\bullet})) \;=\; (f_A \cdot \mathscr{E}) * tr([(Y_{\bullet,\bullet})]_{\sim'}, (f_{\bullet,\bullet})), \tag{35}$$

where (with abuse of notation) $[(Y_{\bullet,\bullet})]_{\sim'}$ here means the image of $(Y_{\bullet,\bullet})$ in $(\Theta_\ell^{-1}((X_{\bullet,\bullet})))/\sim'$.

In particular, if the restriction of $(f_{\bullet,\bullet})$ to A is a scalar multiple of the identity map, then the bijection (33) is an isomorphism of torsors.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \Theta_\ell^{-1}((X_{\bullet,\bullet})) \arrow[r, "(26)", "\simeq"'] \arrow[d, two heads] & \prod_r EXTPAN(\mathfrak{X}^v_{r,r+\ell}, \mathfrak{X}^h_{r-1,r+\ell-1}) \arrow[d, two heads] \\ \left(\Theta_\ell^{-1}((X_{\bullet,\bullet})) \right)/\sim' \arrow[r, "\text{Lem. 3.6.4(b)}", "\simeq"'] & \prod_r Extpan(\mathfrak{X}^v_{r,r+\ell}, \mathfrak{X}^h_{r-1,r+\ell-1}). \end{tikzcd} \tag{36}$$

The torsor structures over the group (34) on the lower level descend from a map

$$\prod_r EXT(A_{r+\ell}, A_r) \times \prod_r EXTPAN(\mathfrak{X}^v_{r,r+\ell}, \mathfrak{X}^h_{r-1,r+\ell-1}) \to \prod_r EXTPAN(\mathfrak{X}^v_{r,r+\ell}, \mathfrak{X}^h_{r-1,r+\ell-1}).$$

We use the symbol $*$ for the operation given by the latter map on the top right object of (36), as well as the operation on $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$ induced by it, and the operations on the lower level of the diagram descended from it. We also use the same notation for the analogous operations for $(X'_{\bullet,\bullet})$.

Let $(Y_{\bullet,\bullet})$ be in $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$ and $\mathscr{E}=(\mathscr{E}_r)$ a tuple in (34). Lift each $\mathscr{E}_r$ to an element of $EXT(A_{r+\ell},A_r)$, which with abuse of notation we also denote by $\mathscr{E}_r$.

Set

$$(Z_{\bullet,\bullet}) := \mathscr{E} * (Y_{\bullet,\bullet}) \in \Theta_\ell^{-1}((X_{\bullet,\bullet}))$$

and $(Z'_{\bullet,\bullet}) = tr((Z_{\bullet,\bullet}),(f_{\bullet,\bullet}))$. Suppose that the blended extension of $\mathfrak{X}^v_{r,r+\ell}$ by $\mathfrak{X}^h_{r-1,r+\ell-1}$ associated to $(Z_{\bullet,\bullet})$ by Construction 3.6.1 is given by the diagram on the left below. Then the blended extension of ${\mathfrak{X}'}^v_{r,r+\ell}$ by ${\mathfrak{X}'}^h_{r-1,r+\ell-1}$ associated to $(Z'_{\bullet,\bullet})$ is given by the diagram on the right below. We have dropped the indices from the $f_{m,n}$ to save space (they are determined by the indices of the objects).

(37)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_r & \xrightarrow{j} & X_{r-1,r+\ell-1} & \xrightarrow{\pi} & X_{r,r+\ell-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\bar\iota} & & \downarrow{\scriptstyle\iota} & & \\
0 & \longrightarrow & A_r & \xrightarrow{\bar j} & Z_{r-1,r+\ell} & \xrightarrow{\bar\pi} & X_{r,r+\ell} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\varpi} & & \downarrow{\scriptstyle\omega} & & \\
 & & & & A_{r+\ell} & = & A_{r+\ell} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_r & \xrightarrow{fjf^{-1}} & X'_{r-1,r+\ell-1} & \xrightarrow{f\pi f^{-1}} & X'_{r,r+\ell-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\bar\iota f^{-1}} & & \downarrow{\scriptstyle f\iota f^{-1}} & & \\
0 & \longrightarrow & A_r & \xrightarrow{\bar j f^{-1}} & Z_{r-1,r+\ell} & \xrightarrow{f\bar\pi} & X'_{r,r+\ell} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f\varpi} & & \downarrow{\scriptstyle f\omega f^{-1}} & & \\
 & & & & A_{r+\ell} & = & A_{r+\ell} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Bringing the indices back to avoid confusion, comparing the second rows of the two diagrams we have

$$(f_{r-1,r})_* \mathscr{X}^h_{r-1,r+\ell} = f^*_{r,r+\ell} {\mathscr{X}'}^h_{r-1,r+\ell}, \tag{38}$$

where $\mathscr{X}^h_{r-1,r+\ell}$ and ${\mathscr{X}'}^h_{r-1,r+\ell}$ refer to the second horizontal extensions in the two diagrams respectively (see Notation 3.3.3). The equality here as well as all the other equalities in the rest of this argument take place in the corresponding Ext^1 groups.

By definition of the torsor structure on the set of isomorphism classes of blended extensions (see §2.1), we have

$$\mathscr{X}^h_{r-1,r+\ell} = \mathscr{Y}^h_{r-1,r+\ell} + \omega^*\mathscr{E}_r$$

in $Ext^1(X_{r,r+\ell},A_r)$. Combining the last two equations we obtain

$$f^*_{r,r+\ell}{\mathscr{X}'}^h_{r-1,r+\ell} = (f_{r-1,r})_*\mathscr{Y}^h_{r-1,r+\ell} + (f_{r-1,r})_*\,\omega^*\mathscr{E}_r = f^*_{r,r+\ell}{\mathscr{Y}'}^h_{r-1,r+\ell} + (f_{r-1,r})_*\,\omega^*\mathscr{E}_r,$$

where $(Y'_{\bullet,\bullet}) = tr((Y_{\bullet,\bullet}),(f_{\bullet,\bullet}))$. (We have used the analogue of (38) for $(Y_{\bullet,\bullet})$.) Thus

$${\mathscr{X}'}^h_{r-1,r+\ell} = {\mathscr{Y}'}^h_{r-1,r+\ell} + (\omega f^{-1}_{r,r+\ell})^*(f_{r-1,r})_*\mathscr{E}_r \tag{39}$$

as pushforward commutes with pullback.

Let

$$(Z''_{\bullet,\bullet}) := (f_A\cdot\mathscr{E}) * tr((Y_{\bullet,\bullet}),(f_{\bullet,\bullet})) = (f_A\cdot\mathscr{E}) * (Y'_{\bullet,\bullet}),$$

so that the right hand side of (35) is the class of $(Z''_{\bullet,\bullet})$ in

$$\big(\,\Theta_\ell^{-1}((X'_{\bullet,\bullet}))\,\big)/\!\sim'.$$

On recalling the definition of $f_A \cdot \mathscr{E}$ and the fact that the map $X'_{r,r+\ell} \twoheadrightarrow A_{r+\ell}$ is $f_{r+\ell-1,r+\ell}\omega f^{-1}_{r,r+\ell}$, we have

$$\mathscr{X}''^h_{r-1,r+\ell} = \mathscr{Y}'^h_{r-1,r+\ell} + (f_{r+\ell-1,r+\ell}\,\omega f^{-1}_{r,r+\ell})^*(f_{r-1,r})_*(f^{-1}_{r+\ell-1,r+\ell})^*\,\mathscr{E}_r \overset{(39)}{=} \mathscr{X}'^h_{r-1,r+\ell}.$$

Thus by Lemma 2.1.2, $(Z'_{\bullet,\bullet})$ and $(Z''_{\bullet,\bullet})$ coincide in

$$\prod_r Extpan(\mathscr{X}'^v_{r,r+\ell}, \mathscr{X}'^h_{r-1,r+\ell-1})$$

and hence are in the same $\sim'$-equivalence class. This completes the proof of the identity.

The statement about the special case when $(f_{\bullet,\bullet})$ is a scalar automorphism of A is immediate from the identity. □

Putting the results of this subsection and the previous one together, we obtain parts (b) and (c) of Theorem 1.2.1:

Proof of Theorem 1.2.1(b,c). Consider an element ϵ of $S'_{\ell-1}(A)$, i.e. a $\sim'$-equivalence class of generalized extensions of level $\ell-1$. Let $(X_{\bullet,\bullet})$ be a representative of the class. Use the bijection (31) and the bijection of Lemma 3.6.4(b) to give the fiber of $S'_\ell(A) \to S'_{\ell-1}(A)$ above ϵ the structure of a torsor for

$$\prod_r Ext^1(A_{r+\ell}, A_r)$$

when this fiber is nonempty. This torsor structure is independent of the choice of $(X_{\bullet,\bullet})$. Indeed, suppose one chooses another representative $(X'_{\bullet,\bullet})$ for ϵ. Let $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ be an isomorphism that is identity on A. By Lemma 3.7.3, transport along $(f_{\bullet,\bullet})$ gives an isomorphism of torsors (33). In view of the commutative diagram (32) of Lemma 3.7.2(b), the induced torsor structures on $\Theta_\ell^{-1}(\epsilon)$ thus coincide.

The assertion in Theorem 1.2.1(c) giving a sufficient condition for surjectivity of the truncation map $S'_\ell(A) \to S'_{\ell-1}(A)$ is immediate from the constructions and Lemma 2.1.1(a). (In fact, a more precise statement about the image of $S'_\ell(A) \to S'_{\ell-1}(A)$ can be obtained from the last sentence of Lemma 3.6.4(b).) □

3.8. Fibers of truncation maps III: Γ-actions. So far, we have studied the fibers of the truncation maps $S'_\ell \to S'_{\ell-1}$. The proof of the remaining assertions of Theorem 1.2.1 involves an additional ingredient, namely the group actions that allow us to pass on from the fibers on the second row of (25) to the fibers on its third row. This is the subject of this subsection. In particular, we will deduce part (e) of Theorem 1.2.1.

Fix $2 \le \ell \le k-1$ and $(X_{\bullet,\bullet}) \in D_{\ell-1}(A)$. The group $Aut((X_{\bullet,\bullet}))$ of the automorphisms of $(X_{\bullet,\bullet})$ (as a generalized extension) acts on the fiber of $\Theta_\ell : D_\ell(A) \to D_{\ell-1}(A)$ above $(X_{\bullet,\bullet})$ by transport. That is, an element $\sigma = (\sigma_{\bullet,\bullet}) \in Aut((X_{\bullet,\bullet}))$ acts by sending $(Y_{\bullet,\bullet}) \in \Theta_\ell^{-1}((X_{\bullet,\bullet}))$ to the transport of $(Y_{\bullet,\bullet})$ along σ. The fact that this transport is also in $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$ is because σ is an automorphism of generalized extensions. Denote the image of $(Y_{\bullet,\bullet})$ under this action by $\sigma\cdot(Y_{\bullet,\bullet})$. Then $\sigma\cdot(Y_{\bullet,\bullet}) = tr((Y_{\bullet,\bullet}),\sigma)$ is obtained from $(Y_{\bullet,\bullet})$ by twisting *only* the arrows between the two lowest diagonals of $(Y_{\bullet,\bullet})$. More explicitly, each horizontal arrow $Y_{r-1,r+\ell} \twoheadrightarrow X_{r,r+\ell}$ (resp. vertical arrow $X_{r-1,r+\ell-1} \hookrightarrow Y_{r-1,r+\ell}$) between the lowest two diagonals gets composed with $\sigma_{r,r+\ell}$ (resp. $\sigma^{-1}_{r-1,r+\ell-1}$). The rest of the diagram remains unchanged. Note that since the action of $Aut((X_{\bullet,\bullet}))$ on $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$ is given by transports, we have $(Y_{\bullet,\bullet}) \sim \sigma\cdot(Y_{\bullet,\bullet})$ for all $(Y_{\bullet,\bullet})$ and σ.

In view of Lemma 3.7.2(b) (applied with $(X'_{\bullet,\bullet}) = (X_{\bullet,\bullet})$ and $(f_{\bullet,\bullet}) = \sigma$), the action of $Aut((X_{\bullet,\bullet}))$ on the fiber $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$ descends to an action on $\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\!\sim'$. This action captures the passing from $\sim'$ to $\sim$ on $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$:

Lemma 3.8.1. *As above, let $2 \leq \ell \leq k-1$ and $(X_{\bullet,\bullet}) \in D_{\ell-1}(A)$. Let $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ be in $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$. Then $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ are $\sim$-equivalent if and only if the classes of $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ mod $\sim'$ are in the same orbit of the action of $Aut((X_{\bullet,\bullet}))$ on*

$$\big(\ \Theta_\ell^{-1}((X_{\bullet,\bullet}))\ \big)/\!\sim' . \tag{40}$$

Proof. Suppose that $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ are isomorphic. Let $(f_{\bullet,\bullet}) : (Y_{\bullet,\bullet}) \to (Y'_{\bullet,\bullet})$ be an isomorphism. Then $(f_{\bullet,\bullet})$ restricts to an automorphism σ of $(X_{\bullet,\bullet})$. Consider the composition of isomorphisms

$$\sigma \cdot (Y_{\bullet,\bullet}) \longrightarrow (Y_{\bullet,\bullet}) \xrightarrow{(f_{\bullet,\bullet})} (Y'_{\bullet,\bullet}),$$

where the first arrow is the isomorphism given by the transport data σ^{-1} (so is identity on the objects of the lowest diagonal and σ^{-1} on $(X_{\bullet,\bullet})$, see §3.7). This composition isomorphism is identity on A, so that $\sigma \cdot (Y_{\bullet,\bullet}) \sim' (Y'_{\bullet,\bullet})$.

Conversely, suppose that the classes of $(Y_{\bullet,\bullet})$ and $(Y'_{\bullet,\bullet})$ in (40) are in the same orbit of the action of $Aut((X_{\bullet,\bullet}))$. Thus there is $\sigma \in Aut((X_{\bullet,\bullet}))$ such that $\sigma \cdot (Y_{\bullet,\bullet}) \sim' (Y'_{\bullet,\bullet})$. Then $(Y_{\bullet,\bullet}) \sim \sigma \cdot (Y_{\bullet,\bullet}) \sim (Y'_{\bullet,\bullet})$. $\square$

The group actions above on the fibers of $D_\ell(A) \to D_{\ell-1}(A)$ descend canonically to group actions on the fibers of $S'_\ell(A) \to S'_{\ell-1}(A)$. Indeed, let $\epsilon' \in S'_{\ell-1}(A)$. For every $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ in $D_{\ell-1}(A)$ that belong to ϵ', by Lemma 3.3.2 there exists a unique isomorphism $(f_{\bullet,\bullet}) : (X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ that is identity on A. This isomorphism induces an isomorphism $Aut((X_{\bullet,\bullet})) \to Aut((X'_{\bullet,\bullet}))$ by conjugation. These distinguished isomorphisms $Aut((X_{\bullet,\bullet})) \to Aut((X'_{\bullet,\bullet}))$ for various pairs $((X_{\bullet,\bullet}), (X'_{\bullet,\bullet}))$ of representatives of ϵ' are compatible with respect to compositions with one another, so that they form a projective system of groups. Set

$$\Gamma(\epsilon') := \lim_{(X_{\bullet,\bullet}) \in \epsilon'} Aut((X_{\bullet,\bullet})).$$

An element of $\Gamma(\epsilon')$ is the data of an automorphism of $(X_{\bullet,\bullet})$ for each representative $(X_{\bullet,\bullet}) \in \epsilon'$, such that the automorphisms for different representatives corresponds to one another under the distinguished isomorphisms between the automorphism groups. For each $(X_{\bullet,\bullet})$, the projection map $\Gamma(\epsilon') \to Aut((X_{\bullet,\bullet}))$ is an isomorphism.

Given any $(X_{\bullet,\bullet})$ and $(X'_{\bullet,\bullet})$ representing ϵ', we have a commutative diagram

$$\begin{tikzcd}
\big(\ \Theta_\ell^{-1}((X_{\bullet,\bullet}))\ \big)/\!\sim' \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & \big(\ \Theta_\ell^{-1}((X'_{\bullet,\bullet}))\ \big)/\!\sim' \arrow[dl, "\simeq"] \\
& \Theta_\ell^{-1}(\epsilon'), &
\end{tikzcd}$$

where the slanted arrows are the natural maps and the top arrow is the bijection given by transport along the distinguished isomorphism $(X_{\bullet,\bullet}) \to (X'_{\bullet,\bullet})$ (see Lemma 3.7.2(b)). Identifying $Aut((X_{\bullet,\bullet}))$ and $Aut((X'_{\bullet,\bullet}))$ via the distinguished isomorphism between them, the top map of this diagram is compatible with their actions (this is easily verified by (29)). We thus obtain a well-defined action of $\Gamma(\epsilon')$ on $\Theta_\ell^{-1}(\epsilon')$, which can be computed by taking any representative $(X_{\bullet,\bullet})$ of ϵ': via the canonical isomorphism $\Gamma(\epsilon') \to \Gamma((X_{\bullet,\bullet}))$ and the left bijection of the diagram above, the action of $\Gamma(\epsilon')$ on $\Theta_\ell^{-1}(\epsilon')$ is the action of $Aut((X_{\bullet,\bullet}))$ on $(\Theta_\ell^{-1}((X_{\bullet,\bullet})))/\sim'$.

We have all the necessary components to establish part (e) of Theorem 1.2.1. Let us restate the result.

Proposition 3.8.2. *For every $\epsilon \in S_{\ell-1}(A)$ and $\epsilon' \in S'_{\ell-1}(A)$ above ϵ, the natural map*

$$\Theta_\ell^{-1}(\epsilon') \to \Theta_\ell^{-1}(\epsilon)$$

(restricted from $S'_\ell(A) \to S_\ell(A)$) is surjective, and it descends to a bijection

$$\Theta_\ell^{-1}(\epsilon')/\Gamma(\epsilon') \;\cong\; \Theta_\ell^{-1}(\epsilon).$$

Proof. Choose a generalized extension $(X_{\bullet,\bullet})$ in ϵ' (and hence ϵ). We have a commutative diagram

$$\begin{tikzcd}
\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\sim' \arrow[r, "\simeq"] \arrow[d, two heads] & \Theta_\ell^{-1}(\epsilon') \arrow[d] \\
\big(\,\Theta_\ell^{-1}((X_{\bullet,\bullet}))\,\big)/\sim \arrow[r, "\simeq"] & \Theta_\ell^{-1}(\epsilon)
\end{tikzcd}$$

where all maps are the natural ones and the horizontal maps are bijective by Lemma 3.7.2. Thus the right vertical arrow is also surjective.

By Lemma 3.8.1, two elements of $(\Theta_\ell^{-1}((X_{\bullet,\bullet})))/\sim'$ coincide in $(\Theta_\ell^{-1}((X_{\bullet,\bullet})))/\sim$ if and only if they are in the same orbit of the action of $Aut((X_{\bullet,\bullet}))$ on $(\Theta_\ell^{-1}((X_{\bullet,\bullet})))/\sim'$. In view of the commutative diagram above and the definition of the action of $\Gamma(\epsilon')$ on $\Theta_\ell^{-1}(\epsilon')$, it follows that two elements of $\Theta_\ell^{-1}(\epsilon')$ coincide in $\Theta_\ell^{-1}(\epsilon)$ if and only if they are in the same $\Gamma(\epsilon')$-orbit. □

We end this discussion with a result about the stabilizers of the Γ-actions, which will be useful for our study of the totally nonsplit case in the next subsection.

Lemma 3.8.3. *Let $(X_{\bullet,\bullet}) \in D_{\ell-1}(A)$ and $(Y_{\bullet,\bullet}) \in \Theta^{-1}((X_{\bullet,\bullet}))$. Then the stabilizer of the $\sim'$-equivalence class of $(Y_{\bullet,\bullet})$ for the action of $Aut((X_{\bullet,\bullet}))$ on (40) is the image of the restriction map*

$$Aut((Y_{\bullet,\bullet})) \hookrightarrow Aut((X_{\bullet,\bullet})).$$

Proof. Let $\sigma \in Aut((X_{\bullet,\bullet}))$. Then σ fixes the $\sim'$-equivalence class of $(Y_{\bullet,\bullet})$ if and only if there exists an isomorphism $f : \sigma \cdot (Y_{\bullet,\bullet}) \to (Y_{\bullet,\bullet})$ that is identity on A, or equivalently, on $(X_{\bullet,\bullet})$ (by Lemma 3.3.2). If f is such an isomorphism, then let $\tilde\sigma$ be the composition $(Y_{\bullet,\bullet}) \to \sigma \cdot (Y_{\bullet,\bullet}) \xrightarrow{f} (Y_{\bullet,\bullet})$, where the first arrow is the isomorphism given by the transport data σ (thus identity on the lowest diagonal and σ on $(X_{\bullet,\bullet})$, see §3.7). Then $\tilde\sigma$ is an automorphism of $(Y_{\bullet,\bullet})$ that restricts to σ on $(X_{\bullet,\bullet})$.

On the other hand, if σ extends to an automorphism $\tilde\sigma$ of $(Y_{\bullet,\bullet})$, let f be the composition $\sigma \cdot (Y_{\bullet,\bullet}) \to (Y_{\bullet,\bullet}) \xrightarrow{\tilde\sigma} (Y_{\bullet,\bullet})$, where the first arrow is the isomorphism given by the transport data σ^{-1}. Then f is identity on $(X_{\bullet,\bullet})$. □

Remark 3.8.4. Let $(X_{\bullet,\bullet}) \in D_{\ell-1}(A)$ and $(Y_{\bullet,\bullet}) \in \Theta^{-1}((X_{\bullet,\bullet}))$. By the previous lemma, the class of $(Y_{\bullet,\bullet})$ in $\Theta^{-1}((X_{\bullet,\bullet}))/\sim'$ is a fixed point of the action of $Aut((X_{\bullet,\bullet}))$ if and only if every automorphism of $(X_{\bullet,\bullet})$ extends to an automorphism of $(Y_{\bullet,\bullet})$. Using the formula of Lemma 3.7.3 one can see that the former statement is also equivalent to the linearity of the action of $Aut((X_{\bullet,\bullet}))$ on $\prod_r Ext^1(A_{r+\ell}, A_r)$ obtained as follows: Use the class of $(Y_{\bullet,\bullet})$ as the base point to get an isomorphism

$$\Theta^{-1}((X_{\bullet,\bullet}))/\sim' \;\simeq\; \prod_r Ext^1(A_{r+\ell}, A_r)$$

(recall that the left hand side is a torsor over the right hand side). Now transport the action of $Aut((X_{\bullet,\bullet}))$ from the left side to the right.

Already when $k = 3$ it is easy to see that the map $Aut((Y_{\bullet,\bullet})) \hookrightarrow Aut((X_{\bullet,\bullet}))$ does not always have to be surjective. Given $(X_{\bullet,\bullet})$ with nonempty $\Theta^{-1}((X_{\bullet,\bullet}))$, it would be interesting to see if there always exists a $(Y_{\bullet,\bullet}) \in \Theta^{-1}((X_{\bullet,\bullet}))$ such that $Aut((Y_{\bullet,\bullet})) \cong Aut((X_{\bullet,\bullet}))$. Note that in the basic case of $k = 3$ (i.e. the case of blended extensions), given $(X_{\bullet,\bullet})$ and arbitrary $(Y_{\bullet,\bullet}) \in \Theta^{-1}((X_{\bullet,\bullet}))$, the question of whether a given element of $Aut((X_{\bullet,\bullet}))$ extends to an automorphism of $(Y_{\bullet,\bullet})$ has appeared previously in the work [4] of Barbieri-Viale and Kahn.[14] Proposition D.1.5 of Appendix D therein gives a necessary and sufficient condition for this. (In the language of Barbieri-Viale and Kahn, the question is about whether a *partial gluing* extends to a *gluing*.)

3.9. The totally nonsplit case. The goal of this subsection is to discuss the last part of Theorem 1.2.1. We will define the relevant notions and give the proof of the result. Note that this special case of Theorem 1.2.1 will play a crucial role in §4.

The following definition generalizes the notion of total nonsplitting to generalized extensions. To recall the definition of total nonsplitting for an extension, see Definition 2.2.1.

Definition 3.9.1. We say a generalized extension $(X_{\bullet,\bullet})$ of positive level $\ell-1$ is weakly totally nonsplit if for every $0\leq r\leq k-\ell$, at least one of the extensions $\mathfrak{X}^v_{r,r+\ell}$ or $\mathfrak{X}^h_{r,r+\ell}$ is totally nonsplit. (These extensions respectively arise from the injective arrow going to $X_{r,r+\ell}$ and the surjective arrow coming from it; see Notation 3.3.3. Note that $X_{r,r+\ell}$ is on the lowest diagonal of $(X_{\bullet,\bullet})$.)

We refer to this notion as the *weak* total nonsplitting because there will be a stronger variant of it that we will introduce later (see §4.4). In level 1, where a generalized extension is the data of an extension $\mathscr{E}_r$ of A_{r+1} by A_r for each r, the generalized extension $(\mathscr{E}_r)$ is weakly totally nonsplit if and only if all the $\mathscr{E}_r$ are totally nonsplit.

What makes the notion of weak total nonsplitting interesting for us is the following property (recall that our category is a filtered tannakian category over a field K of characteristic zero):

Lemma 3.9.2. *Let $(X_{\bullet,\bullet})$ be a weakly totally nonsplit generalized extension of positive level. Then every automorphism of $(X_{\bullet,\bullet})$ is a scalar map, i.e. $Aut((X_{\bullet,\bullet}))\cong K^\times$.*

Proof. Let $\ell-1$ be the level. By restricting to the action on the objects on the lowest diagonal we have an injection

$$Aut((X_{\bullet,\bullet}))\hookrightarrow\prod_r Aut(X_{r,r+\ell}). \tag{41}$$

By Lemma 3.3.5, the image of this map is the set of those elements $(\sigma_{r,r+\ell})$ (with $\sigma_{r,r+\ell}$ an automorphism of $X_{r,r+\ell}$) the entries of which are compatible on the diagonal just above the lowest, i.e. the set of elements $(\sigma_{r,r+\ell})$ such that for each r, the two automorphisms of $X_{r,r+\ell-1}$ induced by $\sigma_{r-1,r+\ell-1}$ and $\sigma_{r,r+\ell}$ respectively via the arrows $X_{r-1,r+\ell-1}\twoheadrightarrow X_{r,r+\ell-1}$ and $X_{r,r+\ell-1}\hookrightarrow X_{r,r+\ell}$ coincide (when they are both available). For every r, either the extension $\mathfrak{X}^v_{r,r+\ell}$ or $\mathfrak{X}^h_{r,r+\ell}$ is totally nonsplit, so that by Lemma 2.2.2 every automorphism of $X_{r,r+\ell}$ is a scalar map. Thus each factor $Aut(X_{r,r+\ell})$ of the codomain of (41) is $K^\times$. Compatibility on the diagonal just above the lowest implies that the image of (41) is the diagonal copy of $K^\times$. ☐

The notion of weakly totally nonsplit generalized extensions clearly descends to isomorphism classes of generalized extensions. We are ready to deduce the final part of Theorem 1.2.1.

Proof of Theorem 1.2.1(f). Let $\epsilon\in S_{\ell-1}(A)$ be weakly totally nonsplit. Let ϵ' be an element of $S'_{\ell-1}(A)$ above ϵ. We need to argue that the action of $\Gamma(\epsilon')$ on $\Theta_\ell^{-1}(\epsilon')$ is trivial. The rest of the statement then follows from the earlier parts of the theorem.

Let $(X_{\bullet,\bullet})\in D_{\ell-1}(A)$ be a representative of ϵ'. By the previous lemma, the automorphism group of $(X_{\bullet,\bullet})$ consists only of the nonzero scalar maps. Thus for every $(Y_{\bullet,\bullet})$ in $\Theta_\ell^{-1}((X_{\bullet,\bullet}))$, the restriction map $Aut((Y_{\bullet,\bullet}))\to Aut((X_{\bullet,\bullet}))$ is surjective, so that by Lemma 3.8.3 the action

[14] I thank Bertrand for bringing this to my attention.

of $Aut((X_{\bullet,\bullet}))$ on $\Theta^{-1}((X_{\bullet,\bullet}))\ /\!\sim'$ fixes the class of $(Y_{\bullet,\bullet})$. Thus the action of $Aut((X_{\bullet,\bullet}))$ on $\Theta_\ell^{-1}((X_{\bullet,\bullet}))/\sim'$ and hence the action of $\Gamma(\epsilon')$ on $\Theta_\ell^{-1}(\epsilon')$ is trivial. □

Remark 3.9.3. Let $\epsilon \in S_{\ell-1}(A)$ be a weakly totally nonsplit element. Then the torsor structures on the fiber $\Theta_\ell^{-1}(\epsilon)$ corresponding to different choices of lifts of ϵ to $S'_{\ell-1}(A)$ are related to each other in a canonical way, as follows. Let ϵ'_1 and ϵ'_2 be two elements of $S'_{\ell-1}(A)$ above ϵ. Use the notation $*_1$ (resp. $*_2$) for the translation operation for the torsor structure on $\Theta_\ell^{-1}(\epsilon)$ corresponding to the choice of ϵ'_1 (resp. ϵ'_2) as the lift of ϵ. Then there exists an automorphism ϕ of $\prod_r Ext^1(A_{r+\ell}, A_r)$ such that for every $\tilde{\epsilon} \in \Theta_\ell^{-1}(\epsilon)$ and $\mathscr{E} \in \prod_r Ext^1(A_{r+\ell}, A_r)$, we have

$$\mathscr{E} *_1 \tilde{\epsilon} = \phi(\mathscr{E})\ *_2\ \tilde{\epsilon}. \tag{42}$$

Indeed, choosing representatives $(X^{(1)}_{\bullet,\bullet})$ and $(X^{(2)}_{\bullet,\bullet})$ in $D_{\ell-1}$ for ϵ'_1 and ϵ'_2, and an isomorphism $(f_{\bullet,\bullet}) : (X^{(1)}_{\bullet,\bullet}) \to (X^{(2)}_{\bullet,\bullet})$, denote the restriction of $(f_{\bullet,\bullet})$ to A by f_A. The desired map ϕ is the automorphism of $\prod_r Ext^1(A_{r+\ell}, A_r)$ induced by f_A by pullbacks and pushforwards (i.e. given by $\phi(\mathscr{E}) = f_A \cdot \mathscr{E}$ in our earlier notation). Note that since ϵ is weakly totally nonsplit, by Lemma 3.9.2 the isomorphism $(f_{\bullet,\bullet})$ is unique up to scaling, so that ϕ will not depend on the choice of $(f_{\bullet,\bullet})$. One can use Theorem 1.2.1(f) and the formula of Lemma 3.7.3 to obtain (42).

4. Motives with maximal unipotent radicals

4.1. Setting and background. In this section we assume that $\mathbf{T}$ is a filtered tannakian category over a field K of characteristic zero such that the pure objects of $\mathbf{T}$ (and hence their direct sums) are semisimple. The prototype examples of this are the category of graded-polarizable mixed Hodge structures over $\mathbb{Q}$, and any reasonable tannakian category of mixed motives over a subfield of $\mathbb{C}$ (e.g. those of Ayoub [3] and Nori [31], Voevodsky's category of mixed Tate motives over a number field, and categories of mixed motives defined via systems of realizations in [18] and [32]). Inspired by the latter set of examples, we often refer to an object of $\mathbf{T}$ as a motive.

Let X be an object of $\mathbf{T}$. Let $\underline{\mathfrak{u}}(X)$ be the object of $\mathbf{T}$ associated with the Lie algebra of the unipotent radical of the tannakian group of X. We will take the background material on the definition of $\underline{\mathfrak{u}}(X)$ for granted, referring the reader to §4.2 and §2 of [22] for a detailed review of this background. We just recall here that $\underline{\mathfrak{u}}(X)$ is the canonical subobject of $W_{-1}\underline{End}(X)$ (where $\underline{End}(X)$ is the internal Hom $\underline{Hom}(X,X)$) with the following property: for every fiber functor ω from $\mathbf{T}$ to the category of vector space over K, if we consider the tannakian group of X with respect to ω as a subgroup of $GL(\omega X)$, then

$$\omega\underline{\mathfrak{u}}(X) \subset \omega W_{-1}\underline{End}(X) = W_{-1}End(\omega X)$$

is the Lie algebra of the kernel of the natural surjection from the tannakian group of X with respect to ω to the tannakian group of $Gr^W X$ with respect to ω. The fact that this kernel is the unipotent radical of the tannakian group of X is because $Gr^W X$ is semisimple.

Definition 4.1.1. We say $\underline{\mathfrak{u}}(X)$ is maximal or X has a maximal unipotent radical if

$$\underline{\mathfrak{u}}(X) = W_{-1}\underline{End}(X).$$

As we mentioned in §1.1, the interest in motives with maximal unipotent radicals comes in part from Grothendieck's period conjecture. If $\mathbf{T}$ is the category of mixed motives over a number field, this conjecture predicts that the transcendence degree of the field generated over $\mathbb{Q}$ by the periods of a motive X should be equal to the dimension of the motivic Galois group of X (i.e., the dimension of the tannakian group of X with respect to the Betti or equivalently, any fiber functor over $\mathbb{Q}$). Since the dimension of the motivic Galois group of X is equal to the dimension of its unipotent radical plus the dimension of the motivic Galois group of $Gr^W X$, in view of

Grothendieck's period conjecture, among all X with a given $Gr^W X$ the field generated by the periods of an X with a maximal unipotent radical should have the largest possible transcendence degree. We refer the reader to André's book [2] or his letter to Bertolin published at the end of [8] for more background on Grothendieck's period conjecture.

The aim of this section is to use the methods of §3 to study motives whose unipotent radicals are maximal and whose associated graded objects are also isomorphic to a given graded object A, particularly in the case where A is *graded-independent* (see Definition 4.3.1). The final result (Theorem 4.4.4) significantly generalizes our work in [22, §6], where with Murty we established the special case of this result when $A = A_1 \oplus A_2 \oplus \mathbb{1}$ with A_1, A_2 pure of negative increasing weights and $Ext^1(\mathbb{1}, A_1) = 0$.

In §4.2 below we briefly discuss the general relationship between two related notions, namely the notion of maximality of the unipotent radical of an object and the notion of total nonsplitting of the extensions that naturally arise from the object. There is little new information in this discussion, and its inclusion is mostly to put the later results in a better context. In §4.3 we define the notion of graded-independence, which is an independence axiom in the spirit of such axioms in [22], and give our maximality criterion for graded-independent motives (Theorem 4.3.2). We then combine this in §4.4 with the weakly totally nonsplit case of Theorem 1.2.1 to give a classification result for motives with maximal unipotent radicals and a given graded-independent associated graded. Finally, in §4.5 we consider the example of 4-dimensional mixed Tate motives over $\mathbb{Q}$ to illustrate the results.

4.2. Maximality and total nonsplitting. The following lemma summarizes some aspects of the relationship between maximality of unipotent radicals and total nonsplitting of extensions coming from the weight filtration. Recall that throughout, $\mathbf{T}$ is a filtered tannakian category over a field of characteristic zero in which pure objects are semisimple. The objects of $\mathbf{T}$ are referred to as motives.

Lemma 4.2.1. *Let X be a motive with more than one weight.*
(a) If $\underline{\mathfrak{u}}(X)$ is maximal, then for every integers $\ell < m < n$ the extension

$$0 \longrightarrow W_m X/W_\ell X \longrightarrow W_n X/W_\ell X \longrightarrow W_n X/W_m X \longrightarrow 0 \tag{43}$$

is totally nonsplit.
(b) If $Gr^W X = Gr^W_m X \oplus Gr^W_n X$ with $m < n$, then $\underline{\mathfrak{u}}(X)$ is maximal if and only if the extension

$$0 \longrightarrow Gr^W_m X \longrightarrow X \longrightarrow Gr^W_n X \longrightarrow 0 \tag{44}$$

is totally nonsplit. (Thus when X has only two weights, the necessary condition of part (a) for maximality of $\underline{\mathfrak{u}}(X)$ is also sufficient.)
(c) Taking $\mathbf{T}$ to be the category of graded-polarizable rational mixed Hodge structures, there exists an object X of $\mathbf{T}$ with 3 weights for which all of the extensions of part (a) are totally nonsplit, but $\underline{\mathfrak{u}}(X)$ is not maximal. (Thus in general, the condition of part (a) for maximality is not sufficient.)

We first discuss parts (b) and (c). The former is by a result of Hardouin ([30, Theorem 2], see also Theorem 2.1 of the unpublished article [29]), which was proved earlier by Bertrand for the special case of D-modules [10, Theorem 1.1]. In the setting of part (b) with two weights $m < n$ (and semisimple $Gr^W X$), Hardouin's result asserts that $\underline{\mathfrak{u}}(X)$ is the smallest subobject of

$$W_{-1}\underline{End}(X) = \underline{Hom}(Gr^W_n X, Gr^W_m X)$$

with the following property: the extension (44), considered as an element of

$$Ext^1(\mathbb{1}, \underline{Hom}(Gr_n^W X, Gr_m^W X))$$

via the isomorphism (12), splits after pushing forward along the quotient map

$$\underline{Hom}(Gr_n^W X, Gr_m^W X) \to \underline{Hom}(Gr_n^W X, Gr_m^W X)/\underline{\mathfrak{u}}(X).$$

On recalling the definition of total nonsplitting (see Definition 2.2.1), the statement in (b) is now immediate.[15]

As for part (c), we refer the reader to §5.2 of [23] for an example with X being a 1-motive.

Turning our attention to part (a), let us first recall a result from [22].

Theorem 4.2.2 (Theorem 4.9.1 of [22])**.** *Let X be an object of $\mathbf{T}$. Let $\mathscr{E}_m(X)$ be the class of the extension*

$$0 \longrightarrow W_m X \longrightarrow X \longrightarrow X/W_m X \longrightarrow 0 \tag{45}$$

considered as an element of

$$Ext^1(\mathbb{1}, \underline{Hom}(X/W_m X, W_m X))$$

via the canonical isomorphism (12)*. Let*

$$\underline{\mathfrak{u}}_m(X) := \underline{\mathfrak{u}}(X) \cap \underline{Hom}(X/W_m X, W_m X),$$

where $\underline{Hom}(X/W_m X, W_m X)$ is considered as a subobject of $W_{-1}\underline{End}(X)$ in the natural way. Then for every subobject L of $\underline{Hom}(X/W_m X, W_m X)$ one has

$$\underline{\mathfrak{u}}_m(X) \subset L$$

if and only if the pushforward $\mathscr{E}_m(X)/L$ of $\mathscr{E}_m(X)$ along the quotient map

$$\underline{Hom}(X/W_m X, W_m X) \to \underline{Hom}(X/W_m X, W_m X)/L$$

is in the image of the obvious map

$$Ext^1_{\langle (X/W_m X)\oplus W_m X\rangle}\left(\mathbb{1}, \underline{Hom}(X/W_m X, W_m X)/L\right) \hookrightarrow Ext^1\left(\mathbb{1}, \underline{Hom}(X/W_m X, W_m X)/L\right).$$

Here and later in the paper, the notation $\langle Y\rangle$ means the tannakian subcategory generated by an object Y of $\mathbf{T}$ (i.e., the smallest full tannakian subcategory of $\mathbf{T}$ containing Y and closed under subquotients). The notation $Ext^1_{\langle Y\rangle}$ means the Ext^1 group for the category $\langle Y\rangle$.

Referring to the setting of the theorem, note in particular that the result implies that if a subobject L of $\underline{Hom}(X/W_m X, W_m X)$ has the property that the pushforward $\mathscr{E}_m(X)/L$ splits, then L contains $\underline{\mathfrak{u}}_m(X)$. Thus one obtains the following corollary:

Corollary 4.2.3. *With the notation and setting as in Theorem 4.2.2, if*

$$\underline{\mathfrak{u}}_m(X) = \underline{Hom}(X/W_m X, W_m X),$$

then $\mathscr{E}_m(X)$ (or equivalently, the extension (45)*) is totally nonsplit.*

We are ready to deduce part (a) of Lemma 4.2.1.

[15] Both Hardouin and Bertrand consider extensions of $\mathbb{1}$ by a semisimple object L. The version of the result that allows for extensions of a semisimple object N by a semisimple object L (which is what is being used here) can be found in [21, Corollary 3.4.1].

Proof of Lemma 4.2.1(a). Let $\underline{\mathfrak{u}}(X)=W_{-1}\underline{End}(X)$. The inclusion $\langle W_nX/W_\ell X\rangle\subset\langle X\rangle$ induces a surjection from the tannakian group of X to the tannakian group of $W_nX/W_\ell X$, which in turn induces a surjection $\underline{\mathfrak{u}}(X)\to\underline{\mathfrak{u}}(W_nX/W_\ell X)$. This surjection fits in a commutative diagram

$$\begin{tikzcd}
\underline{\mathfrak{u}}(X) \arrow[r, two heads] \arrow[d, equal] & \underline{\mathfrak{u}}(W_nX/W_\ell X) \arrow[d, hook] \\
W_{-1}\underline{End}(X) \arrow[r, two heads] & W_{-1}\underline{End}(W_nX/W_\ell X),
\end{tikzcd}$$

where the vertical arrows are the canonical inclusions and the bottom horizontal arrow is the map that after applying a fiber functor ω, it sends an element of $W_{-1}End(\omega X)$ to its induced endomorphism of $W_n\omega X/W_\ell\omega X$. It follows that $\underline{\mathfrak{u}}(W_nX/W_\ell X)$ is also maximal. The assertion now follows from the previous corollary (applied to $W_nX/W_\ell X$ instead of X). □

Remark 4.2.4. The hypothesis of semisimplicity of pure objects of $\mathbf{T}$ is not needed for Theorem 4.2.2 (and is not assumed in Theorem 4.9.1 of [22]). Subsequently, Corollary 4.2.3 and Lemma 4.2.1(a) are true in arbitrary filtered tannakian categories over fields of characteristic zero. However, Lemma 4.2.1(b) needs the hypothesis of semisimplicity of pure objects.

4.3. Maximality of unipotent radicals for graded-independent motives. In this subsection we study maximality of unipotent radicals of *graded-independent* motives, a class of motives defined below which are the focus of the rest of the paper. We will prove a necessary and sufficient condition for maximality of unipotent radicals of such motives.

We start with the definition of the notion of graded-independence.

Definition 4.3.1. Let X be a nonzero motive. Denote the weights of X by $p_1<\cdots<p_k$. Consider the k objects

$$C_r:=\underline{Hom}(Gr^W_{p_{r+1}}X,Gr^W_{p_r}X)\qquad(1\le r\le k-1)\tag{46}$$

and

$$C_0:=\bigoplus_{j-i>1}\underline{Hom}(Gr^W_{p_j}X,Gr^W_{p_i}X).\tag{47}$$

We say that the motive X is graded-independent if for every distinct r,r' $(0\le r,r'\le k-1)$, the two objects C_r and $C_{r'}$ do not have any nonzero isomorphic subquotients (or equivalently, subobjects since Gr^WX is semisimple).

The property is an "independence axiom" in the spirit of such axioms in [22]. In general, the reason such properties are of interest is twofold. Firstly, they force $Gr^W\underline{\mathfrak{u}}(X)$ to decompose in a way that makes it easier to study. Secondly, they are not far too restrictive, in the sense that they are satisfied in some very interesting situations. For instance, the property above (as well as all the independence axioms in [22]) is satisfied as long as the weights of X are sufficiently spread out so that the numbers p_i-p_j are all distinct as the integers i,j vary in $1\le i<j\le k$.

We now state our maximality criterion:

Theorem 4.3.2. *Let X be a graded-independent motive. Let $p_1<\cdots<p_k$ be the weights of X. Set $p_0:=p_1-1$ (so that $W_{p_0}(X)=0$). Then the following two statements are equivalent:*

(i) *$\underline{\mathfrak{u}}(X)$ is maximal.*

(ii) *For every integer $1\le r\le k-1$, the extension*

$$0\longrightarrow Gr^W_{p_r}X\longrightarrow\frac{W_{p_{r+1}}X}{W_{p_{r-1}}X}\longrightarrow Gr^W_{p_{r+1}}X\longrightarrow 0\tag{48}$$

is totally nonsplit.

Proof. The fact that (i) implies (ii) is clear and in fact, does not require the graded-independence condition; see Lemma 4.2.1(a). We will prove that (ii) implies (i).

It is enough to show that

$$Gr^W \underline{\mathfrak{u}}(X) = Gr^W W_{-1}\underline{End}(X).$$

The graded-independence condition (on recalling that $\langle Gr^W X\rangle$ is semisimple) guarantees that the subobject $Gr^W\underline{\mathfrak{u}}(X)$ of

$$Gr^W W_{-1}\underline{End}(X) = \bigoplus_{i<j} \underline{Hom}(Gr^W_j X, Gr^W_i X)$$

decomposes according to the decomposition of $Gr^W W_{-1}\underline{End}(X)$ as the direct sum of the k objects (46) and (47) of Definition 4.3.1. For each $1 \le r \le k-1$, consider the composition

$$\underline{\mathfrak{u}}(X) \twoheadrightarrow \underline{\mathfrak{u}}(W_{p_{r+1}}X/W_{p_{r-1}}X) \hookrightarrow \underline{Hom}(Gr^W_{p_{r+1}}X, Gr^W_{p_r}X),$$

where the first map is induced by the inclusion of $\langle W_{p_{r+1}}X/W_{p_{r-1}}X\rangle$ in $\langle X\rangle$ and the second map is the natural inclusion. Since (48) is totally nonsplit, by Lemma 4.2.1(b) this second arrow is an equality, so that the composition is surjective. Applying Gr^W we get a surjection of $Gr^W\underline{\mathfrak{u}}(X)$ onto $\underline{Hom}(Gr^W_{p_{r+1}}X, Gr^W_{p_r}X)$. In view of the decomposition of $Gr^W\underline{\mathfrak{u}}(X)$ as the direct sum of its intersections with the k objects of Definition 4.3.1 and the fact that these k objects do not have any nonzero isomorphic subquotients, it follows that $Gr^W\underline{\mathfrak{u}}(X)$ contains

$$\underline{Hom}(Gr^W_{p_{r+1}}X, Gr^W_{p_r}X)$$

for all $1 \le r \le k-1$. Since $Gr^W W_{-1}\underline{End}(X)$ is generated as a Lie algebra by these $k-1$ objects, the result follows. □

Remark 4.3.3. The graded-independence hypothesis is crucial for condition (ii) of the previous result to guarantee that $\underline{\mathfrak{u}}(X)$ is maximal. See Lemma 4.2.1(c).

4.4. Motives with maximal unipotent radicals and a prescribed graded-independent associated graded. In this subsection we combine our maximality criterion from §4.3 with our work in §3.

Fix $k \ge 2$. Let $A_1, \dots, A_k$ be nonzero pure motives, with A_i of weight p_i and $p_1 < \cdots < p_k$. Set

$$A := \bigoplus_{1\le i\le k} A_i.$$

We will use the notation and language of §3. Thus $S(A)$ denotes the set of isomorphism classes of motives whose associated graded is isomorphic to A, and for each $1 \le \ell \le k-1$ by $S_\ell(A)$ we denote the set of $\sim$-equivalence classes (i.e. isomorphism classes) of generalized extensions of level ℓ of A. The truncation map from level ℓ to level $\ell - 1$ is denoted by Θ_ℓ.

Notation 4.4.1. Let $S^*(A)$ be the subset of $S(A)$ consisting of the isomorphism classes of motives X with maximal $\underline{\mathfrak{u}}(X)$ and $Gr^W X$ isomorphic to A.

Our goal is to give a characterization of the set $S^*(A)$ when A is graded-independent.

Earlier we defined the notion of a weakly totally nonsplit generalized extension (Definition 3.9.1). We now define a stronger variant of the notion.

Definition 4.4.2. We say a generalized extension $(X_{\bullet,\bullet})$ of any level is totally nonsplit if the extensions $\mathfrak{X}^v_{m,n}$ and $\mathfrak{X}^h_{m,n}$ (see Notation 3.3.3) are totally nonsplit for every pair (m,n) in the eligible range. (Here, (m,n) need not correspond to an entry on the lowest diagonal.)

In level 1, the two notions of total nonsplitting and weak total nonsplitting coincide. In any level, a totally nonsplit generalized extension is weakly totally nonsplit. It is not difficult to see that the converse is not true in general. The notion of a totally nonsplit generalized extension descends to the isomorphism classes of such extensions.

It is clear from the definition that if a generalized extension is totally nonsplit, then so are its truncations. In the graded-independent case, the converse is also true:

Lemma 4.4.3. *Let A (= the direct sum of the A_i, as above) be graded-independent and $(X_{\bullet,\bullet})$ a generalized extension of A of level $\ell \geq 2$. If the truncation $\Theta_\ell((X_{\bullet,\bullet}))$ is totally nonsplit, then so is $(X_{\bullet,\bullet})$.*

Proof. We show that if $(X_{i,j})_{j-i\leq 2}$ (i.e the truncation of $(X_{\bullet,\bullet})$ to level 1) is totally nonsplit, then so is $(X_{\bullet,\bullet})$; this will establish the result since total nonsplitting of $\Theta_\ell((X_{\bullet,\bullet}))$ implies total nonsplitting of the truncation to level 1.

Suppose that $(X_{i,j})_{j-i\leq 2}$ is totally nonsplit. By definition, this means that each of the extensions

$$0 \longrightarrow A_i \longrightarrow X_{i-1,i+1} \longrightarrow A_{i+1} \longrightarrow 0 \tag{49}$$

is totally nonsplit. Consider an object $X_{r-1,r+\ell}$ on the lowest diagonal of $(X_{\bullet,\bullet})$. By Lemma 3.3.1 we have a canonical isomorphism from $Gr^W X_{r-1,r+\ell}$ to the direct sum of A_r, A_{r+1}, $\ldots$, $A_{r+\ell}$. Denoting this canonical isomorphism by ϕ, from Lemma 3.5.3(b) we know that the two generalized extensions (i) the part of $(X_{\bullet,\bullet})$ to the above and right of $X_{r-1,r+\ell}$ (i.e. consisting of the $X_{i,j}$ with $i \geq r-1$ and $j \leq r+\ell$) and (ii) $ext(X_{r-1,r+\ell},\phi)$ (i.e. the generalized extension associated with $(X_{r-1,r+\ell},\phi)$, see §3.2) are $\sim'$-equivalent. It follows that after identifying

$$Gr^W X_{r-1,r+\ell} \cong \bigoplus_{r\leq i\leq r+\ell} A_i$$

via ϕ, for each i with $r \leq i < r+\ell$ the class of the extension

$$0 \longrightarrow Gr^W_{p_i} X_{r-1,r+\ell} \longrightarrow W_{p_{i+1}} X_{r-1,r+\ell}/W_{p_{i-1}} X_{r-1,r+\ell} \longrightarrow Gr^W_{p_{i+1}} X_{r-1,r+\ell} \longrightarrow 0$$

coming from the weight filtration on $X_{r-1,r+\ell}$ is equal to the class of the extension (49). In particular, the former extension is totally nonsplit for all $r \leq i < r+\ell$. Since A is graded-independent, so is $X_{r-1,r+\ell}$, so that by Theorem 4.3.2, $X_{r-1,r+\ell}$ has a maximal unipotent radical. It now follows from Lemma 4.2.1(a) that all of the extensions coming from the weight filtration on $X_{r-1,r+\ell}$ are totally nonsplit, so that $ext(X_{r-1,r+\ell},\phi)$ is totally nonsplit. Being isomorphic to $ext(X_{r-1,r+\ell},\phi)$, the part of $(X_{\bullet,\bullet})$ to the above and right of $X_{r-1,r+\ell}$ is thus also totally nonsplit. This is true for all r, hence $(X_{\bullet,\bullet})$ is totally nonsplit. □

We are ready to give our classification result for graded-independent motives with maximal unipotent radicals.

Theorem 4.4.4. *For each $1 \leq \ell \leq k-1$, denote the set of all totally nonsplit elements of $S_\ell(A)$ by $S^*_\ell(A)$. Recall that $S^*(A)$ is the subset of $S(A)$ consisting of the isomorphism classes of objects with maximal $\underline{\mathfrak{u}}$.*
(a) We have a succession of maps

$$S^*_{k-1}(A) \xrightarrow{\Theta_{k-1}} S^*_{k-2}(A) \xrightarrow{\Theta_{k-2}} S^*_{k-3}(A) \xrightarrow{\Theta_{k-3}} \cdots \xrightarrow{\Theta_3} S^*_2(A) \xrightarrow{\Theta_2} S^*_1(A)$$

given by truncation. There is a natural bijection

$$S^*_1(A) \cong \left\{ (\mathscr{E}_r) \in \prod_r Ext^1(A_{r+1},A_r) : \text{each } \mathscr{E}_r \text{ is totally nonsplit} \right\} \Big/ Aut(A), \tag{50}$$

with the action of $Aut(A)$ given by pushforwards and pullbacks.
(b) Suppose that A is graded-independent. Then the following statements are true:

(i) *There is a canonical bijection*
$$S^*(A) \cong S^*_{k-1}(A).$$

(ii) *For every $\ell \geq 2$, every nonempty fiber of $\Theta_\ell : S^*_\ell(A) \to S^*_{\ell-1}(A)$ is a torsor over*
$$\prod_r Ext^1(A_{r+\ell}, A_r).$$

(iii) *For every $\ell \geq 2$, if the Ext^2 groups*
$$Ext^2(A_{r+\ell}, A_r)$$
vanish for all $1 \leq r \leq k-\ell$, then $\Theta_\ell : S^*_\ell(A) \to S^*_{\ell-1}(A)$ is surjective.

Proof. (a) The truncation of a totally nonsplit generalized extension is totally nonsplit, so the truncation maps between the $S_\ell(A)$ restrict to maps between the $S^*_\ell(A)$. The bijection (50) is the restriction of the bijection of Lemma 3.5.1(b).
(b) The canonical bijection between $S^*(A)$ and $S^*_{k-1}(A)$ is the restriction of the bijection

$$S(A) \to S_{k-1}(A) \tag{51}$$

of Lemma 3.5.3(e), which is given by sending the isomorphism class of X to the isomorphism class of the generalized extension $ext(X, \phi)$ associated with a pair (X, ϕ), where ϕ is any isomorphism $Gr^W X \to A$. The fact that (51) restricts to a map $S^*(A) \to S^*_{k-1}(A)$ is by Lemma 4.2.1(a). The fact that this restricted map is surjective is by Theorem 4.3.2.

By Lemma 4.4.3, the fiber of $S^*_\ell(A) \to S^*_{\ell-1}(A)$ above any element $\epsilon \in S^*_{\ell-1}(A)$ is the same as the fiber of $S_\ell(A) \to S_{\ell-1}(A)$ above ϵ. The assertions in (ii) and (iii) now follow from parts (f) and (c) of Theorem 1.2.1. □

We note that the special case of Theorem 4.4.4 when $k = 3$, $A_3 = \mathbb{1}$, and $Ext^1(\mathbb{1}, A_1) = 0$ was proved in [22, §6] (see §6.7 therein).

4.5. Example: Mixed Tate motives with four weights and maximal unipotent radicals. As mentioned above, the case of Theorem 4.4.4 with 3 weights was handled in [22] under some extra hypotheses. As an example, there we gave a homological classification of 3-dimensional graded-independent mixed Tate motives over $\mathbb{Q}$ with three weights and maximal unipotent radicals ([22, §6.8]). In this final subsection, as an example that illustrates Theorem 4.4.4 in a case with more than 3 weights we take $\mathbf{T}$ to be the category of mixed Tate motives over $\mathbb{Q}$ (say, of Voevodsky) and give a classification of isomorphism classes of all 4-dimensional graded-independent mixed Tate motives over $\mathbb{Q}$ with four weights and maximal unipotent radicals. As we shall see, this will lead to some interesting questions about periods, building on those that arose in [22].

We may assume that the highest weight of our motives is 0. Thus we are interested in the isomorphism classes of motives X over $\mathbb{Q}$ whose associated graded is isomorphic to

$$A := \mathbb{Q}(a+b+c) \oplus \mathbb{Q}(a+b) \oplus \mathbb{Q}(a) \oplus \mathbb{1}, \tag{52}$$

where a, b and c are positive integers. Such a motive is graded-independent if and only if a, b, c are distinct, $a+b \neq c$ and $b+c \neq a$. In view of Lemma 3.5.1 and the fact that the automorphism group of each $\mathbb{Q}(n)$ is canonically isomorphic to $\mathbb{Q}^\times$ we easily see that

$$\begin{aligned} S_1(A) &\cong Ext^1(\mathbb{Q}(a+b), \mathbb{Q}(a+b+c))/\mathbb{Q}^\times \times Ext^1(\mathbb{Q}(a), \mathbb{Q}(a+b))/\mathbb{Q}^\times \times Ext^1(\mathbb{1}, \mathbb{Q}(a))/\mathbb{Q}^\times \\ &\cong Ext^1(\mathbb{1}, \mathbb{Q}(c))/\mathbb{Q}^\times \times Ext^1(\mathbb{1}, \mathbb{Q}(b))/\mathbb{Q}^\times \times Ext^1(\mathbb{1}, \mathbb{Q}(a))/\mathbb{Q}^\times, \end{aligned}$$

where in all of these the action of $\mathbb{Q}^\times$ is via its action as the automorphism group of the object in the first entry of the corresponding Ext group via pushforward, which coincides with the action of $\mathbb{Q}^\times$ as the group of units of the scalar field of the vector space structure on the Ext group. Taking the orbits of nonsplit (= totally nonsplit, in this particular situation) extensions in each factor above we obtain the subset $S_1^*(A)$ of $S_1(A)$ consisting of the totally nonsplit elements.

Before proceeding any further, let us recall the characterization of the Ext groups for the category of mixed Tate motives over $\mathbb{Q}$ (see [19], for instance). The Ext^2 groups all vanish. We have

$$\dim_{\mathbb{Q}} Ext^1(\mathbb{1}, \mathbb{Q}(n)) = \begin{cases} 1 & \text{if } n \text{ is odd and } > 1 \\ 0 & \text{if } n \text{ is even or } \leq 0 \end{cases}$$

and

$$Ext^1(\mathbb{1}, \mathbb{Q}(1)) \cong \mathbb{Q}^\times \otimes \mathbb{Q}. \tag{53}$$

All of these descriptions remain the same in the category of mixed Tate motives over $\mathbb{Z}$, with the exception of the description of $Ext^1(\mathbb{1}, \mathbb{Q}(1))$, which is zero for the category of mixed Tate motives over $\mathbb{Z}$.

For any odd integer $n > 1$, the middle objects of all nonsplit extensions of $\mathbb{1}$ by $\mathbb{Q}(n)$ are isomorphic, as all nonzero extension classes are in the same $\mathbb{Q}^\times$-orbit. We refer to this middle object (which is unique up to isomorphism) as the motive of $\zeta(n)$ and denote it by Z_n. Thanks to Deligne [18] (also see [19]), we know that after choosing suitable bases for Betti and de Rham realizations, the period matrix of Z_n is

$$\begin{pmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n) \\ 0 & 1 \end{pmatrix}.$$

The extensions of $\mathbb{1}$ by $\mathbb{Q}(1)$ are given by Kummer motives. Under the isomorphism (53), for any rational $r > 1$ the extension class corresponding to $r \otimes 1$ arises from the weight filtration on the relative homology $H_1(\mathbb{G}_m, \{1, r\})$ (or equivalently, the 1-motive $[\mathbb{Z} \xrightarrow{1 \mapsto r} \mathbb{G}_m]$, see [16]). We denote this motive by L_r and refer to it as the motive of $\log(r)$; it has a period matrix of the form

$$\begin{pmatrix} (2\pi i)^{-1} & (2\pi i)^{-1}\log(r) \\ 0 & 1 \end{pmatrix}.$$

A complete set of (non-equivalent) representatives for the nonzero orbits of $Ext^1(\mathbb{1}, \mathbb{Q}(1))/\mathbb{Q}^\times$ consists of the extension classes corresponding to the elements $r \otimes 1$, as r runs through the set of rationals > 1 that are not of the form s^n for any $s \in \mathbb{Q}$ and integer $n > 1$. The L_r, as r runs through said set, give a complete set of representatives for isomorphism classes of non-semisimple motives with associated graded isomorphic to $\mathbb{Q}(1) \oplus \mathbb{1}$.

To simplify the notation, we shall allow a few instances of abuse of notation and terminology. Given an object X whose associated graded is isomorphic to $\mathbb{Q}(n) \oplus \mathbb{1}$, there is a well-defined element of $Ext^1(\mathbb{1}, \mathbb{Q}(n))/\mathbb{Q}^\times$ associated with it. We will use the same notation for the object and this associated element. We will also use the same notation for a non-pure motive and its Tate twists (i.e. for X and $X(n)$, as long as X is not pure). The Tate twist in question will always be clear from our context. Finally, we might speak about (say) Z_n as an extension, in which case we will mean an extension of $\mathbb{1}$ by $\mathbb{Q}(n)$ with Z_n as the middle object, or a Tate twist of this extension.

We now return to the problem of classifying the isomorphism classes of motives with maximal unipotent radicals and associated graded isomorphic to (52), where a, b, c are distinct positive integers, $a + b \neq c$ and $b + c \neq a$. If we want, we may also restrict ourselves to one of the cases $c > a$ or $a > c$, as one case can be transformed to the other via dualization followed by an appropriate Tate twist.

By Theorem 4.4.4 we have truncation maps

$$S^*(A) \cong S_3^*(A) \twoheadrightarrow S_2^*(A) \twoheadrightarrow S_1^*(A), \tag{54}$$

which are surjective because the Ext^2 groups vanish. Denoting the set of nonsplit extensions of $\mathbb{1}$ by $\mathbb{Q}(n)$ by $Ext^1(\mathbb{1}, \mathbb{Q}(n))^*$, we have

$$S_1^*(A) \cong Ext^1(\mathbb{1}, \mathbb{Q}(c))^*/\mathbb{Q}^\times \times Ext^1(\mathbb{1}, \mathbb{Q}(b))^*/\mathbb{Q}^\times \times Ext^1(\mathbb{1}, \mathbb{Q}(a))^*/\mathbb{Q}^\times. \tag{55}$$

Since $Ext^1(\mathbb{1}, \mathbb{Q}(n))$ vanishes when n is even and is nonzero otherwise for $n > 0$, the set $S_1^*(A)$ and hence $S^*(A)$ is nonempty if and only if a, b, c are all odd (in which case, the conditions $a + b \neq c$ and $b + c \neq a$ will be automatic).

Fix odd (distinct positive) a, b, c.

Case I: Suppose $1 \notin \{a, b, c\}$. Then $S_1^*(A)$ is a singleton. The set $S_2^*(A)$ is a torsor over

$$Ext^1(\mathbb{1}, \mathbb{Q}(a + b)) \times Ext^1(\mathbb{Q}(a), \mathbb{Q}(a + b + c)),$$

which is zero by the description of Ext^1 groups and our parity conditions. In other words, $S_2^*(A)$ is also a singleton. It consists of the class of a generalized extension of the form

$$\begin{array}{ccccc} \mathbb{Q}(a+b+c) & & & & \\ Z_c & \mathbb{Q}(a+b) & & & \\ Z_{c,b} & Z_b & \mathbb{Q}(a) & & \\ & Z_{b,a} & Z_a & \mathbb{1}, & \end{array} \tag{56}$$

where we have dropped the arrows from the writing and as mentioned earlier, are not keeping track of Tate twists in the notation for the non-pure motives. The motives called $Z_{b,a}$ and $Z_{c,b}$ are respectively, the motives (unique up to isomorphism) that fit in a blended extension of Z_a by Z_b and a blended extension of Z_b by Z_c. The motive $Z_{b,a}$ has a period matrix of the form

$$\begin{pmatrix} (2\pi i)^{-a-b} & (2\pi i)^{-a-b}\zeta(b) & (2\pi i)^{-a-b} z_{b,a} \\ & (2\pi i)^{-a} & (2\pi i)^{-a}\zeta(a) \\ & & 1 \end{pmatrix},$$

where $z_{b,a}$ (not uniquely defined, as it can change according to the choice of bases) is another period. By Theorem 4.3.2 (or its special case [22, Corollary 6.7.1]), the objects on the lowest diagonal of the generalized extension (56) have maximal unipotent radicals, and hence the motivic Galois group of each has dimension 4. Thus Grothendieck's period conjecture predicts that $z_{b,a}$, $\zeta(b)$, $\zeta(a)$, and π form an algebraically independent set. Similarly, the four numbers $z_{c,b}$, $\zeta(c)$, $\zeta(b)$, and π should be algebraically independent.

We now consider completions of the generalized extension class above to one of level 3. The fiber of $S^*(A) \to S_2^*(A)$ above the class of (56), which is simply all of $S^*(A)$ because $S_2^*(A)$ is a singleton, is a torsor over

$$Ext^1(\mathbb{1}, \mathbb{Q}(a + b + c)) \simeq \mathbb{Q}\,.$$

In particular, we have a non-canonical bijection $S^*(A) \simeq \mathbb{Q}$.

A discussion of the periods is in order. After choosing suitable bases for Betti and de Rham realizations of a motive in the isomorphism class $\epsilon \in S^*(A)$, the period matrix of ϵ is of the form

$$\begin{pmatrix} (2\pi i)^{-a-b-c} & (2\pi i)^{-a-b-c}\zeta(c) & (2\pi i)^{-a-b-c} z_{c,b} & (2\pi i)^{-a-b-c} z_{c,b,a}(\epsilon) \\ & (2\pi i)^{-a-b} & (2\pi i)^{-a-b}\zeta(b) & (2\pi i)^{-a-b} z_{b,a} \\ & & (2\pi i)^{-a} & (2\pi i)^{-a}\zeta(a) \\ & & & 1 \end{pmatrix}.$$

(Again, $z_{c,b,a}(\epsilon)$ is only well-defined to the extent allowed by its dependence on the choice of bases.)

The motivic Galois group of ϵ has a maximal unipotent radical, so that it has dimension 7. Thus Grothendieck's period conjecture predicts that $z_{c,b,a}(\epsilon), z_{c,b}, z_{b,a}$, the zeta values in the matrix, and π are algebraically independent. Since all of the discussion above can be thought of as having taken place in the category of mixed Tate motives over $\mathbb{Z}$ (as we did not use the motives of logarithms in the process), by a theorem of Brown [12] all of the above unknown periods are in the algebra generated by multiple zeta values and $1/(2\pi i)$.

By Theorem A.1 of the appendix, for any $\epsilon \in S^*(A)$ we have

$$Ext^1_{\langle\epsilon\rangle}(\mathbb{1}, \mathbb{Q}(a+b+c)) = 0.$$

Here, $\langle\epsilon\rangle$ means the subcategory generated by any object in ϵ, or equivalently, by all objects in ϵ. It follows that $\zeta(a+b+c)$ should also be algebraically independent from the 7 aforementioned periods $z_{c,b,a}(\epsilon)$, $z_{c,b}$, etc. Indeed, this is because Z_{a+b+c} is not in $\langle\epsilon\rangle$, so that the surjection from the motivic Galois group of $\epsilon \oplus Z_{a+b+c}$ (where with abuse of notation, ϵ refers to a motive in the isomorphism class) to the motivic Galois group of ϵ is not an isomorphism. The two groups have the same maximal reductive quotients and hence the unipotent radical of the former must be strictly larger than the latter's, so that the dimension of the motivic Galois group of $\epsilon \oplus Z_{a+b+c}$ is strictly larger than that of ϵ.

Case II: Suppose $1 \in \{a, b, c\}$. This is the more interesting case, as it involves the motives L_r of logarithms and hence the periods that arise may not be multiple zeta values or even their cyclotomic analogues.

We will consider the case $b = 1$. The other two cases, which are related to one another by duality and Tate twists, can be considered similarly.

Again we start forming our generalized extensions from the smallest level, working backwards through the maps of (54). The set $S^*_1(A)$ is no longer a singleton; instead, it is in a one-to-one correspondence with the set of isomorphism classes of Kummer motives. More precisely, from (55) we have

$$S^*_1(A) \cong \{Z_c\} \times \{L_r\}_r \times \{Z_a\},$$

where the L_r, as described earlier, form a set of representatives for the isomorphism classes of Kummer motives. Once we fix r, the lifting to $S^*_2(A)$ involves no choices (as in Case I). There is a unique element of $S^*_2(A)$ with Z_c, L_r, and Z_a on its level 1 diagonal:

$$\begin{array}{cccc} \mathbb{Q}(a+1+c) & & & \\ Z_c & \mathbb{Q}(a+1) & & \\ M'_{c,r} & L_r & \mathbb{Q}(a) & \\ & M_{a,r} & Z_a & \mathbb{1}. \end{array} \tag{57}$$

Here, $M_{a,r}$ (resp. $M'_{c,r}$) is the object (unique up to isomorphism) that fits as the middle object of a blended extension of Z_a by L_r (resp. L_r by Z_c). The object $M_{a,r}$ is the unique mixed Tate motive over $\mathbb{Q}$ with associated graded isomorphic to $\mathbb{Q}(a+1) \oplus \mathbb{Q}(a) \oplus \mathbb{1}$ and a maximal unipotent radical such that its corresponding extension of $\mathbb{Q}(a)$ by $\mathbb{Q}(a+1)$ is a Tate twist of the motive of $\log(r)$. There is a similar description for $M'_{c,r}$ (now with associated graded a twist of $\mathbb{Q}(c+1) \oplus \mathbb{Q}(1) \oplus \mathbb{1}$). By their uniqueness properties, $M'_{c,r}$ is isomorphic to a Tate twist of the dual to $M_{c,r}$. We refer the reader to [22, §6.8] for a more detailed discussion of these motives. (The notation here for $M_{a,r}$ and $M'_{c,r}$ is consistent with that in *loc. cit.*, except for a slight difference in the indices.)

The period matrix of $M_{a,r}$ with respect to suitable choices of bases is of the form

$$\begin{pmatrix} (2\pi i)^{-a-1} & (2\pi i)^{-a-1}\log(r) & (2\pi i)^{-a-1}p_{a,r} \\ & (2\pi i)^{-a} & (2\pi i)^{-a}\zeta(a) \\ & & 1 \end{pmatrix}.$$

The entry $p_{a,r}$ is a period which assuming Grothendieck's conjecture, together with $\zeta(a)$, $\log(r)$, and π form an algebraically independent set. Here, due to the presence of the motive of $\log(r)$ in the construction, the motive $M_{a,r}$ is not a mixed Tate motive over $\mathbb{Z}$, but rather one over $\mathbb{Z}[1/r]$ (i.e. with good reduction outside r). In the cases $r = 2, 3, 6$, Deligne has proved in [20] that the category of mixed Tate motives over $\mathcal{O}_{\mathbb{Q}(\mu_r)}[1/r]$ (i.e. the full subcategory of the category of mixed Tate motives over $\mathbb{Q}(\mu_r)$ consisting of objects that are unramified outside r) is generated by the motivic fundamental group of $\mathbb{G}_m - \mu_r$. This implies that the unknown period $p_{a,r}$ in these cases is in the $\mathbb{Q}(\mu_r)$-algebra generated by $1/(2\pi i)$ and cyclotomic multiple zeta values (see [20] and [19]). However, as Deligne explains in the Introduction of [20], these cases of r are in fact exceptional: if r is a positive integer that has a prime divisor ≥ 5, Goncharov [26] has shown that the motivic fundamental group of $\mathbb{G}_m - \mu_r$ does not generate the category of mixed Tate motives over $\mathcal{O}_{\mathbb{Q}(\mu_r)}[1/r]$. The upshot of these remarks is that when $r \notin \{2, 3, 6\}$, the nature of $p_{a,r}$ and other unknown periods that appear below seems more mysterious.

The fiber of $S^*(A) \to S^*_2(A)$ above the class of (57) is a torsor over

$$Ext^1(\mathbb{1}, \mathbb{Q}(a+1+c)) \simeq \mathbb{Q}\,.$$

Thus, having fixed r, we obtain a collection of non-isomorphic objects parametrized (non-canonically) by the Ext group above. Each $\epsilon \in S^*(A)$ above (57) (after making the relevant choices) has a period matrix of the form

$$\begin{pmatrix} (2\pi i)^{-a-1-c} & (2\pi i)^{-a-1-c}\zeta(c) & (2\pi i)^{-a-1-c}p'_{c,r} & p_{a,r,c}(\epsilon) \\ & (2\pi i)^{-a-1} & (2\pi i)^{-a-1}\log(r) & (2\pi i)^{-a-1}p_{a,r} \\ & & (2\pi i)^{-a} & (2\pi i)^{-a}\zeta(a) \\ & & & 1 \end{pmatrix}.$$

Assuming Grothendieck's period conjecture, the numbers $p_{a,r,c}(\epsilon)$, $p_{a,r}$, $p'_{c,r}$ (closely related to $p_{c,r}$), $\zeta(c)$, $\zeta(a)$, $\log(r)$, and π are algebraically independent.

As pointed out earlier, the nature of the unknown periods is rather mysterious. It would be interesting to somehow compute them. We should point out that a geometric construction of these motives (and more generally, of non-semisimple mixed Tate motives with a prescribed associated graded with few weights) seems out of reach at the moment. See [13] for more details.

Theorem A.1 again implies that for any $\epsilon \in S^*(A)$ above (57), there are no nontrivial extensions of $\mathbb{1}$ by $\mathbb{Q}(a+1+c)$ in the category generated by ϵ. Thus as in Case I, Grothendieck's period conjecture predicts that $\zeta(a+1+c)$ and the 7 numbers $p_{a,r,c}(\epsilon)$, $p_{a,r}$, etc. are algebraically independent. It will be interesting to understand the role that $\zeta(a+1+c)$ plays in the formation of $p_{a,r,c}(\epsilon)$ as ϵ varies in the fiber of $S^*(A) \to S^*_2(A)$ above (57). Does it reflect the role that Z_{a+1+c} plays for this fiber via the torsor structure?

Appendix A. A vanishing result for Ext groups in categories generated by motives with maximal unipotent radicals

In this appendix we prove another result about motives with maximal unipotent radicals. This result was used in the discussion of periods in §4.5. We shall work in the general setting of §4 prior to §4.5: $\mathbf{T}$ is a filtered tannakian category over a field K of characteristic zero in which pure objects are semisimple. As before, we refer to objects of $\mathbf{T}$ as motives.

Suppose that a motive X has k weights $p_1 < \cdots < p_k$. Suppose that $\underline{\mathfrak{u}}(X)$ is maximal. Then in particular, the extensions (48) are nonsplit, so that the tannakian subcategory $\langle X\rangle$ of $\mathbf{T}$ generated by X has a nontrivial extension of $Gr^W_{p_{r+1}}(X)$ by $Gr^W_{p_r}(X)$ for each $1 \leq r \leq k-1$. The following result concerns the Ext^1 groups in $\langle X\rangle$ between the non-consecutive graded pieces of X.

Theorem A.1. *Let X be a graded-independent motive with a maximal unipotent radical. Let $p_1 < \cdots < p_k$ be the weights of X. Then for every i, j with $i \leq j-2$ we have*

$$Ext^1_{\langle X\rangle}(Gr^W_{p_j}X, Gr^W_{p_i}X) = 0.$$

The proof uses the following characterization of Ext^1 groups in categories of representations of linear algebraic groups:

Proposition A.2. *Let G be a linear algebraic group over K (a field of characteristic zero). Denote the category of finite-dimensional representations of G over K by $\mathbf{Rep}(G)$. Denote the Ext^1 groups for $\mathbf{Rep}(G)$ by Ext^1_G.*

Suppose that $G = U \rtimes R$ where U is unipotent and R is reductive. Let $\mathfrak{u}$ be the Lie algebra of U, considered as an object of $\mathbf{Rep}(G)$ through the adjoint action, and let $\mathfrak{u}^{ab}$ be the abelianization of $\mathfrak{u}$, also considered as an object of $\mathbf{Rep}(G)$ via the induced action (note that $\mathfrak{u}^{ab}$ is semisimple, as the action of G on $\mathfrak{u}^{ab}$ factors through an action of R). Then for every semisimple object N of $\mathbf{Rep}(G)$ with $Hom(\mathbb{1}, N) = 0$, there is a canonical isomorphism

$$Ext^1_G(\mathbb{1}, N) \cong Hom(\mathfrak{u}^{ab}, N).$$

(In both instances, $Hom := Hom_{\mathbf{Rep}(G)} = Hom_{\mathbf{Rep}(R)}$.)

The special case of this statement where R is the multiplicative group $\mathbb{G}_m$ has been used frequently in the literature in the context of mixed Tate motives (e.g. see [19, §A.13]). The more general case stated here is also well known to experts, although unfortunately I am not aware of a reference where it is explicitly written (at least, in this form). Nonetheless, in the interest of saving space, here we skip the proof. The result can be obtained from group cohomology using the analogue for linear algebraic groups of the Hochschild-Serre spectral sequence (see equation (39) of [28, §16])[16]. An explicit version of the argument that does not require familiarity with group cohomology can be found in the appendix of the slightly more expanded version of this article available on arXiv (arXiv:2307.15487v2).

Taking Proposition A.2 for granted, we now prove Theorem A.1. Take G to be the tannakian group of X with respect to a fiber functor ω with values in the category of vector spaces over K. Take R to be the tannakian group of $Gr^W X$ (which is a semisimple object, by assumption) with respect to ω, and U the unipotent radical of G. Then U is the kernel of the natural map $G \twoheadrightarrow R$, and the Lie algebra $\mathfrak{u}$ of U is the image of $\underline{\mathfrak{u}}(X)$ under ω. Moreover, the G-action on $\mathfrak{u}$ corresponding to $\underline{\mathfrak{u}}(X)$ under the equivalence of categories $\langle X\rangle \xrightarrow{\omega} \mathbf{Rep}(G)$ is given by the adjoint action. Choosing a Levi factor of G we may identify $G = U \rtimes R$.

Let $\underline{\mathfrak{u}}^{ab}(X)$ be the abelianization of $\underline{\mathfrak{u}}(X)$; it is a semisimple quotient of $\underline{\mathfrak{u}}(X)$, and the quotient map $\underline{\mathfrak{u}}(X) \to \underline{\mathfrak{u}}^{ab}(X)$ becomes the quotient map $\mathfrak{u} \to \mathfrak{u}^{ab}$ after applying ω. In view of Proposition A.2 and the equivalence of categories $\langle X\rangle \xrightarrow{\omega} \mathbf{Rep}(G)$, for every semisimple object N of $\langle X\rangle$ with $Hom(\mathbb{1}, N) = 0$ there is a canonical isomorphism

$$Ext^1_{\langle X\rangle}(\mathbb{1}, N) \cong Hom(\underline{\mathfrak{u}}^{ab}(X), N). \tag{58}$$

(Hom on the right hand side is both for $\langle X\rangle$ and $\mathbf{T}$.)

We are ready to deduce the result. By our assumptions, $\underline{\mathfrak{u}}(X) = W_{-1}\underline{End}(X)$. Thus

$$Gr^W[\underline{\mathfrak{u}}(X), \underline{\mathfrak{u}}(X)] = Gr^W W_{-2}\underline{End}(X) = \bigoplus_{i \leq j-2} \underline{Hom}(Gr^W_{p_j}X, Gr^W_{p_i}X).$$

Applying the functor Gr^W to the sequence

$$0 \longrightarrow [\underline{\mathfrak{u}}(X), \underline{\mathfrak{u}}(X)] \longrightarrow \underline{\mathfrak{u}}(X) \longrightarrow \underline{\mathfrak{u}}^{ab}(X) \longrightarrow 0,$$

[16] I thank Richard Hain for bringing this to my attention.

on recalling that $\underline{\mathfrak{u}}^{ab}(X)$ is semisimple we obtain

$$\underline{\mathfrak{u}}^{ab}(X) \cong Gr^W \underline{\mathfrak{u}}^{ab}(X) \cong \bigoplus_j \underline{Hom}(Gr^W_{p_j} X, Gr^W_{p_{j-1}} X).$$

Hence, thanks to the graded-independence hypothesis, for every i, j with $i \leq j-2$ there are no nonzero morphisms from $\underline{\mathfrak{u}}^{ab}(X)$ to $\underline{Hom}(Gr^W_{p_j} X, Gr^W_{p_i} X)$. For every such i, j we thus have

$$Ext^1_{\langle X\rangle}(Gr^W_{p_j} X, Gr^W_{p_i} X) \cong Ext^1_{\langle X\rangle}(\mathbb{1}, \underline{Hom}(Gr^W_{p_j} X, Gr^W_{p_i} X)) = 0$$

by (58).

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Department of Mathematics and Statistics, University of Winnipeg, Winnipeg MB, Canada
Email address: p.eskandari@uwinnipeg.ca