The Grothendieck period conjecture and mixed motives with maximal unipotent radicals

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (Re(s) > 1)$$

- Euler (around 1740):

$$\zeta(2) = \pi^2/6 \qquad \zeta(4) = \pi^4/90 \qquad \zeta(6) = \pi^6/945$$

$$\vdots$$

$$\boxed{\zeta(2k) = \pi^{2k} \cdot (\text{a rational number})}$$

- Lambert (1760): π is irrational.
- Lindemann (1882): π is transcendental.

$\boxed{\text{Corollary: } \zeta(2k) \text{ is transcendental.}}$

A different picture : Odd zeta values

- Euler spent a lot of time on odd zeta values too. In 1785 he speculated that perhaps there are $\alpha, \beta \in \mathbb{Q}$ such that

$$\zeta(3) = \alpha(\log 2)^3 + \beta\pi^2 \log 2.$$

- To date, very little known about transcendence/irrationality of odd zeta values.
- Apéry (1978): $\zeta(3)$ is irrational.
- We don't even Know irrationality of $\zeta(5)$. Nothing is known about transcendence of any odd zeta value.
- Likely expectation in modern times: $\{\pi, \zeta(3), \zeta(5), \zeta(7), \dots\} \cup \{\log p : p \text{ prime}\}$ should be algebraically independent.

Grothendieck period conj.

Grothendieck period conjecture

Alg. Geometry

Various non-alg. geometric linearizations

Var./$\mathbb{Q}$

X

Betti (singular) cohomology $H_B^n(X)$

de Rham cohomology $H_{dR}^n(X)$

l-adic cohomology $H_l^n(X)$

$\vdots$

Periods

Grothendieck period conjecture

Alg. Geometry | Various non-alg. geometric linearizations

Var./$\mathbb{Q}$ $\xrightarrow{\text{linearization}}$ Motives

X $\qquad$ $H^n(X)$

(realizations)

Betti (singular) cohomology $H^n_B(X)$

de Rham cohomology $H^n_{dR}(X)$

l-adic cohomology $H^n_l(X)$

$\vdots$

$\rightsquigarrow$ Periods

Nowadays, we finally have non-conjectural geometrically constructed tannakian categories of motives (Ayoub '14, Nori '00s).

- Tannakian: similar properties to categories of representations: abelian, tensor pro., nice duals, admits a fiber functor
- Key fact: Any tan. Cat. over K is equiv. to Rep(G) for some affine group scheme G over K. More precisely:

$\boxed{\text{an af. gr. sch. over K called the fund. gr. of } \mathbf{T} \text{ wrt } \omega}$

$\mathbf{T}$ tan. over K, any fiber functor $\mathbf{T} \xrightarrow{\omega} \mathrm{Vec}_K$ gives an equiv. $\mathbf{T} \to \mathrm{Rep}(\underline{Aut}^{\otimes}(\omega))$

Def: Let X be a motive over a subfield of $\mathbb{C}$. The **motivic Galois group** of X is

$$G^{mot}(X) := \text{fund. gr. of } \langle X\rangle \text{ w.r.t. Betti fiber functor } (\underline{\mathrm{Aut}}^{\otimes}(\omega_B|_{\langle X\rangle}))$$

↪ an alg. group over $\mathbb{Q}$; can be identified as a subgroup of $GL(X_B)$

Have an equiv. of cat.'s $\langle X\rangle \longrightarrow Rep(G^{mot}(X))$ given by ω_B.

Grothendieck period conjecture (mid-late '60s): For every motive X over a number field,

$\dim G^{mot}(X)$ = transcendence degree of field generated by periods of X.

Example. $\mathbb{Q}(-1) := H^1(\mathbb{C}^\times)$; periods $2\pi i\mathbb{Q}$

$H_1^B(\mathbb{C}^\times)$ = spanned by 

$H^1_{dR}(\mathbb{C}^\times)$ spanned by dz/z

$$G^{mot}(\mathbb{Q}(-1)) = \mathbb{G}_m$$

GPC for $\mathbb{Q}(-1) \Leftrightarrow$ transcendence of π

Unipotent radicals of motivic Galois groups

Cat. of motives has a functorial filtration W. like the wt filt. on polarizable MHS's.

Sup. X = a motive.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U^{mot}(X) & \longrightarrow & G^{mot}(X) & \longrightarrow & \overbrace{G^{mot}(Gr^W(X))}^{\text{reductive}} & \longrightarrow & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \longrightarrow & W_{-1}GL(X_B) & \longrightarrow & W_0GL(X_B) & \longrightarrow & GL(Gr^W(X_B)) & \longrightarrow & 1 \\ & & \text{Unipotent radical} & & \begin{array}{c}\text{parabolic subgr.}\\ \text{for W}\end{array} & & & & \end{array}$$

$\boxed{\text{Semisimple}}$

$U^{mot}(X) =$ unipotent radical of $G^{mot}(X)$

$$\mathfrak{u}(X) := Lie(U^{mot}(X)) \subset W_{-1}End(X_B) \xrightarrow{\text{Tan. Formalism}} \boxed{\underline{\mathfrak{u}}(X) \subset W_{-1}\underline{End}(X)}$$

A motive

- Unipotent radicals of motivic Galois groups have been studied previously by Deligne, Bertrand, Hardouin, Bertolin, Jossen, K. Murty, E., etc.
- unipotent radicals of motivic Galois groups. $\longleftrightarrow$ Ext groups in categories of motives

GPC and alg. indep. of zeta values and logarithms

- <u>Theorem</u> (Borel '72, Soulé '78, Voevodsky/Levine): Ext groups for mixed Tate motives

$$\dim Ext^1_{MTM(\mathbb{Q})}(1,\mathbb{Q}(n)) = \begin{cases} 1 & n=3,5,7,\ldots \\ 0 & n=2,4,6,\ldots \end{cases}$$

$\longleftarrow$ Same for $MTM(\mathbb{Z})$

$$Ext^1_{MTM(\mathbb{Q})}(1,\mathbb{Q}(1)) \cong \mathbb{Q}^{\times} \otimes \mathbb{Q}$$

$$Ext^1_{MTM(\mathbb{Z})}(1,\mathbb{Q}(1)) = 0$$

L_a, "motive of log(a)", motive over $\mathbb{Z}[1/a]$

$$H_1(\mathbb{C}^{\times},\{1,a\}) \longleftrightarrow a \otimes 1$$

per. mat. $\begin{bmatrix} (2\pi i)^{-1} & (2\pi i)^{-1}\log(a) \\ 0 & 1 \end{bmatrix}$

Ext^2 groups all vanish.

- Theorem (Deligne '89): For odd n >1, the nontrivial extension of 1 by $\mathbb{Q}(n)$ has a period matrix

$$\begin{bmatrix} (2\pi i)^{-n} & (2\pi i)^{-n}\zeta(n) \\ 0 & 1 \end{bmatrix}.$$

Call this motive Z_n "the motive of $\zeta(n)$"

- **Warm up exercise:** GPC and alg. indep. of $\{\pi, \zeta(n)\}$ and $\{\pi, \log a\}$ (n odd, >1)

Suppose $0 \longrightarrow \mathbb{Q}(n) \longrightarrow X \longrightarrow 1 \longrightarrow 0$ is nonsplit.

$$\rightsquigarrow \begin{array}{ccccccccc} 1 & \longrightarrow & U^{mot}(X) & \longrightarrow & G^{mot}(X) & \longrightarrow & G^{mot}(GrX) & \longrightarrow & 1 \\ & & \cap & & & & \| & & \\ & & \mathbb{G}_a \cong \left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\right\} & & & & \mathbb{G}_m & & \end{array}$$

X nonsemisimple $\Rightarrow U^{mot}(X) = \mathbb{G}_a \Rightarrow \dim G^{mot}(X) = 2$

Corollary: GPC for Z_n (resp. L_a) $\Leftrightarrow$ alg. indep. of $\{\pi, \zeta(n)\}$ (resp. $\{\pi, \log(a)\}$).

(Well-known)

<u>Proposition</u>: GPC implies alg. indep. of $\{\zeta(n) : n \underset{>1}{\text{odd}}\} \cup \{\pi\} \cup \{\log p : p \text{ prime}\}$.

Pf: Let's focus on alg. indep. of {log p} over $\mathbb{Q}(\pi, \zeta(3), \zeta(5), \dots)$ first.

Set $Z_{\leq 2k+1} := \bigoplus_{n \leq 2k+1} Z_n$. Enough to show that

$$\dim G^{mot}(Z_{\leq 2k+1} \bigoplus \oplus_{n\leq N} L_{p_n}) < \dim G^{mot}(Z_{\leq 2k+1} \bigoplus \oplus_{n \leq N+1} L_{p_n})$$

We have:

$$\begin{tikzcd} 1 \arrow[r] & U^{mot}(Z_{\leq 2k+1} \bigoplus \oplus_{n\leq N+1} L_{p_n}) \arrow[r] \arrow[d, two heads] & G^{mot}(Z_{\leq 2k+1} \bigoplus \oplus_{n\leq N+1} L_{p_n}) \arrow[r] \arrow[d, two heads, "(*)"] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & U^{mot}(Z_{\leq 2k+1} \bigoplus \oplus_{n\leq N} L_{p_n}) \arrow[r] & G^{mot}(Z_{\leq 2k+1} \bigoplus \oplus_{n\leq N} L_{p_n}) \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \end{tikzcd}$$

Enough to show (*) is not $\cong$. Enough to have $L_{p_{N+1}} \notin \langle \underbrace{Z_{\leq 2k+1} \bigoplus \oplus_{n\leq N} L_{p_n}}_{\text{Every object unramified at } p_{N+1}} \rangle$

Ramified at p_{N+1}

Remains to show alg. indep. of $\{\zeta(n) : n \text{ odd} > 1\} \cup \{\pi\}$.

Recall $Z_{\leq 2k+1} = \bigoplus_{n \leq 2k+1} Z_n$. Again easy to see that it is enough to argue that

$$Z_{2k+1} \notin \langle Z_{\leq 2k-1} \rangle$$

<u>Lemma</u>: Let $G = U \rtimes R$ be an alg. group over a field of char. 0, with R reductive and U unipotent. Let $\mathfrak{u}^{ab}$ be the abelianization of the Lie alg. of U, considered as a G-rep. Then for every semisimple object N of Rep(G) with $Hom_G(1,N)=0$, we have $Ext^1_G(1,N) \cong Hom_G(\mathfrak{u}^{ab}, N)$.

Apply with $G = G^{mot}(Z_{\leq 2k-1})$, $N = \mathbb{Q}(2k+1)$:

$$Ext^1_{\langle Z_{\leq 2k-1} \rangle}(1, \mathbb{Q}(2k+1)) = Hom(\underline{\mathfrak{u}}^{ab}(Z_{\leq 2k-1}), \mathbb{Q}(2k+1)) = 0$$

Weight considerations

$(\underline{\mathfrak{u}}(Z_{\leq 2k-1}) \subset W_{-1}\underline{End}(Z_{\leq 2k-1}))$

$\square$

Towards motives with maximal unipotent radicals

To see the alg. indep. of $\pi, \log 5, \zeta(3)$ here we used a 4-dim'l motive with $\mathrm{Gr}^W \simeq \mathbb{Q}(3) \oplus \mathbb{Q}(1) \oplus 1^2$.

Can we do this with a smaller motive?

1. Is there a 3-dim'l motive that has Z_3 and L_5 as subquotients?

   Answered by Grothendieck's theory of blended extensions
   (Extensions panachées)

2. If so, would the GPC for it imply the alg. indep. of $\pi, \log 5, \zeta(3)$?

$\rightsquigarrow$ Motives with maximal unipotent radicals

Blended extensions

Invented by Grothendieck to study filtrations $0 \subsetneq X_1 \subsetneq X_2 \subsetneq X_3 = X$ with $X_1 \simeq A_1$, $X_2/X_1 \simeq A_2$, $X_3/X_2 \simeq A_3$

$$
X \quad \rightsquigarrow \quad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d, equal] & X_2 \arrow[r] \arrow[d] & X_2/X_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & X \arrow[r] \arrow[d] & X/X_1 \arrow[r] \arrow[d] & 0 \\
 & & X/X_2 \arrow[r, equal] \arrow[d] & X/X_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Blended extensions

Invented by Grothendieck to study filtrations $0 \subsetneq X_1 \subsetneq X_2 \subsetneq X_3 = X$ with $X_1 \simeq A_1$, $X_2/X_1 \simeq A_2$, $X_3/X_2 \simeq A_3$

$$X \rightsquigarrow \begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d, "A_2"] & \\ 0 \arrow[r] & X_1 \arrow[r] \arrow[d, equal, "A_1"'] & X_2 \arrow[r] \arrow[d] & X_2/X_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X_1 \arrow[r] & X \arrow[r] \arrow[d] & X/X_1 \arrow[r] \arrow[d] & 0 \\ & & X/X_2 \arrow[r, equal, "A_3"'] \arrow[d] & X/X_2 \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

$Gr(X) \xrightarrow{\phi} \oplus A_r$

Fix extensions $\mathscr{L}$ and $\mathscr{N}$:

$\mathscr{L}$: $\quad 0 \longrightarrow A_1 \longrightarrow L \longrightarrow A_2 \longrightarrow 0$

$\mathscr{N}$: $\quad 0 \longrightarrow A_2 \longrightarrow N \longrightarrow A_3 \longrightarrow 0$

A blended extension of $\mathscr{N}$ by $\mathscr{L}$ is a comm. diagram of the form, with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

- EXTPAN($\mathscr{N}, \mathscr{L}$) := col. of all bl. ext's of $\mathscr{N}$ by $\mathscr{L}$
- Extpan($\mathscr{N}, \mathscr{L}$) := iso. classes of bl. ext's $\mathscr{N}$ by $\mathscr{L}$ (Commuting maps that are identity on L and N)

Fix extensions $\mathscr{L}$ and $\mathscr{N}$:

$$\mathscr{L}: \quad 0 \longrightarrow A_1 \longrightarrow L \longrightarrow A_2 \longrightarrow 0$$

$$\mathscr{N}: \quad 0 \longrightarrow A_2 \longrightarrow N \longrightarrow A_3 \longrightarrow 0$$

A blended extension of $\mathscr{N}$ by $\mathscr{L}$ is a comm. diagram of the form, with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] \arrow[dr] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

- $\mathrm{EXTPAN}(\mathscr{N}, \mathscr{L}) :=$ col. of all bl. ext's of $\mathscr{N}$ by $\mathscr{L}$
- $\mathrm{Extpan}(\mathscr{N}, \mathscr{L}) :=$ iso. classes of bl. ext's $\mathscr{N}$ by $\mathscr{L}$ (Commuting maps that are identity on L and N)

Prop: a) $\mathrm{Extpan}(\mathscr{N}, \mathscr{L})$ is nonempty iff $\mathscr{L} \circ \mathscr{N}$ vanishes in $Ext^2(A_3, A_1)$.

Fix extensions $\mathscr{L}$ and $\mathscr{N}$:

$$\mathscr{L}: \quad 0 \longrightarrow A_1 \longrightarrow L \longrightarrow A_2 \longrightarrow 0$$

$$\mathscr{N}: \quad 0 \longrightarrow A_2 \longrightarrow N \longrightarrow A_3 \longrightarrow 0$$

A blended extension of $\mathscr{N}$ by $\mathscr{L}$ is a comm. diagram of the form, with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & A_3 \arrow[r, equal] \arrow[d] & A_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

- EXTPAN($\mathscr{N}, \mathscr{L}$) := col. of all bl. ext's of $\mathscr{N}$ by $\mathscr{L}$
- Extpan($\mathscr{N}, \mathscr{L}$) := iso. classes of bl. ext's $\mathscr{N}$ by $\mathscr{L}$ (Commuting maps that are identity on L and N)

Prop: a) Extpan($\mathscr{N}, \mathscr{L}$) is nonempty iff $\mathscr{L} \circ \mathscr{N}$ vanishes in $Ext^2(A_3, A_1)$.

b) Extpan($\mathscr{N}, \mathscr{L}$) is canonically a torsor over $Ext^1(A_3, A_1)$.

Back to our question 1: Take

$$\mathscr{N}: \quad 0 \longrightarrow \mathbb{Q}(1) \longrightarrow L_p \longrightarrow 1 \longrightarrow 0 \qquad \text{(p prime)}$$

$$\mathscr{L}: \quad 0 \longrightarrow \mathbb{Q}(n+1) \longrightarrow Z_n(1) \longrightarrow \mathbb{Q}(1) \longrightarrow 0 \qquad \text{(n odd >1)}$$

$\boxed{Ext^2(1,\mathbb{Q}(n+1)) = 0}$

$\underbrace{\text{There is}}$ $M_{n,p}$, $\underbrace{\text{unique up to iso.}}$, fitting in

$\boxed{Ext^1(1,\mathbb{Q}(n+1)) = 0}$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Q}(n+1) \arrow[r] \arrow[d, equal] & Z_n(1) \arrow[r] \arrow[d] & \mathbb{Q}(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Q}(n+1) \arrow[r] & M_{n,p} \arrow[r] \arrow[d] & L_p \arrow[r] \arrow[d] & 0 \\
 & & 1 \arrow[r, equal] \arrow[d] & 1 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$$Gr^W M_{n,p} \simeq \overbrace{\mathbb{Q}(n+1) \oplus \mathbb{Q}(1)}^{Z_n} \oplus 1, \qquad \overbrace{\mathbb{Q}(1) \oplus 1}^{L_p}$$

per. mat.

$$\begin{bmatrix} (2\pi i)^{-n-1} & (2\pi i)^{-n-1}\zeta(n) & (2\pi i)^{-n-1}\pi_{n,p} \\ 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log p \\ 0 & 0 & 1 \end{bmatrix}$$

Moving on to the second question:

Def: Say a motive X has a maximal unipotent radical if

$$\underline{\mathfrak{u}}(X) = W_{-1}\underline{End}(X).$$

Def: An extension (class) $\mathcal{E}$ of X by Y is totally nonsplit if its class in $Ext^1(1, \underline{Hom}(X, Y))$ does not come from any proper subject of $\underline{Hom}(X, Y)$.

- For extensions of 1 by simple X, totally nonsplit = nonsplit.
- X has a max. uni. rad. $\Longrightarrow$ $\left\{\begin{array}{l}\text{All the all ext'ns} \\ 0 \to W_mX/W_\ell X \to W_nX/W_\ell X \to W_nX/W_\ell X \to 0 \quad (\ell < m < n) \\ \text{are tot. nonsplit.}\end{array}\right.$

$\boxed{\text{Converse false in general.}}$

Convese true if X has 2 weights (Bertrand/Hardouin).

Theorem (E.- K. Murty) Let X be a (not necessarily mixed Tate) motive with exactly 3 wts $a_1 < a_2 < a_3$. Suppose:

(i) The extensions of $Gr^W_{a_2}X$ by $Gr^W_{a_1}X$ and $Gr^W_{a_3}X$ by $Gr^W_{a_2}X$ coming from X are tot. nonsplit.

(ii) $\underline{Hom}(Gr^W_{a_2}X, Gr^W_{a_1}X)$ and $\underline{Hom}(Gr^W_{a_3}X, Gr^W_{a_2}X)$ have no nonzero isomorphic subobjects.

Then X has a maximal unipotent radical.

Recall: given odd n>1 and p, $\exists! M_{n,p}$ w.

$$Gr^W M_{n,p} \simeq \overbrace{\mathbb{Q}(n+1) \oplus \mathbb{Q}(1)}^{Z_n} \oplus 1$$

(the brace L_p spans $\mathbb{Q}(1) \oplus 1$)

Its per. mat.: $\begin{bmatrix} (2\pi i)^{-n-1} & (2\pi i)^{-n-1}\zeta(n) & (2\pi i)^{-n-1}\pi_{n,p} \\ 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log p \\ 0 & 0 & 1 \end{bmatrix}$

Cor: $M_{n,p}$ has a max. uni. rad.

$$\dim G^{mot}(M_{n,p}) = \dim W_{-1}\underline{End}(M_{n,p}) + \dim \mathbb{G}_m = 4$$

GPC for $M_{n,p}$ $\Leftrightarrow$ Alg. indep. of $\{\pi, \log p, \zeta(n), \pi_{n,p}\}$

Open question: What is $\pi_{n,p}$?

(A per. of MTM($\mathbb{Z}[1/p]$))