The Grothendieck period conjecture and motives with maximal unipotent radicals - II

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Fields Institute, March 2024

- **T** = a Tannakian cat. over a field of char. 0, equipped with a "weight" filtration W such that pure objects are semisimple.

(Think: polarizable Mixed Hodge structures over $\mathbb{Q}$ or a Tannakian category of (mixed) motives over a subfield of $\mathbb{C}$)

- "Motive" = an object of **T**
- $X \in \mathrm{obj}(\mathbf{T}) \rightsquigarrow \mathfrak{u}(X)$, a submotive of $W_{-1}\underline{End}(X)$

  Lie algebra of the unipotent radical of the motivic Galois group of X

- $\mathfrak{u}(X) \longleftrightarrow$ Extensions in $\langle X \rangle^{\otimes}$ (Deligne, Bertrand, Brown, EM, Goncharov, Hain, Hardouin, Jossen,...)

Interested in:

i.e. $\mathfrak{u}(X) = W_{-1}\underline{End}(X)$

1) When does X have a <u>maximal</u> unipotent radical?
2) Given semisimple A, classify X with $Gr(X) \simeq A$ and max. $\mathfrak{u}(X)$, up to isomorphism.

Ideally, not a part of data!

Motivation: Grothendieck period conj.

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2b) Given semisimple A, classify X with $Gr(X) \simeq A$ and max. $\mathfrak{u}(X)$, up to isomorphism.

Ideally, not a part of data!

2a) Given semisimple A, classify X with $Gr(X) \simeq A$ up to isomorphism.

A related notion: total nonsplitting

- Def: An extension (class) $\mathscr{E}$ of $\mathbf{1}$ by Y is totally nonsplit if for every $Y' \subsetneq Y$, the pushforward $\mathscr{E}/Y'$ does not split.

(So for extensions of 1 by simple Y, totally nonsplit = nonsplit.)

- Def: An extension (class) $\mathscr{E}$ of X by Y is totally nonsplit if its class in $Ext^1(1, \underline{Hom}(X, Y))$ is totally nonsplit.

- X has a max. uni. rad. $\Longrightarrow$ All the all ext'ns $0 \to W_m X / W_\ell X \to W_n X / W_\ell X \to W_n X / W_\ell X \to 0$ $(\ell < m < n)$ are totally nonsplit.

Converse is true if X has 2 weights (Bertrand/Hardouin).

Converse is false in general (even for 3 weights).

Recall from last talk: Task 1 for 3 weights

<u>Theorem (E.- K. Murty)</u> Let X be a motive with 3 wts $a_1 < a_2 < a_3$. Suppose:

(i) The extensions of $Gr_{a_2}X$ by $Gr_{a_1}X$ and $Gr_{a_3}X$ by $Gr_{a_2}X$ coming from X are totally nonsplit.

(ii) $Hom\big(\underline{Hom}(Gr_{a_2}X, Gr_{a_1}X), \underline{Hom}(Gr_{a_3}X, Gr_{a_2}X)\big) = 0.$

Then X has a maximal unipotent radical.

<u>Ex:</u> Given odd n>1 and p, $\exists! M_{n,p}$ w.

$$GrM_{n,p} \simeq \underbrace{\mathbb{Q}(n+1) \oplus \mathbb{Q}(1)}_{Z_n \text{ (Motive of } \zeta(n))} \oplus 1, \quad \underbrace{\mathbb{Q}(1) \oplus 1}_{L_p \text{ (Motive of log(p))}}$$

Its per. mat.: $\begin{bmatrix} (2\pi i)^{-n-1} & (2\pi i)^{-n-1}\zeta(n) & (2\pi i)^{-n-1}\pi_{n,p} \\ 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log p \\ 0 & 0 & 1 \end{bmatrix}$

Cond.'s (i) and (ii) hold.

$M_{n,p}$ has a max. uni. rad.

dim $G^{mot}(M_{n,p})$ = 3+1=4.

GPC for $M_{n,p} \iff$ Alg. indep. of $\{\pi, \log p, \zeta(n), \pi_{n,p}\}$

Open question: What is $\pi_{n,p}$?

(A per. of MTM($\mathbb{Z}[1/p]$))

Recall from last time: Task 2 for 3 weights

Key tool: Grothendieck's Blended extensions

(Extensions panachées)

Invented by Grothendieck to study filtrations $0 \subsetneq X_1 \subsetneq X_2 \subsetneq X_3 = X$ with $X_1 \simeq A_1$, $X_2/X_1 \simeq A_2$, $X_3/X_2 \simeq A_3$

$$
X \quad \rightsquigarrow \quad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_2/X_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X & \longrightarrow & X/X_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X/X_2 & = & X/X_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

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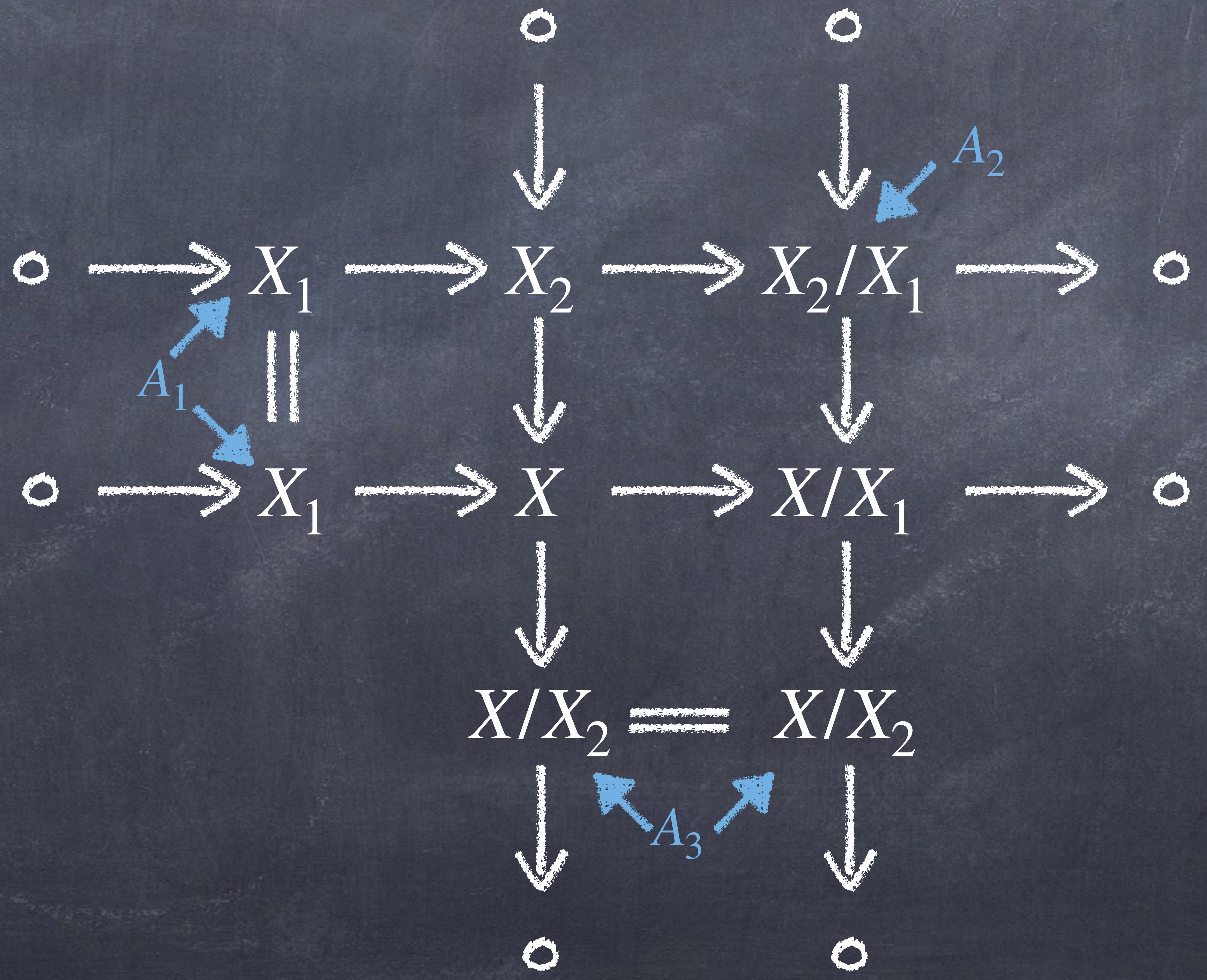
$X \rightsquigarrow$



$Gr(X) \xrightarrow{\phi} \oplus A_r$

Fix extensions $\mathscr{L}$ and $\mathscr{N}$:

$$\mathscr{L}: \quad 0 \longrightarrow A_1 \longrightarrow L \longrightarrow A_2 \longrightarrow 0$$

$$\mathscr{N}: \quad 0 \longrightarrow A_2 \longrightarrow N \longrightarrow A_3 \longrightarrow 0$$

A blended extension of $\mathscr{N}$ by $\mathscr{L}$ is a comm. diagram of the form, with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & L & \longrightarrow & A_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & X & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A_3 & = & A_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

- $\mathrm{EXTPAN}(\mathscr{N}, \mathscr{L})$:= collection of all bl. ext's of $\mathscr{N}$ by $\mathscr{L}$
- $\mathrm{Extpan}(\mathscr{N}, \mathscr{L})$:= iso. classes of bl. ext's $\mathscr{N}$ by $\mathscr{L}$ (Commuting maps that are identity on L and N)

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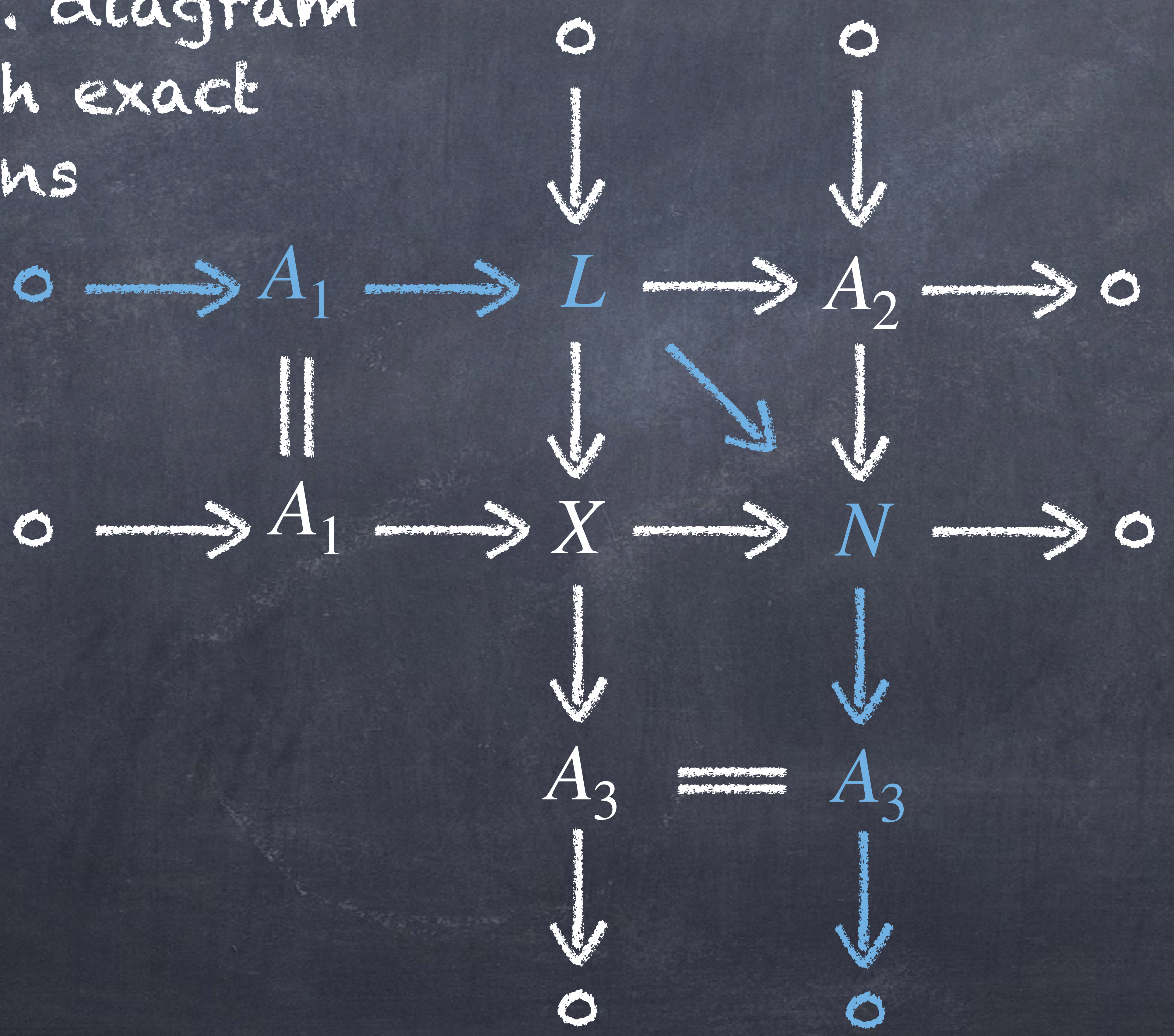

- EXTPAN($\mathscr{N}, \mathscr{L}$) := collection of all bl. ext's of $\mathscr{N}$ by $\mathscr{L}$
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Prop: a) Extpan($\mathscr{N}, \mathscr{L}$) is nonempty iff $\mathscr{L} \circ \mathscr{N}$ vanishes in $Ext^2(A_3, A_1)$.

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$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
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Prop: a) Extpan($\mathscr{N}, \mathscr{L}$) is nonempty iff $\mathscr{L} \circ \mathscr{N}$ vanishes in $Ext^2(A_3, A_1)$.

b) When nonempty, Extpan($\mathscr{N}, \mathscr{L}$) is canonically a torsor over $Ext^1(A_3, A_1)$.

Back to our earlier example:

$$\mathcal{N}: \quad 0 \longrightarrow \mathbb{Q}(1) \longrightarrow L_p \longrightarrow 1 \longrightarrow 0 \qquad \text{(p prime)}$$

$$\mathcal{L}: \quad 0 \longrightarrow \mathbb{Q}(n+1) \longrightarrow Z_n(1) \longrightarrow \mathbb{Q}(1) \longrightarrow 0 \qquad \text{(n odd >1)}$$

$Ext^2(1,\mathbb{Q}(n+1)) = 0$

There is a unique $M_{n,p}$ fitting in

$Ext^1(1,\mathbb{Q}(n+1)) = 0$

(Unique up to blended ext., hence unique up to iso. of motives.)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}(n+1) & \longrightarrow & Z_n(1) & \longrightarrow & \mathbb{Q}(1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}(n+1) & \longrightarrow & M_{n,p} & \longrightarrow & L_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & = & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Motives with any number of wts - report on a recent work

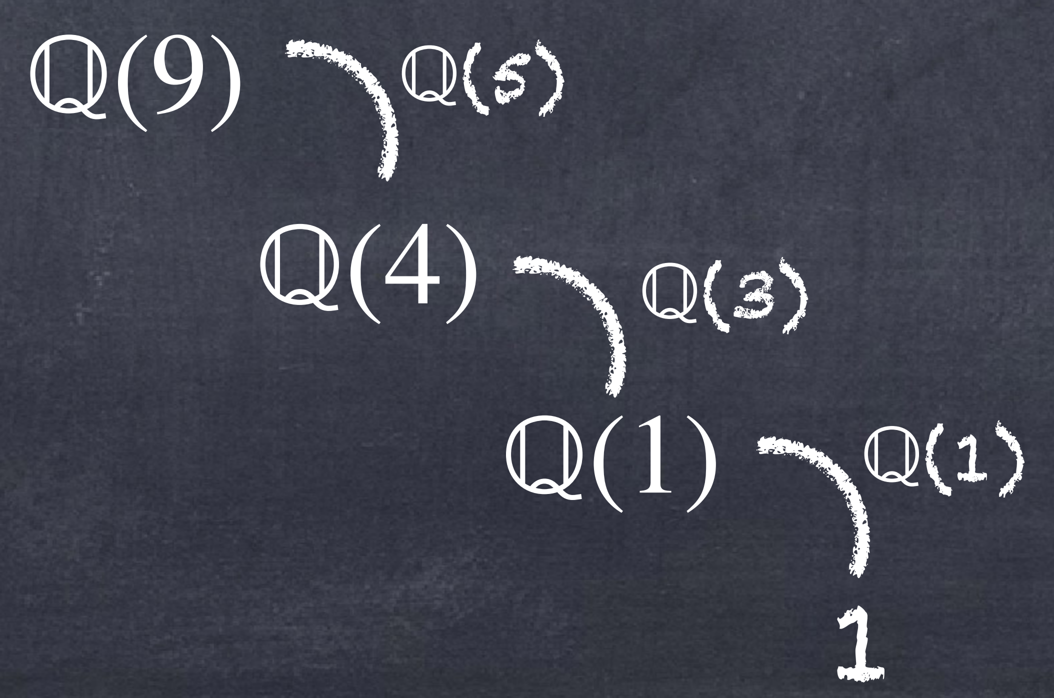
Task 1

Def: We say a motive X with k weights $a_1 < \cdots < a_k$ is graded-independent (GI) if there are no nonzero morphisms between any two of

$$\underline{Hom}(Gr_{a_j}X, Gr_{a_{j-1}}X) \quad (1 < j \leq k) \quad \text{and} \quad \bigoplus_{j-i>1} \underline{Hom}(Gr_{a_j}X, Gr_{a_i}X)$$

Examples:
- All motives with 2 weights are GI.
- For motives with 3 weights, this is just cond. (ii) of previous theorem.
- The condition is guaranteed if the weights are sufficiently "spread out".

e.g.



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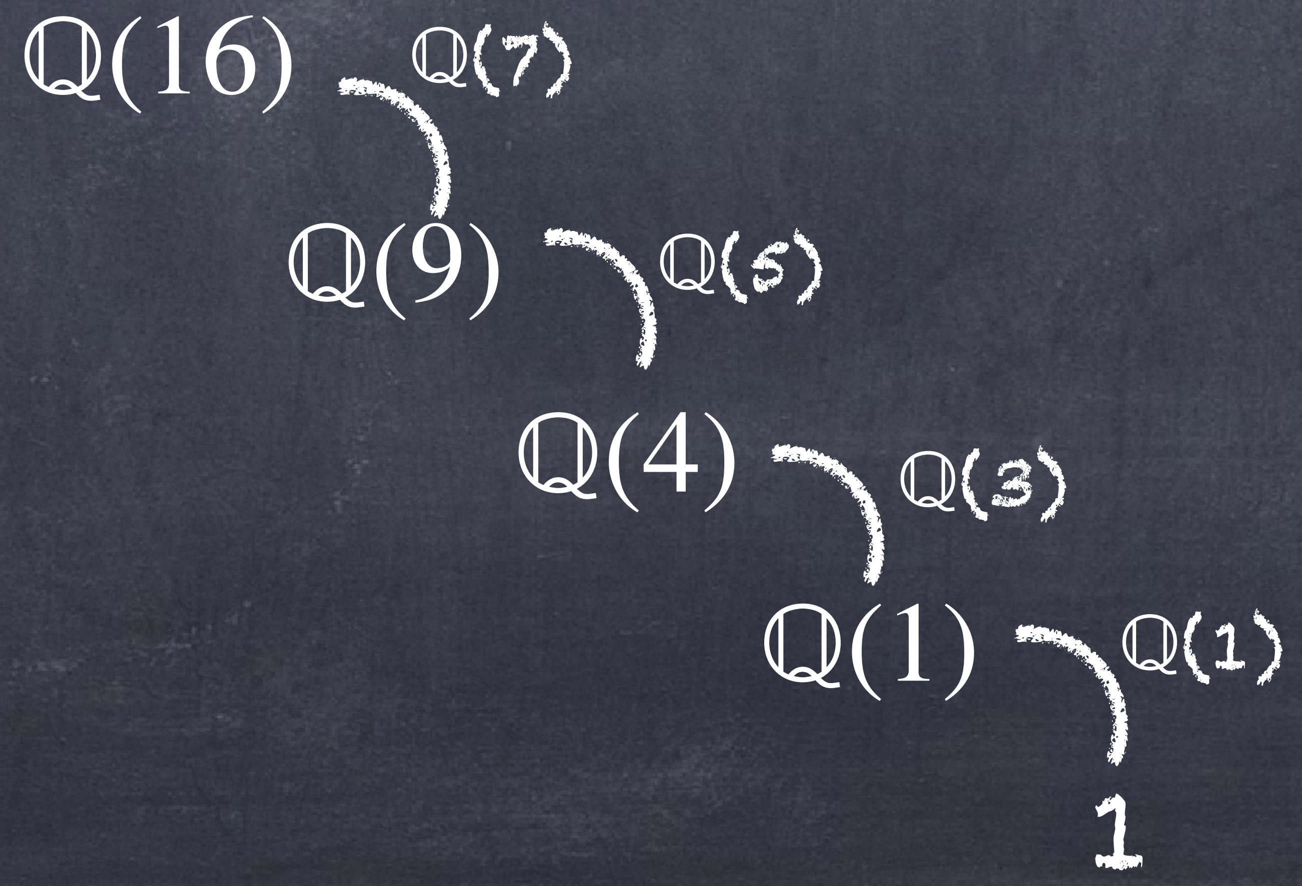

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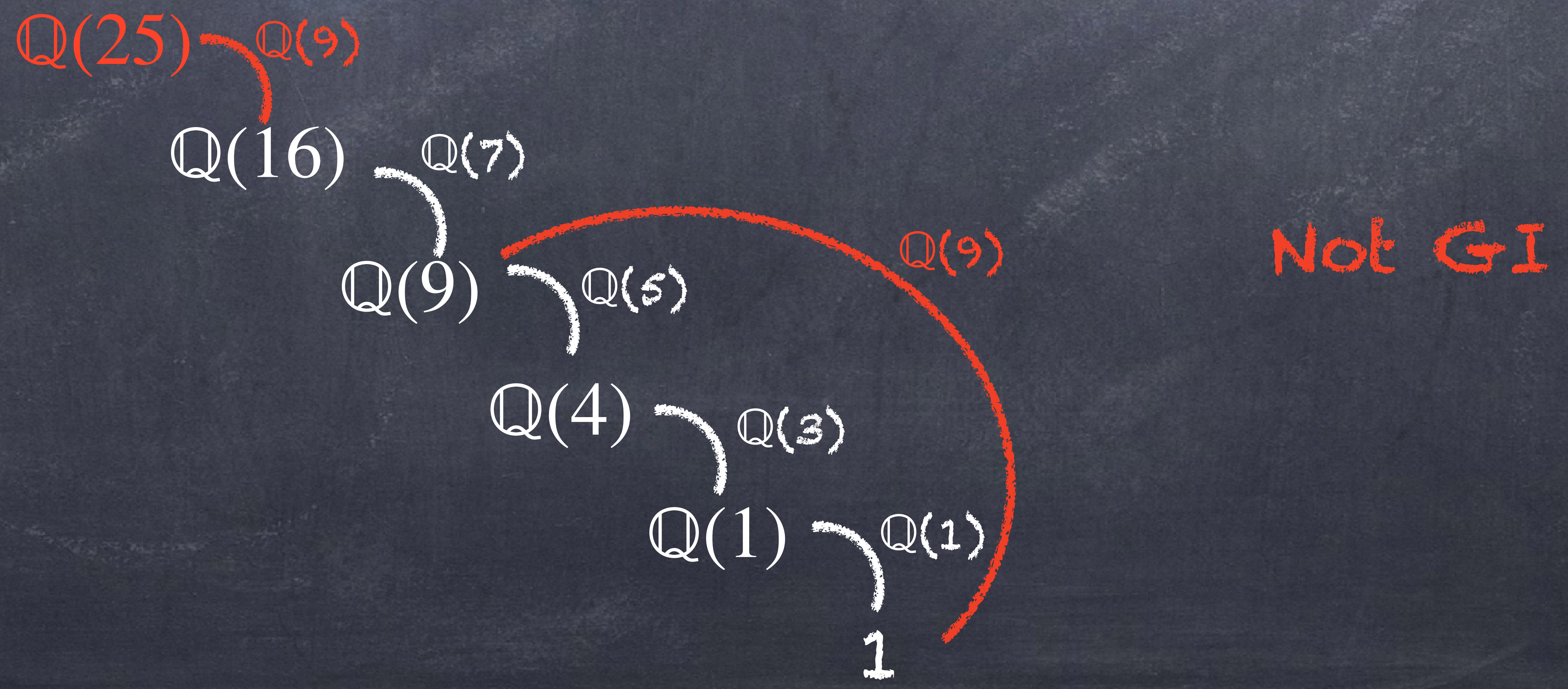

Theorem. Let X be a GI motive with k weights $a_1 < \cdots < a_k$. Suppose that each of the extensions

$$0 \longrightarrow Gr_{a_{j-1}}X \longrightarrow W_{a_j}X/W_{a_{j-2}}X \longrightarrow Gr_{a_j}X \longrightarrow 0$$

is tot. nonsplit. Then X has a maximal unipotent radical.

Example. Suppose Gr X $\simeq \mathbb{Q}(9) \oplus \mathbb{Q}(4) \oplus \mathbb{Q}(1) \oplus 1$

If successive extensions of X are L_p, $Z_3(1)$, and $Z_5(4)$ then X has a max. uni. rad.

More generally, if Gr X $\simeq \mathbb{Q}(n+m+1) \oplus \mathbb{Q}(n+1) \oplus \mathbb{Q}(1) \oplus 1$ (m,n distinct, odd, >0)

with nonsplit successive extensions then X has a max. uni. radical.

Task 2:

Fix nonzero pure objects $A_1, \ldots, A_k$ of weights $a_1 < \cdots < a_k$. Set $A = \bigoplus A_r$

Set S(A):= {objects X with Gr(X) $\simeq$ A}/iso. , S'(A):=$\{(X, GrX \xrightarrow[\simeq]{\phi} A)\}$/equiv.

$(X, \phi) \sim (X', \phi')$ iff $\exists f : X \xrightarrow{\cong} X'$ s.t.

$$\begin{array}{ccc} Gr^W X & \xrightarrow{\phi} & A \\ & {\scriptstyle Gr^W f} \searrow \quad \nearrow {\scriptstyle \phi'} & \\ & Gr^W X' & \end{array}$$

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X

wts $a_1 < \cdots < a_k$

$X_r := W_{a_r} X$

$0 =: X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_k = X$

$X_n / X_m =: X_{n,m}$

$$\rightsquigarrow \quad \begin{array}{ccccccccc}
X_{1,0} & & & & & & & & \\
\downarrow & & & & & & & & \\
X_{2,0} & \twoheadrightarrow & X_{2,1} & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
X_{3,0} & \twoheadrightarrow & X_{3,1} & \twoheadrightarrow & X_{3,2} & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & \vdots & \ddots & & & \\
\downarrow & & \downarrow & & & & & & \\
X_{k-1,0} & \twoheadrightarrow & X_{k-1,1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & X_{k-1,k-2} & & \\
\downarrow & & \downarrow & & & & \downarrow & & \\
X_{k,0} & \twoheadrightarrow & X_{k,1} & \twoheadrightarrow & \cdots & \twoheadrightarrow & X_{k,k-2} & \twoheadrightarrow & X_{k,k-1}
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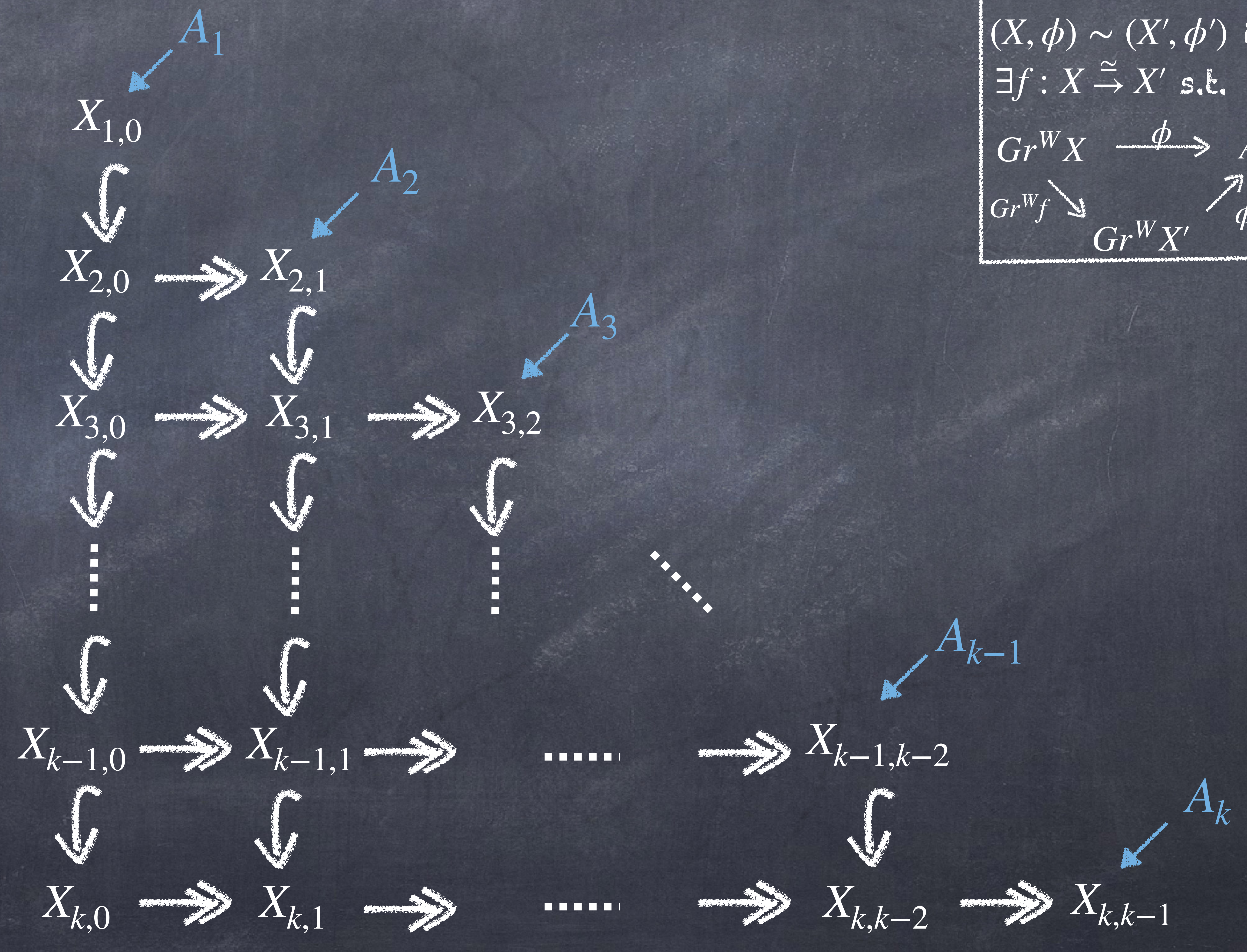

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X

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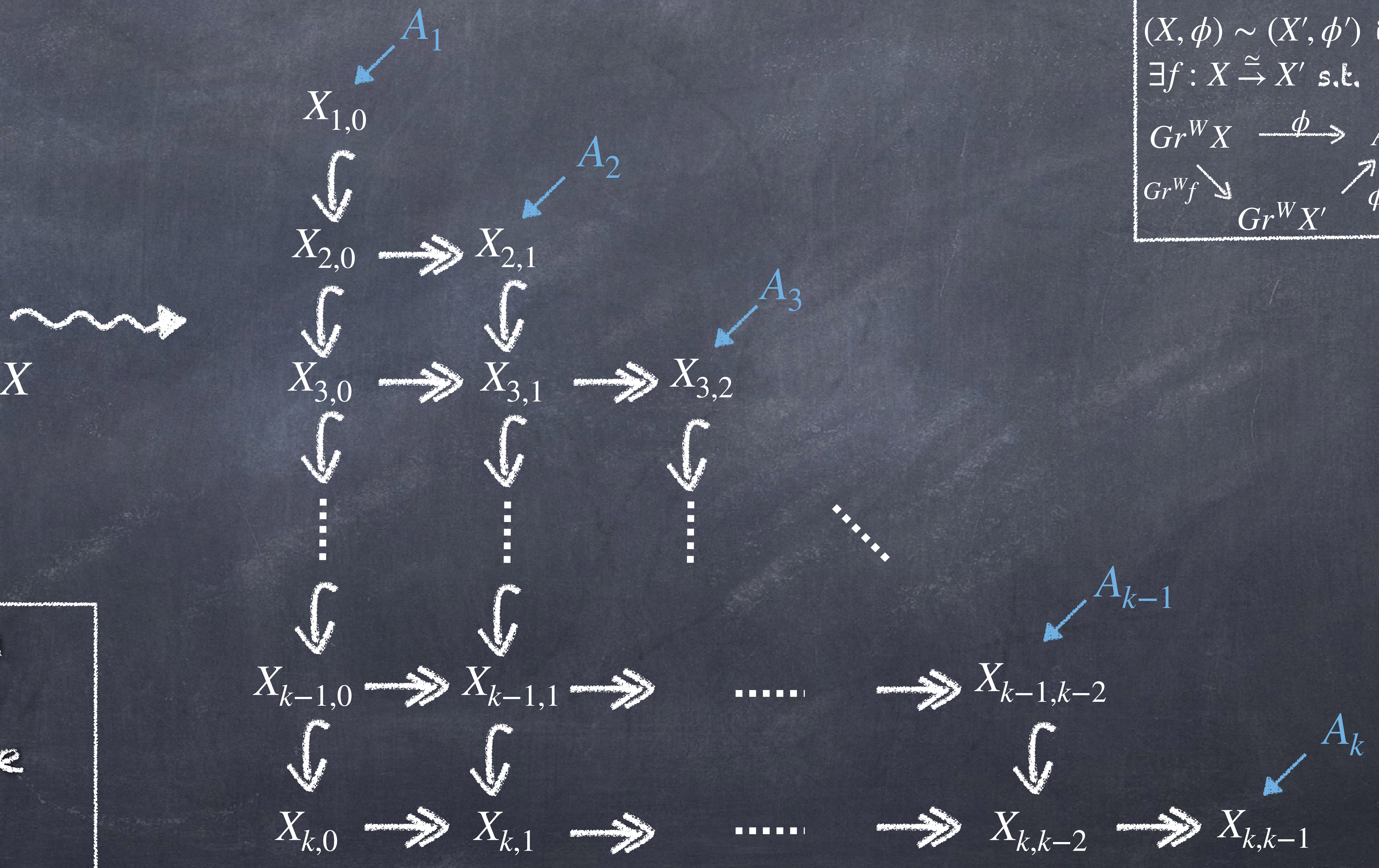
$X_r := W_{a_r} X$

$0 =: X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_k = X$

$X_n / X_m =: X_{n,m}$

$Gr^W X \xrightarrow{\phi, \simeq} A$



Idea: To get all X, form all such diagrams one diagonal at a time. Take appropriate equiv. rel.'s into account.

Example: k=4

A_1

A_2 →

A_3

A_4

A_1

$X_{2,0}$ A_2

$X_{3,1}$ A_3

$X_{4,2}$ A_4

Choices $\cong$

$$\prod_r EXT(A_{r+1}, A_r)$$

Example: k=4

$$\begin{matrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{matrix} \quad \longrightarrow \quad \begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ & X_{3,1} & A_3 & \\ & & X_{4,2} & A_4 \end{matrix} \quad \longrightarrow \quad \begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ X_{3,0} & X_{3,1} & A_3 & \\ & & X_{4,2} & A_4 \end{matrix}$$

Choices $\cong$

$$\prod_r EXT(A_{r+1}, A_r)$$

Choices for adding $X_{3,0} \cong$

EXTPAN($X_{3,1}$,$X_{2,0}$)

If nonempty, torsor over $EXT(A_3, A_1)$

Example: k=4

$$A_1$$
$$A_2$$
$$A_3$$
$$A_4$$

$\rightarrow$

$$A_1$$
$$X_{2,0} \quad A_2$$
$$X_{3,1} \quad A_3$$
$$X_{4,2} \quad A_4$$

Choices $\cong$

$$\prod_r EXT(A_{r+1}, A_r)$$

$\rightarrow$

$$A_1$$
$$X_{2,0} \quad A_2$$
$$X_{3,0} \quad X_{3,1} \quad A_3$$
$$X_{4,1} \quad X_{4,2} \quad A_4$$

Choices $\cong$

$$\prod_r EXTPAN(X_{r+2,r}, X_{r+1,r-1})$$

Empty or Torsor over

$$\prod_r EXT(A_{r+2}, A_r)$$

Example: k=4

$$\begin{matrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{matrix} \quad \rightarrow \quad \begin{matrix} A_1 & & & & \\ X_{2,0} & A_2 & & & \\ & X_{3,1} & A_3 & & \\ & & X_{4,2} & A_4 \end{matrix}$$

Choices $\cong$

$$\prod_r EXT(A_{r+1}, A_r)$$

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Choices $\cong$

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Empty or Torsor over

$$\prod_r EXT(A_{r+2}, A_r)$$

$$\rightarrow \quad \begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ X_{3,0} & X_{3,1} & A_3 & \\ X_{4,0} & X_{4,1} & X_{4,2} & A_4 \end{matrix}$$

Choices $\cong$

$$EXTPAN(X_{4,1}, X_{3,0})$$

Empty or Torsor over

$$EXT(A_4, A_1)$$

Example: k=4

$$
\begin{matrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{matrix}
\;\longrightarrow\;
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\;\longrightarrow\;
\begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ X_{3,0} & X_{3,1} & A_3 & \\ X_{4,0} & X_{4,1} & X_{4,2} & A_4 \end{matrix}
$$

Choices $\cong$ $\prod_r EXT(A_{r+1}, A_r)$

Choices $\cong$ $\prod_r EXTPAN(X_{r+2,r}, X_{r+1,r-1})$

Empty or Torsor over

$\prod_r EXT(A_{r+2}, A_r)$

Choices $\cong$ $EXTPAN(X_{4,1}, X_{3,0})$

Empty or Torsor over

$EXT(A_4, A_1)$

In each step, need to mod out by appropriate equivalence relations.

To formalize this approach:

Def: A *generalized extension* (of level k-1) of A is the data of an (abstract) commuting diagram of motives

$$\begin{array}{ccccccccccc}
A_1 & & & & & & & & & & \\
\downarrow & & & & & & & & & & \\
X_{2,0} & \twoheadrightarrow & A_2 & & & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
X_{3,0} & \twoheadrightarrow & X_{3,1} & \twoheadrightarrow & A_3 & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
\vdots & & \vdots & & \vdots & \ddots & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
X_{k-1,0} & \twoheadrightarrow & X_{k-1,1} & \twoheadrightarrow & & \cdots & & \twoheadrightarrow & A_{k-1} & & \\
\downarrow & & \downarrow & & & & & & \downarrow & & \\
X_{k,0} & \twoheadrightarrow & X_{k,1} & \twoheadrightarrow & & \cdots & & \twoheadrightarrow & X_{k,k-2} & \twoheadrightarrow & A_k
\end{array}$$

such that each

$$0 \longrightarrow A_m \longrightarrow X_{n,m-1} \longrightarrow X_{n,m} \longrightarrow 0$$

is exact.

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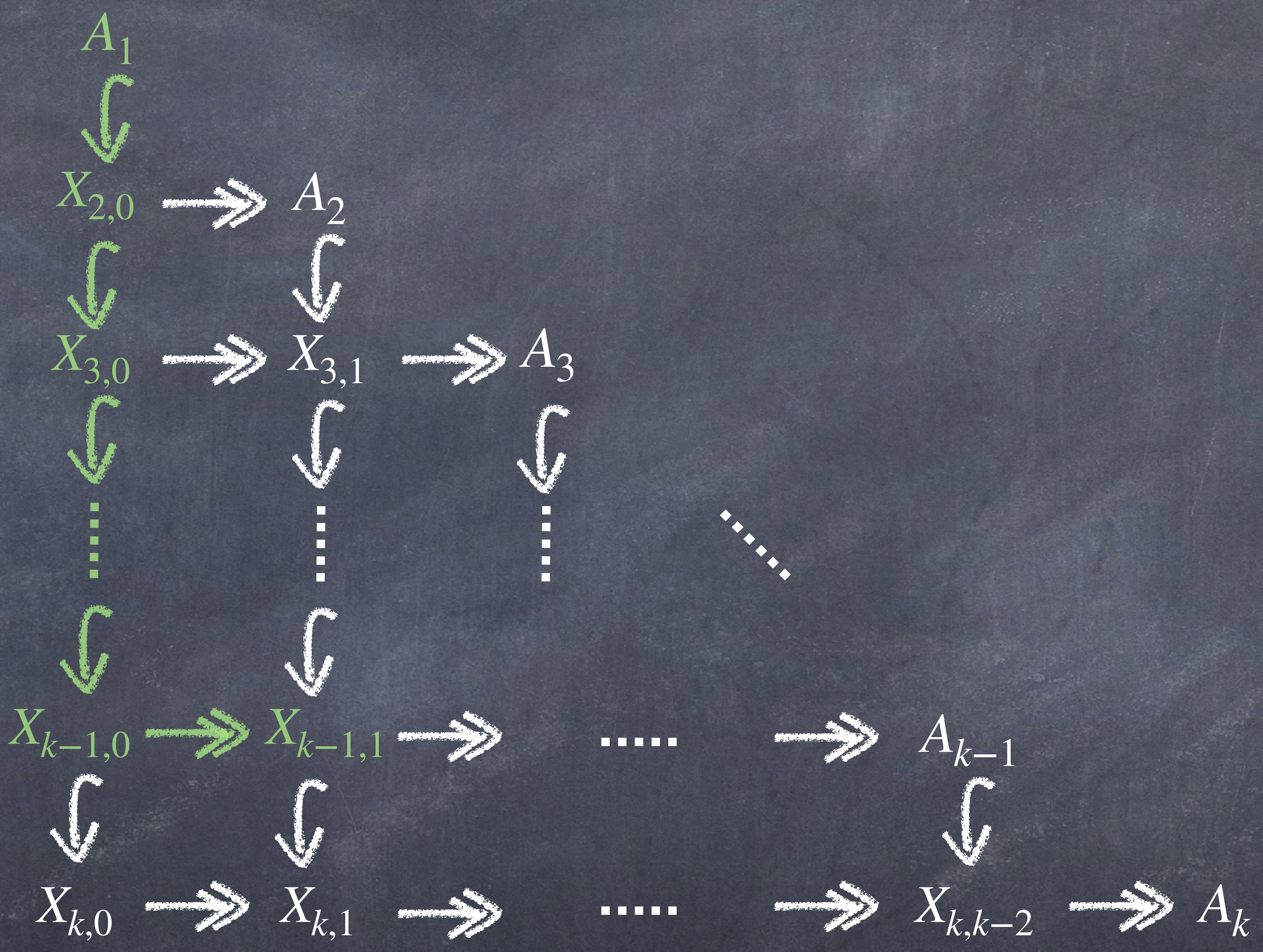

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$$0 \longrightarrow A_m \longrightarrow X_{n,m-1} \longrightarrow X_{n,m} \longrightarrow 0$$

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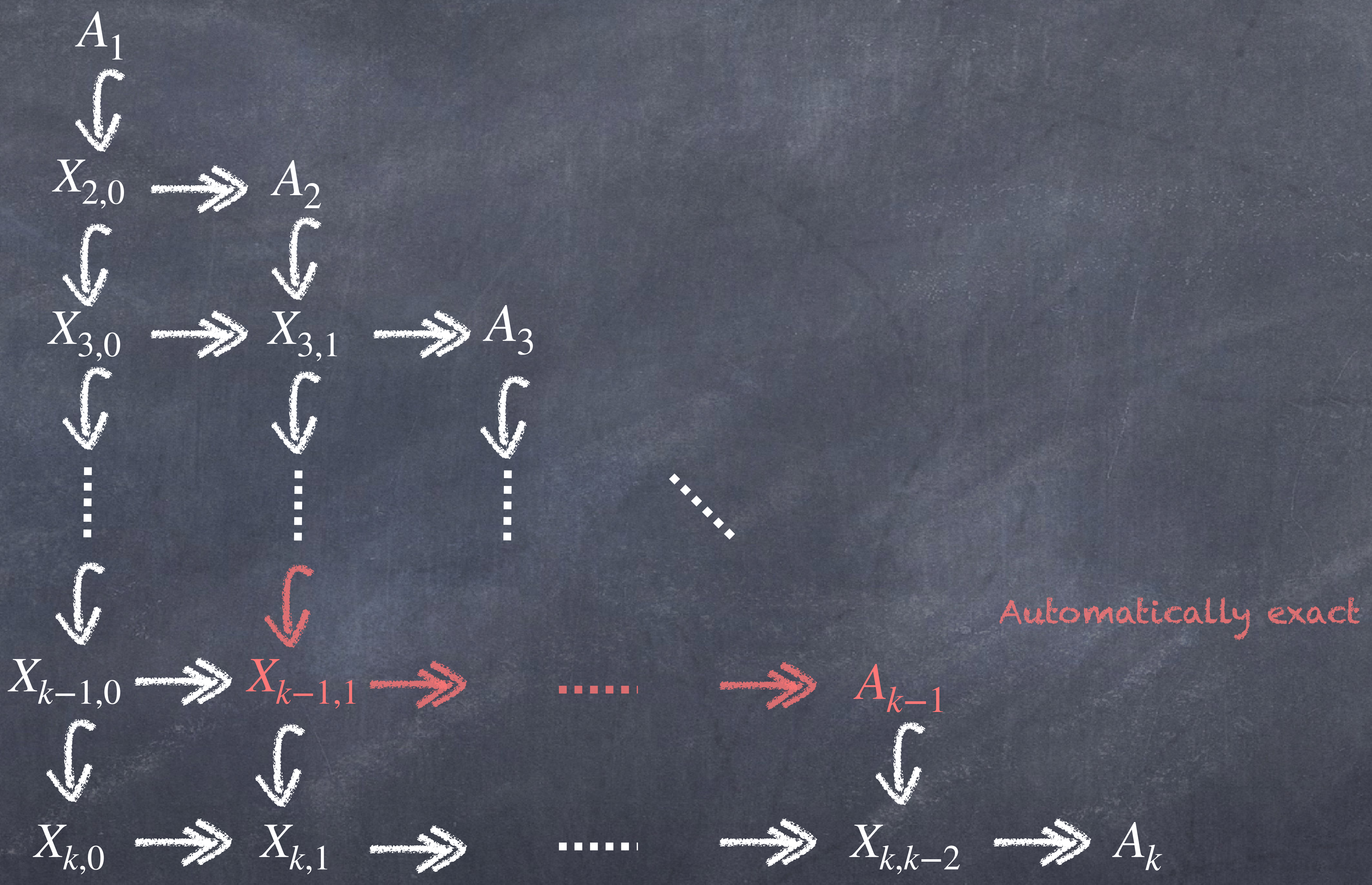

such that each

$$0 \longrightarrow A_m \longrightarrow X_{n,m-1} \longrightarrow X_{n,m} \longrightarrow 0$$

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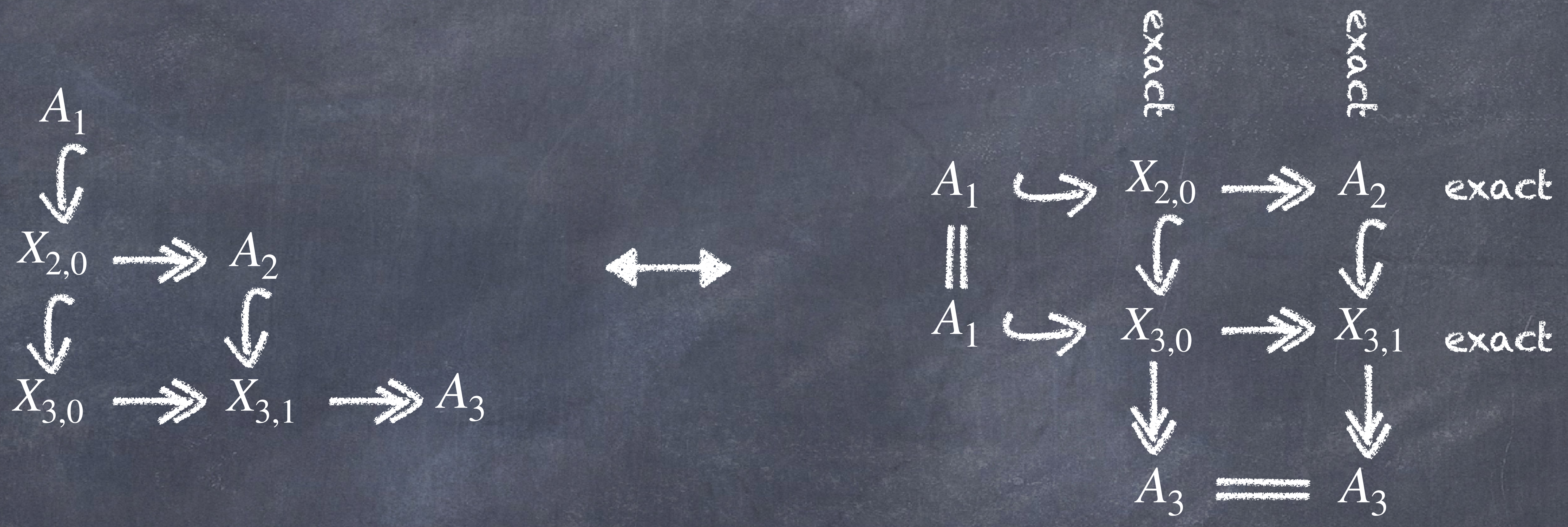
Examples:

- k=2 A gen ext of A is just an extension of A_2 by A_1

- k=3



A gen ext of A is just a blended ext of an element of $\mathrm{EXT}(A_3, A_2)$ by an element of $\mathrm{EXT}(A_2, A_1)$

Def: A gen. ext. of level l of A is a similar data as before but with only l diagonals below A.

$k=4$

$$\begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ & X_{3,1} & A_3 & \\ & & X_{4,2} & A_4 \end{matrix} \qquad \begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ X_{3,0} & X_{3,1} & A_3 & \\ & X_{4,1} & X_{4,2} & A_4 \end{matrix} \qquad \begin{matrix} A_1 & & & \\ X_{2,0} & A_2 & & \\ X_{3,0} & X_{3,1} & A_3 & \\ X_{4,0} & X_{4,1} & X_{4,2} & A_4 \end{matrix}$$

$l=1 \qquad\qquad l=2 \qquad\qquad l=3$

$D_l(A)$:= collection of all gen. ext.'s of level l of A

There are truncation maps $D_l(A) \longrightarrow D_{l-1}(A)$.

$S_l(A) := {}^{D_l(A)}\!/_{\text{commuting isomorphisms}}$ $\quad S'_l(A) := {}^{D_l(A)}\!/_{\text{commuting isomorphisms identity on } A}$

Trunc. $\rightsquigarrow$ $S'_l(A) \longrightarrow S'_{l-1}(A)$

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$S_l(A) \longrightarrow S_{l-1}(A)$.

Recall: $S(A)=\{X \text{ with } Gr(X) \simeq A\}/\text{iso}$

$S'(A)=\{(X, Gr(X) \xrightarrow{\simeq} A)\}/\text{equiv}$

Theorem. (a) We have

$$S'(A) \cong S'_{k-1}(A) \to S'_{k-2}(A) \to \cdots \to S'_2(A) \to S'_1(A) = \prod_r Ext^1(A_{r+1}, A_r)$$

s.t. the fiber of $S'_\ell \to S'_{\ell-1}$ is either empty or canonically a torsor over

$$\prod_r Ext^1(A_{r+\ell}, A_r). \qquad (1)$$

(b) This descends to

$$S(A) \cong S_{k-1}(A) \to S_{k-2}(A) \to \cdots \to S_2(A) \to S_1(A) = \prod_r Ext^1(A_{r+1}, A_r) \big/ Aut(A)$$

s.t. the fibers are... In particular, the fibers above tot. nonsp. elements remain unchanged.

(c) $S'_\ell \to S'_{\ell-1}$ is surjective if $Ext^2(A_{r+\ell}, A_r) = 0$ for each r.

Task 2a

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Task 2a

* (d) Suppose A is GI. Let $S^*(A) \subset S(A)$ be the set of iso. classes with max. uni. rad's. Then maps of (b) restrict to maps

$$S^*(A) \cong S^*_{k-1}(A) \longrightarrow S^*_{k-2}(A) \longrightarrow \cdots\cdots \longrightarrow S^*_2(A) \longrightarrow S^*_1(A) = \text{tot. nonsp. orbits}$$

s.t. each nonempty fiber of $S^*_\ell \longrightarrow S^*_{\ell-1}$ is a torsor over (1).

Example. Mixed Tate motives with maximal uni. rad's and $\mathrm{Gr} \simeq \mathbb{Q}(9) \oplus \mathbb{Q}(4) \oplus \mathbb{Q}(1) \oplus 1$

(Similar picture for $\mathrm{Gr}\, X \simeq \mathbb{Q}(n+m+1) \oplus \mathbb{Q}(n+1) \oplus \mathbb{Q}(1) \oplus 1$ for any m,n distinct, odd, >0)

$\mathbb{Q}(9)$					$\mathbb{Q}(9)$					$\mathbb{Q}(9)$			
Z_5	$\mathbb{Q}(4)$				Z_5	$\mathbb{Q}(4)$				Z_5	$\mathbb{Q}(4)$		
	Z_3	$\mathbb{Q}(1)$		$Z_{5,3}$	Z_3	$\mathbb{Q}(1)$			$Z_{5,3}$	Z_3	$\mathbb{Q}(1)$		
		L_a	1	$M_{3,a}$	L_a	1		X	$M_{3,a}$	L_a	1		

$\Cap$

$S_1^* \cong \{L_a\}/_{\text{iso.}}$

$Ext^1(A_{r+2}, A_r) = 0 \Rightarrow$ unique lift in S_2^*

{Lifts to S^*} a torsor over $Ext^1(1, \mathbb{Q}(9))$ (Gen. by Z_9)

$$\begin{bmatrix} (2\pi i)^{-9} & (2\pi i)^{-9}\zeta(5) & (2\pi i)^{-9} z_{5,3} & (2\pi i)^{-9}\lambda(X) \\ 0 & (2\pi i)^{-4} & (2\pi i)^{-4}\zeta(3) & (2\pi i)^{-4}\pi_{3,a} \\ 0 & 0 & (2\pi i)^{-1} & (2\pi i)^{-1}\log a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

per. mat. of X

Max. of uni. rad. $\Rightarrow \dim G^{mot} = 7$
$\Rightarrow$ Assuming GPC,
$\{\pi, \zeta(5), \zeta(3), \log a, z_{5,3}, \pi_{3,a}, \lambda(X)\}$
is alg. independent.

Questions about $\lambda(X)$:

- Evaluation (X is a motive in $\mathrm{MTM}(\mathbb{Z}[1/a])$.)
- Its dependence on $\zeta(9)$ as X varies in the family

Theorem. Let X be a GI motive with weights $a_1 < \cdots < a_k$ and max. uni. radical. Then

$$Ext^1_{\langle X \rangle}(Gr_{a_j}X, Gr_{a_i}X) = 0 \qquad \text{if } j-i>1.$$

Back to the previous example, for every X in the family,

$$Ext^1_{\langle X \rangle}(1, \mathbb{Q}(9)) = 0.$$

$\Longrightarrow$ Z_9 is not in $\langle X \rangle$.

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Thank you!