Quadratic periods of meromorphic forms on punctured Riemann surfaces

Payman Eskandari

ABSTRACT. We give three proofs of a relation involving classical and quadratic periods of meromorphic differentials on a punctured elliptic curve. The first proof is based on an old argument of Gunning. The second proof considers how quadratic periods vary in the Legendre family of elliptic curves. The final proof exploits connections to the Hodge theory of the fundamental group and is suitable for generalization to arbitrary Riemann surfaces. The obstacle for such generalization is a lack of a simple description of the Hodge filtration on the space of iterated integrals of length ≤ 2 on a punctured Riemann surface of arbitrary genus in terms of meromorphic differentials.

1. Introduction

This article aims to study the quadratic periods of a punctured Riemann surface. We shall be interested in the quadratic periods of *meromorphic* forms. More precisely, let X be a compact Riemann surface of genus > 0, and fix distinct points $\infty, e \in X$. Ideally, our goal is to find explicit relations involving the classical and quadratic periods of meromorphic forms on $X - \{\infty\}$ with base point e; the latter are by definition Chen-type iterated integrals of the form $\int_\beta \alpha\alpha'$, where $\beta \in \pi_1(X - \{\infty\}, e)$ and α, α' are meromorphic forms on X with only possible pole at ∞. The interest in quadratic periods of meromorphic forms, as opposed to values of other spanning sets of the space of closed iterated integrals of length 2 (e.g. those involving harmonic forms- see Kaenders [**14**, Theorem 1.4]) is motivated by number theory: If X, ∞, e, α and α' are all defined over a subfield $F \subset \mathbb{C}$, the values of $\int \alpha\alpha'$ are among the periods of the (conjectural) motive attached to $\pi_1(X - \{\infty\}, e)$.

Before we say more about the contents of the paper let us fix the following notation. For any pointed manifold (M, a), let $L_n(M, a)$ denote the space of closed (i.e. homotopy functional) iterated integrals of length $\leq n$ on M with base point a (see Paragraph 5.3). Chen's de Rham theorem asserts that

$$L_n(M, a) = \left(\frac{\mathbb{C}[\pi_1(M, a)]}{I^{n+1}} \right)^\vee,$$

where I is the augmentation ideal (i.e. the kernel of the map $\mathbb{C}[\pi_1(M, a)] \to \mathbb{C}$ defined by $\gamma \mapsto 1$).

Most of this paper focuses on the case of an elliptic curve. We shall call our curve E in this case, and reserve the symbol X for discussions in which the genus is arbitrary (positive). Let α_1, α_2 be meromorphic forms with only possible pole at ∞, representing a basis of $H^1(E)$. The space $L_2(E - \{\infty\}, e)$ has the following basis (over $\mathbb{C}$):

$$1, \int \alpha_i \,, \int \alpha_i \alpha_j \quad (i, j \leq 2).$$

The subspace generated multiplicatively by $L_1(E-\{\infty\}, e)$ is the span of

$$1, \int \alpha_i \,, \int \alpha_i\alpha_i = \frac{1}{2}(\int \alpha_i)^2 \,, \int \alpha_1 \int \alpha_2 = \int \alpha_1\alpha_2 + \alpha_2\alpha_1.$$

This coincides with the subspace $L_2(E, e) \subset L_2(E-\{\infty\}, e)$. The iterated integral $\int \alpha_1\alpha_2 - \alpha_2\alpha_1$ however, which is homotopy functional on $E - \{\infty\}$ but not on E, is more mysterious. We shall prove the following result:

THEOREM 1. Suppose α_1 is holomorphic on E, and α_2 has a (single) pole of order 2 at ∞. Let β_1 and β_2 be loops in $E - \{\infty\}$ based at e whose homology classes form a basis of $H_1(E, \mathbb{Z})$. Let $\mathrm{Per}_{\mathbb{Z}}(\alpha_1) := \mathbb{Z} \int_{\beta_1} \alpha_1 + \mathbb{Z} \int_{\beta_2} \alpha_1$. Then

$$(1) \qquad \frac{1}{\int_{\beta_1} \alpha_1 \int_{\beta_2} \alpha_2 - \int_{\beta_2} \alpha_1 \int_{\beta_1} \alpha_2} \left(\int_{\beta_1} \alpha_1 \int_{\beta_2} (\alpha_1\alpha_2 - \alpha_2\alpha_1) - \int_{\beta_2} \alpha_1 \int_{\beta_1} (\alpha_1\alpha_2 - \alpha_2\alpha_1) \right) \equiv 2 \int_e^\infty \alpha_1$$

mod $\mathrm{Per}_{\mathbb{Z}}(\alpha_1)$.

Note that the integral on the right hand side of (1) is over any path in E from e to ∞, and thus its value is only well-defined mod $\mathrm{Per}_{\mathbb{Z}}(\alpha_1)$.

We shall give three proofs of Theorem 1. We believe these proofs (in particular the second and the third) are more important than the result itself, as they leave much to be explored. Below we say a few words about each approach.

First approach (Section 3): The first proof is based on an argument of Gunning in **[7]**, where he defines quadratic periods of *holomorphic* differentials on a compact Riemann surface. After observing the non-abelian nature of these periods (i.e. that they do not give homomorphisms from the fundamental group) Gunning proves that for hyperelliptic curves, if the base point is a ramification point, these periods are quadratic expressions in the classical periods. We observe that his argument also applies to meromorphic differentials with divisor $\geq -2\infty$ and hence deduce (1).

Second approach (Section 4): We consider the quadratic periods in the Legendre family of elliptic curves $\{E_\lambda\}$ with a suitable choice of base point. Much like the calculations that lead to the classical Picard-Fuchs differential equations, one can see that the quadratic periods of $\int \alpha_1\alpha_2 - \alpha_2\alpha_1$ with $\alpha_1 = \frac{dx}{y}$ and $\alpha_2 = \frac{xdx}{y}$ are constant in the family. The result is then obtained as one allows λ go towards the degeneration points.

This argument points to the direction of a general algebraic theory of Gauss-Manin connections for iterated integrals on families of pointed varieties, perhaps in the spirit of **[15]**.† To our knowledge such a theory has not been worked out, though the required technology might already exist in the literature. The closest fit in the present literature seems to be the articles **[12]** of Hain and Zucker and **[9]** of Hain, where (among other things) they show that the local system of homotopy groups associated with a family of pointed smooth complex varieties underlines a variation of mixed Hodge structures (see Section 4 of **[12]** for the precise setting and statement). However, the methods they use are transcendental, and in particular, do not say if the Gauss-Manin connection on iterated integrals is defined over the field of definition of the pointed family.‡ There are also some works specifically about that quadratic periods of holomorphic differentials in certain

† As further evidence for existence of such a theory we point out that using similar calculations to the ones carried out in this paper, we have found explicit Fuchsian differential equations with coefficients in $\mathbb{Q}(\lambda)$ satisfied by the periods of iterated integrals of length ≤ 3 on the Legendre family.

‡ Note that following the ideas of Wojtkowiak **[19]** one can define an F-lattice (the "de Rham" lattice) inside the space of closed iterated integrals on a pointed variety defined over a subfield $F \subset \mathbb{C}$.

special families of curves satisfy ordinary differential equations (see [**18**] and [**3**]), but the methods in those are again transcendental and far from algebraic.

Third approach (Section 5): Our last argument explores connections to the Hodge theory of the fundamental group of a variety. Let us fix some notation before we proceed. With abuse of notation, for a smooth complex variety U we denote the associated complex manifold by the same symbol. For a pointed smooth complex variety (U, a), (again with abuse of notation) we shall denote Hain's mixed Hodge structure on $L_n(U, a)$ also by $L_n(U, a)$.

Let us go back to the general case of an arbitrary compact Riemann surface X. Writing H^1 for $H^1(X) = H^1(X - \{\infty\})$, one has a short exact sequence of mixed Hodge structures

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{L_1}{L_0}(X-\{\infty\}, e) & \overset{\text{inclusion}}{\longrightarrow} & \frac{L_2}{L_0}(X-\{\infty\}, e) & \longrightarrow & \frac{L_2}{L_1}(X-\{\infty\}, e) & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & H^1 & & & & (H^1)^{\otimes 2} & & \end{array}$$

(see Section 5 for the details). Denote this by $\mathbb{E}_2^\infty \in \mathrm{Ext}(H^1 \otimes H^1, H^1)$. Let ξ be a Hodge class in $H^1 \otimes H^1$. Thinking of ξ as a morphism $\mathbb{Z}(-1) \to H^1 \otimes H^1$, we can pull back $\mathbb{E}_e^\infty$ along ξ and get a point

$$\xi^{-1}(\mathbb{E}_e^\infty) \in \mathrm{Ext}(\mathbb{Z}(-1), H^1) \cong \mathrm{Jac},$$

where Jac is the Jacobian of X. This point was calculated by Kaenders in [**14**, Theorem 1.2] for $\xi = \xi_{\Delta(X)}$ the $H^1 \otimes H^1$ component of the class of the diagonal of X. He showed that†

$$\xi_{\Delta(X)}^{-1}(\mathbb{E}_e^\infty) = (-2g)\infty + 2e + K, \tag{2}$$

where K is the canonical divisor of X. In the case of an elliptic curve, we give a simple description of the Hodge filtration on $L_2(E - \{\infty\}, e)$ in terms of meromorphic differentials (Lemma 3). Then (1) follows from (2).

We should point out that en route to prove (2), Kaenders established a relation similar to (1) for arbitrary X [**14**, Theorem 1.4], with the quadratic iterated integrals involved being of the form $\int \eta\eta'$, where η and η' are harmonic forms on X. His argument however does not seem to directly give a simple relation (as in (1)) for quadratic periods of meromorphic forms, even in the case of an elliptic curve.

The last section of the article discusses the possibility of generalizing the third approach to arbitrary curves. The obstacle is a lack of a simple characterization of the Hodge filtration on $L_2(X - \{\infty\}, e)$ in terms of meromorphic forms.

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2. Recollections on iterated integrals

In this section we recall the definition of iterated integrals and a few basic facts about them. Let $\omega_1, \ldots, \omega_n$ be (smooth, complex-valued) 1-forms on a connected manifold M and $\gamma : [0, 1] \to M$ be a path. Chen defines

$$\int_\gamma \omega_1 \cdots \omega_n = \int_{0 \le t_1 \le \cdots \le t_n \le 1} \gamma^*(\omega_1) \wedge \cdots \wedge \gamma^*(\omega_n).$$

†The extension k_{pq} in [**14**] is $-\xi_{\Delta(X)}^{-1}(\mathbb{E}_p^q)$ in our notation.

The number n is referred to as the length. By convention, if $n = 0$ the integral is defined to be 1. The definition is extended in the obvious way to the expressions of the form $\int_\gamma \sum w$, where each w is a word in 1-forms on M.

Iterated integrals behave in the following way with respect to composition and inverses of paths.

$$\int_{\gamma_1\gamma_2} \omega_1 \cdots \omega_n = \sum_{r=0}^{n} \int_{\gamma_1} \omega_1 \cdots \omega_r \cdot \int_{\gamma_2} \omega_{r+1} \cdots \omega_n \tag{3}$$

$$\int_{\gamma^{-1}} \omega_1 \cdots \omega_n = (-1)^n \int_{\gamma} \omega_n \cdots \omega_1 \tag{4}$$

The following formulas have analogues for iterated integrals of arbitrary length, but we shall only need them for length 2.

$$\int_\gamma \omega(df) = f(\gamma(1)) \int_\gamma \omega - \int_\gamma f\omega \tag{5}$$

$$\int_\gamma (df)\omega = \int_\gamma f\omega - f(\gamma(0)) \int_\gamma \omega \tag{6}$$

The proofs of the formulas (3)-(6) are all straightforward. For instance, the formula (3) is obtained from the decomposition

$$\{ (t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq 1 \} = \bigcup_{r=0}^{n} \{ (t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_r \leq \frac{1}{2} \leq t_{r+1} \leq \cdots \leq t_n \leq 1 \} .$$

The two formulas (5) and (6) are easily proved using integration by parts.

Iterated integrals are functorial: If $\phi : N \to M$ is a smooth map, the ω_i are smooth 1-forms on M and γ is a path on N,

$$\int_{\phi_*(\gamma)} \omega_1 \cdots \omega_n = \int_\gamma \phi^*(\omega_1) \cdots \phi^*(\omega_n).$$

Fix a base point a in the manifold M. Let Ω_a be the loop space at a. By the iterated integral $\int \sum w$, where each w is a word in 1-forms on M, we mean the function

$$\gamma \mapsto \int_\gamma \sum w$$

on Ω_a. We say $\int \sum w$ has length $\leq n$ if each w has length $\leq n$. Note that as illustrated by (5) and (6), two iterated integrals that look different might actually be equal (as functions on Ω_a). We call an iterated integral $\int \sum w : \Omega_a \to \mathbb{C}$ *closed* if it is a homotopy functional, i.e. if its value at a loop $\gamma \in \Omega_a$ only depends on the homotopy class of γ. One should keep in mind that even if all the 1-forms involved in $\int \sum w$ are closed, the iterated integral may not be closed. One has the following well-known lemma about closed iterated integrals of length ≤ 2.

LEMMA 1. Let ω_1, ω_2 be closed 1-forms on M. Suppose there is a 1-form ν on M such that $\omega_1 \wedge \omega_2 + d\nu = 0$. Then the iterated integral $\int \omega_1\omega_2 + \nu$ is closed (for any choice of base point a).

PROOF. Let $p : \tilde{M} \to M$ be a universal covering space of M. Suppose γ_1, γ_2 be homotopic paths in M based at a. Pick a point $\tilde{a} \in \tilde{M}$ above a. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be the lifts of γ_1 and γ_2 starting at $\tilde{a}$. Let $\tilde{b}$ be the common end point of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Let f be a function on $\tilde{M}$ such that $df = p^*(\omega_1)$, chosen such that $f(\tilde{a}) = 0$. Then by functoriality

$$\int_{\gamma_i} \omega_1\omega_2 + \nu = \int_{\tilde{\gamma}_i} (p^*(\omega_1)) \cdot (p^*(\omega_2)) + p^*(\nu)$$

$$= \int_{\tilde{\gamma}_i} df \cdot (p^*(\omega_2)) + p^*(\nu)$$

$$\text{(7)} \qquad \overset{\text{by (6)}}{=} \int_{\tilde{\gamma}_i} fp^*(\omega_2) + p^*(\nu).$$

Note that

$$d\left(fp^*(\omega_2) + p^*(\nu) \right) = p^* \left(\omega_1 \wedge \omega_2 + d\nu \right) = 0.$$

Being a closed 1-form on $\tilde{M}$, the differential form $fp^*(\omega_2) + p^*(\nu)$ is also exact. The result now follows from (7) in view of the fact that the $\tilde{\gamma}_i$ have the same end points.

□

Thus for instance, if M is a Riemann surface and ω_1, ω_2 are both holomorphic or anti-holomorphic, $\int \omega_1\omega_2$ is closed (as $\omega_1 \wedge \omega_2 = 0$).

We end this section by recalling an important consequence of the formula (3). By linearity one can evaluate iterated integrals on the group ring $\mathbb{C}[\Omega_a]$. It follows from (3) that for $\gamma_i \in \Omega_a$,

$$\text{(8)} \qquad \int_{(\gamma_1 - 1)\cdots(\gamma_m - 1)} \omega_1 \cdots \omega_n = \begin{cases} \prod_{i=1}^{n} \int_{\gamma_i} \omega_i & \text{(if } m = n) \\ 0 & \text{(if } m > n). \end{cases}$$

3. An argument of Gunning and the first proof of Theorem 1

Quadratic periods were first defined by Gunning in [7]. After observing the non-abelian nature of these periods, he proves the following result [7, Theorem 1].

THEOREM 2 (Gunning). Let X be a hyperelliptic curve and ω, ω' be holomorphic 1-forms on X. Suppose e is a ramification point of X. Then there exist generators $\{\beta_j\}$ of $\pi_1(X, e)$ such that

$$\int_{\beta_j} \omega\omega' = \int_{\beta_j} \omega'\omega.$$

Gunning's argument makes essential use of the fact that the hyperelliptic involution acts on the space of holomorphic forms by multiplication by -1. It is easy to see that the same remains valid if one considers the action of the hyperelliptic involution on meromorphic forms whose divisor is $\geq -2\infty$, where ∞ is a ramification point.

LEMMA 2. Let X be a hyperelliptic (resp. elliptic) curve and ∞ be a ramification (resp. arbitrary) point of X. Let $\phi : X \to X$ be the hyperelliptic involution (resp. the involution that fixes ∞). Then $\phi^*(\omega) = -\omega$ for every meromorphic differential with divisor $\geq -2\infty$.

PROOF. Note that the space of meromorphic differentials with divisor $\geq -2\infty$ is invariant under ϕ^*. If ω is holomorphic on X, then $\omega + \phi^*(\omega)$ is a holomorphic differential fixed by ϕ, i.e. is a holomorphic differential on $\phi\backslash X \simeq P^1$, and hence is zero. This proves the lemma in the case that ω is holomorphic.

By the Riemann-Roch theorem the space of meromorphic forms with divisor $\geq -2\infty$ has dimension $g+1$ (g = the genus). Thus to finish the proof of the lemma it is now enough to verify the assertion on any 1-form with pole divisor 2∞. Let $f : X \to P^1$ be the composition of the natural map $X \to \phi\backslash X$ with an isomorphism $\phi\backslash X \simeq P^1$ which sends $\overline{\infty} \mapsto \infty \in P^1$, where $\overline{\infty}$ is the image of ∞ in $\phi\backslash X$. Then f is a meromorphic function on X with a single pole of order 2 at ∞. Moreover, it is clear from the construction that $\phi^* f = f$. Let ω be a holomorphic differential on X that does not vanish at ∞. Then $f\omega$ has a single pole of order 2 at ∞ and $\phi^*(f\omega) = -f\omega$. □

Thanks to Lemma 2 we have the following version of Gunning's theorem.

PROPOSITION 1. Let (X, ∞, e) be such that there is an involution $\phi : X \to X$ that fixes e and ∞, and the quotient curve is isomorphic to P^1. Then there is a set of generators $\{\beta_j\}$ of $\pi_1(X - \{\infty\}, e)$ forming a basis of $H_1(X, \mathbb{Z})$ such that for every meromorphic forms ω, ω' with divisor $\geq -2\infty$,

$$\int_{\beta_i} \omega\omega' = \int_{\beta_i} \omega'\omega. \tag{9}$$

PROOF. With Lemma 2 in hand, the proof is identical to Gunning's proof of Theorem 2. We will include it regardless, partly for the sake of completeness, and partly because Gunning does not use the language of iterated integrals in **[7]**. Denote the genus of X by g. Call the remaining fixed points of the involution $p_1, \ldots, p_{2g}$. Identify $\phi\backslash X$ with P^1 via a fixed isomorphism. Denote the image of a point $x \in X$ under the quotient map $X \to \phi\backslash X = P^1$ by $\overline{x}$. Let $\{\tau_i\}$ be disjoint paths in $P^1 - \{\overline{\infty}\}$, with τ_i from $\overline{e}$ to $\overline{p}_i$. For each i, let τ_i^1 and τ_i^2 be the two lifts of τ_i in X. Then τ_i^1 and $\tau_i^2 = \phi_* \tau_i^1$ both go from e to p_i, and aside from their common end points do not intersect. Let $\beta_i = \tau_i^1 (\tau_i^2)^{-1}$. Then the β_i freely generate $\pi_1(X - \{\infty\}, e)$ (see for instance, Section 3 of **[7]**) and thus their images in $H_1(X, \mathbb{Z})$ form a basis. In view of the properties of iterated integrals,

$$\begin{aligned} \int_{\beta_i} \omega\omega' &= \int_{\tau_i^1 (\tau_i^2)^{-1}} \omega\omega' \\ &= \int_{\tau_i^1} \omega\omega' - \int_{\tau_i^1} \omega \int_{\tau_i^2} \omega' + \int_{\tau_i^2} \omega'\omega. \end{aligned}$$

Similarly,

$$\int_{\beta_i} \omega'\omega = \int_{\tau_i^1} \omega'\omega - \int_{\tau_i^1} \omega' \int_{\tau_i^2} \omega + \int_{\tau_i^2} \omega\omega'.$$

Putting these together with $\phi_*(\tau_i^1) = \tau_i^2$ and Lemma 2, in view of the functoriality of iterated integrals we get (9). □

We can now give the first proof of Theorem 1. It is easy to see using (3) that mod $\mathrm{Per}_{\mathbb{Z}}(\alpha_1)$, the left hand side of (1) is independent of the choice of β_1, β_2. Also, a straightforward calculation using (3) and (4) shows that it is enough to prove Theorem 1 for a particular choice of e; the result for arbitrary base point will then follow. Take e to be a fixed point of the involution ϕ of E that fixes ∞. Identify $\phi\backslash E$ and P^1 via a fixed isomorphism. Let β_1, β_2 be as in Proposition 1 (applied to (E, ∞, e)). Then the left hand side of (1) is zero. Composing $E \to \phi\backslash E = P^1$ with an automorphism of P^1 that sends $\overline{\infty} \mapsto \infty$ and $\overline{e} \mapsto 0$ (where $\overline{x}$ denotes the image of x under $E \to \phi\backslash E$ as before), we get a meromorphic function on E with divisor $2(e-\infty)$. Thus by Abel's theorem the right hand side of (1) belongs to $\mathrm{Per}_{\mathbb{Z}}(\alpha_1)$, establishing Theorem 1.

4. Quadratic periods in the Legendre family and the second proof of Theorem 1

For each $\lambda \in \mathbb{C}-\{0,1\}$ let E_λ be the elliptic curve defined by the the affine equation

$$y^2 = x(x-1)(x-\lambda).$$

The E_λ are the well-known Legendre family of elliptic curves. We may assume that our curve E is E_{λ_0} for some λ_0, the point ∞ is the point at infinity, the 1-forms α_1 and α_2 are $\frac{dx}{y}$ and $\frac{xdx}{y}$, and the base point e is the point $(0,0)$ in the affine part. As in our first proof, in order to prove Theorem 1 it is enough to show that there are $\beta_1(\lambda_0), \beta_2(\lambda_0) \in \pi_1(E_{\lambda_0}-\{\infty\}, e)$ which form a basis of integral homology and moreover satisfy

$$\int_{\beta_i(\lambda_0)} \alpha_1\alpha_2 - \alpha_2\alpha_1 = 0. \tag{10}$$

(Note that $2(e-\infty)$ is a principal divisor and thus the right hand side of (1) is in $\mathrm{Per}_{\mathbb{Z}}(\alpha_1)$.) The function x makes each E_λ a double cover of P^1, ramified at the points $e=(0,0)$, $(1,0)$, $(\lambda,0)$, and ∞. Let τ_1 and $\tau_2(\lambda_0)$ be disjoint paths in $\mathbb{C}$ from 0 to 1 and λ_0. Then with the superscript notation as in the proof of Proposition 1 for the double cover $x : E_{\lambda_0} \to P^1$, set $\beta_1(\lambda_0) = \tau_1^1(\tau_1^2)^{-1}$ and $\beta_2(\lambda_0) = \tau_2(\lambda_0)^1(\tau_2(\lambda_0)^2)^{-1}$. For λ in a small neighborhood U of λ_0, let $\tau_2(\lambda)$ be a path in $\mathbb{C}$ from 0 to λ that varies continuously as λ varies. Set $\beta_1(\lambda) = \tau_1^1(\tau_1^2)^{-1}$ and $\beta_2(\lambda) = \tau_2(\lambda)^1(\tau_2(\lambda)^2)^{-1}$, where the superscripts signify the two lifts for $x : E_\lambda \to P^1$ and the labelling has been done "consistently" on U (i.e. such that $\beta_i(\lambda)$ varies continuously in $\mathcal{E} = \cup E_\lambda$). Define

$$f_i(\lambda) = \int_{\beta_i(\lambda)} \alpha_1\alpha_2 - \alpha_2\alpha_1.$$

Then the f_i are holomorphic on U and can be analytically continued along paths in $\mathbb{C}-\{0,1\}$ (by continuously deforming the paths in $\mathbb{C}$ and lifting to E_λ), hence are multi-valued functions on this space. Since $\int \alpha_1\alpha_2 - \alpha_2\alpha_1$ is a homotopy functional on each $E_\lambda - \{\infty\}$ (∞ the point at infinity), similar to the case of classical periods one can push differentiation inside the integral and get

$$f_i'(\lambda) = \frac{1}{2}\int_{\beta_i'(\lambda)} \frac{\alpha_1}{x-\lambda}\cdot\alpha_2 + \alpha_1\cdot\frac{\alpha_2}{x-\lambda} - \frac{\alpha_2}{x-\lambda}\cdot\alpha_1 - \alpha_2\cdot\frac{\alpha_1}{x-\lambda}, \tag{11}$$

where $\beta_i'(\lambda)$ is homotopic to $\beta_i(\lambda)$ in $E_\lambda - \{\infty\}$, but does not pass through $(\lambda,0)$. As 1-forms on $E_\lambda - \{\infty, (\lambda,0)\}$ one has

$$\frac{\alpha_1}{x-\lambda} = \frac{1}{1-\lambda}\alpha_1 - \frac{1}{\lambda(1-\lambda)}\alpha_2 + \frac{2}{\lambda(\lambda-1)}d\left(\frac{y}{x-\lambda}\right)$$

and

$$\frac{\alpha_2}{x-\lambda} = \frac{1}{1-\lambda}\alpha_1 - \frac{1}{1-\lambda}\alpha_2 + \frac{2}{1-\lambda}d\left(\frac{y}{x-\lambda}\right).$$

Substituting these in (11), a straightforward calculation using (5), (6) and $\frac{y}{x-\lambda}(0,0) = 0$ shows that $f_i'(\lambda) = 0$. Thus in fact, the f_i are constant (and single-valued) on $\mathbb{C}-\{0,1\}$. Letting $\lambda \to 0$, we see that $f_2 \equiv 0$. Letting $\lambda \to 1$ we get $f_2 \to f_1$. Thus $f_1 \equiv 0$ as well, concluding our second proof of Theorem 1.

5. Hodge theory of π_1 of a punctured curve

5.1. Extensions in the category of mixed Hodge structures. Let us start with some notation. Given a mixed Hodge structure A, by $A_{\mathbb{Z}}$ (resp. $A_{\mathbb{Q}}$ or $A_{\mathbb{C}}$) we mean the underlying $\mathbb{Z}$-module (resp. rational or complex vector space). As usual, $W_\cdot$ and $F^\cdot$ denote the weight and Hodge filtrations. For each n, define

$$J^n(A) := \frac{A_{\mathbb{C}}}{F^n A_{\mathbb{C}} + A_{\mathbb{Z}}}.$$

If A is pure of odd weight $2n-1$, set $JA := J^n A$. We use the notation $\underline{\mathrm{Hom}}$ for internal hom in the category of mixed Hodge structures. More explicitly, $\underline{\mathrm{Hom}}(A, B)$ is the mixed Hodge structure on the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(A_{\mathbb{Z}}, B_{\mathbb{Z}})$, with the weight and Hodge filtrations as follows:

$$W_n \mathrm{Hom}_{\mathbb{Q}}(A_{\mathbb{Q}}, B_{\mathbb{Q}}) = \{f : A_{\mathbb{Q}} \to B_{\mathbb{Q}} : \ f(W_l A_{\mathbb{Q}}) \subset W_{n+l} B_{\mathbb{Q}} \text{ for all } l\}$$
$$F^p \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}}, B_{\mathbb{C}}) = \{f : A_{\mathbb{C}} \to B_{\mathbb{C}} : \ f(F^l A_{\mathbb{C}}) \subset F^{p+l} B_{\mathbb{C}} \text{ for all } l\}.$$

Note that if A and B are pure of weights a and b, then $\underline{\mathrm{Hom}}(A, B)$ is pure of weight $b-a$.

We shall need a result of Carlson on classifying extensions of mixed Hodge structures. Let A and B be mixed Hodge structures. By $\mathrm{Ext}(A, B)$ we mean the group of extensions of A by B in the category of mixed Hodge structures. Suppose the highest weight of B is less than the lowest weight of A. Carlson in **[1]** gives an isomorphism

$$\mathrm{Ext}(A, B) \cong J^0 \underline{\mathrm{Hom}}(A, B)\ ,$$

as follows: Given an extension $\mathbb{E}$ represented by the short exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0\ ,$$

the corresponding element in $J^0\underline{\mathrm{Hom}}(A, B)$ to $\mathbb{E}$ is the class of $\rho_{\mathbb{Z}} \circ \sigma_F$, where σ_F is a Hodge section of $E_{\mathbb{C}} \to A_{\mathbb{C}}$ and $\rho_{\mathbb{Z}}$ is an integral retraction of $B_{\mathbb{C}} \to E_{\mathbb{C}}$. (By a Hodge section we mean a section that preserves the Hodge filtrations, and by an integral retraction we mean a retraction that is induced by a map between the underlying $\mathbb{Z}$-modules.)

In what follows we shall identify $\mathrm{Ext}(A, B)$ and $J^0\underline{\mathrm{Hom}}(A, B)$ via the isomorphism of Carlson.

5.2. Review of Abel-Jacobi maps. Let Y be a smooth complex projective variety. We denote by $\mathcal{Z}_i(Y)$ the group of i-dimensional algebraic cycles on Y. Let $CH_i(Y)$ be the Chow group of i-dimensional algebraic cycles on Y (i.e. $\mathcal{Z}_i(Y)$ modulo rational equivalence). We denote the homologically trivial subgroups of $\mathcal{Z}_i(Y)$ and $CH_i(Y)$ respectively by $\mathcal{Z}_i^{\mathrm{hom}}(Y)$ and $CH_i^{\mathrm{hom}}(Y)$. Throughout, with abuse of notation we use the same symbol for an algebraic cycle and its class in the Chow group. In this paragraph we briefly recall the definition of Griffith's Abel-Jacobi map

$$AJ\ : CH_i^{\mathrm{hom}}(Y) \longrightarrow \frac{\left(F^{i+1}H^{2i+1}(Y)\right)^{\vee}}{H_{2i+1}(Y, \mathbb{Z})}\ . \tag{12}$$

We refer the reader to **[20]** for details and proofs. Let Z be a homologically trivial i-dimensional algebraic cycle on Y. Then there is a chain C whose boundary is Z. Given an element $c \in F^{i+1}H^{2i+1}(Y)$, one can choose a representative $\omega \in F^{i+1}\mathcal{E}^{2i+1}(Y)$, where $\mathcal{E}^\cdot(Y)$ is the complex of complex-valued smooth differential forms on Y and $F^\cdot$ is the Hodge filtration on this complex. Set

$$\int_C c := \int_C \omega.$$

This is well-defined as $\int_C \omega$ does not depend on the choice of the representative $\omega \in F^{i+1}\mathcal{E}^{2i+1}(Y)$ of c. Thus we have an element $\int_C$ in $\big(F^{i+1}H^{2i+1}(Y)\big)^\vee$. It is easy to see that the class of this element in $\dfrac{\big(F^{i+1}H^{2i+1}(Y)\big)^\vee}{H_{2i+1}(Y,\mathbb{Z})}$ only depends on Z; in fact, one can show that this class only depends on Z modulo rational equivalence. The Abel-Jacobi map (12) is then defined by

$$Z \mapsto \text{the class of} \int_C .$$

Let $\mathbb{Z}(0)$ be the unique Hodge structure of weight zero on $\mathbb{Z}$. The restriction map $H^{2i+1}_{\mathbb{C}}(Y)^\vee \to \big(F^{i+1}H^{2i+1}(Y)\big)^\vee$ induces an isomorphism

$$J\underline{\mathrm{Hom}}(H^{2i+1}(Y),\mathbb{Z}(0)) \longrightarrow \frac{\big(F^{i+1}H^{2i+1}(Y)\big)^\vee}{H_{2i+1}(Y,\mathbb{Z})}$$

(given by $[f] \mapsto [f\,|_{F^{i+1}H^{2i+1}(Y)}]$). We shall identify the two spaces via this isomorphism. Thus we may consider the target of the Abel-Jacobi map (12) to be $J\underline{\mathrm{Hom}}(H^{2i+1}(Y),\mathbb{Z}(0))$.

5.3. Background on Hodge theory of the fundamental group. In this paragraph we briefly recall certain results of Chen [**2**] and Hain [**10**], [**8**]. Let M be a connected manifold. Choose a base point a. We denote by $L_n(M,a)$ the space of closed iterated integrals of length $\le n$ on the pointed manifold (M,a); thus an element of $L_n(M,a)$ is a function on the loop space at a of the form $\int \sum w$, where each w is a word of length $\le n$ in the 1-forms on M, such that the value $\int_\gamma \sum w$ only depends on the homotopy class of the loop γ. By extending linearly we consider elements of $L_n(M,a)$ as functionals on the group ring $\mathbb{C}[\pi_1(M,a)]$. Let $I \subset \mathbb{C}[\pi_1(M,a)]$ be the augmentation ideal. By (8) the elements of $L_n(M,a)$ vanish on I^{n+1}. A theorem of Chen [**2**, Theorem 5.3] asserts that in fact, every functional $\mathbb{C}[\pi_1(M,a)] \to \mathbb{C})$ that vanishes on I^{n+1} is given by an iterated integral of length $\le n$, so that

$$L_n(M,a) = \left(\frac{\mathbb{C}[\pi_1(M,a)]}{I^{n+1}}\right)^\vee .$$

Let U be a smooth complex variety. We denote the associated complex manifold also by U. Let $a \in U$. Hain showed that there is a natural mixed Hodge structure on the integral lattice

$$\left(\frac{\mathbb{Z}[\pi_1(U,a)]}{I^{n+1}}\right)^\vee$$

(π_1 the topological fundamental group and I the augmentation ideal). We shall use the same notation for this mixed Hodge structure as its underlying complex vector space, i.e. $L_n(U,a)$. To describe the weight and Hodge filtrations on $L_n(U,a)$, one realizes U as the complement of a normal crossing divisor D in a smooth projective variety Y. Let $\mathcal{E}^1(Y \log D)$ be the space of smooth 1-forms on Y with at most logarithmic singularity along D. One can show that every element of $L_n(U,a)$ can be expressed as an iterated integral of length $\le n$ formed solely using differentials in $\mathcal{E}^1(Y \log D)$. Then the weight and Hodge filtrations on $L_n(U,a)$ are as follows:

- The weight filtration: $W_m(L_n)$ is the space of those closed iterated integrals that can be expressed as a sum of (not necessarily closed) iterated integrals of the form $\int \omega_1 \cdots \omega_r$, with $r \leq n$ and $\omega_i \in \mathcal{E}^1(Y \log D)$, such that at most $m-r$ of the ω_i are not smooth along D. Note that this in particular implies that $W_m(L_n) \subset L_m$ and $W_{2n}(L_n) = L_n$. One can prove that this filtration is indeed defined over $\mathbb{Q}$.

- The Hodge filtration: $F^p(L_n)$ is the space of those closed iterated integrals that can be expressed as a sum of (not necessarily closed) iterated integrals of the form $\int \omega_1 \cdots \omega_r$, where $r \leq n$ and $\omega_i \in \mathcal{E}^1(Y \log D)$, such that at least p of the ω_i are of type (1,0).

The construction of the mixed Hodge structure on L_n and the proofs of the facts listed above can be found in [**10**] (see Theorem (5.1) therein and its proof). Alternatively, the reader can refer to [**8**] (in particular, Corollary (2.4.4), Theorem (3.2.1), and Lemma (5.6.3) therein) for a different treatment.

5.4. Let X be a compact Riemann surface of arbitrary genus $g > 0$ and ∞, e distinct points in X. Let $I \subset \mathbb{Z}[\pi_1(X - \{\infty\}, e)]$ be the augmentation ideal. Consider the short exact sequence

$$0 \longrightarrow \frac{I^n}{I^{n+1}} \overset{\text{inclusion}}{\longrightarrow} \frac{\mathbb{Z}[\pi_1(X-\{\infty\},e)]}{I^{n+1}} \longrightarrow \frac{\mathbb{Z}[\pi_1(X-\{\infty\},e)]}{I^n} \longrightarrow 0. \tag{13}$$

Since $\pi_1(X - \{\infty\}, e)$ is free,

$$\left(\frac{I}{I^2}\right)^{\otimes n} \cong \frac{I^n}{I^{n+1}}$$

via

$$[\gamma_1 - 1] \otimes \cdots \otimes [\gamma_n - 1] \mapsto [(\gamma_1 - 1) \cdots (\gamma_n - 1)].$$

Combining with $\frac{I}{I^2} \cong H_1(X - \{\infty\}, \mathbb{Z})$, we can rewrite (13) as

$$0 \longrightarrow H_1(X-\{\infty\},\mathbb{Z})^{\otimes n} \longrightarrow \frac{\mathbb{Z}[\pi_1(X-\{\infty\},e)]}{I^{n+1}} \longrightarrow \frac{\mathbb{Z}[\pi_1(X-\{\infty\},e)]}{I^n} \longrightarrow 0.$$

Dualizing we get a short exact sequence

$$0 \longrightarrow \left(\frac{\mathbb{Z}[\pi_1(X-\{\infty\},e)]}{I^n}\right)^{\vee} \overset{\text{inclusion}}{\longrightarrow} \left(\frac{\mathbb{Z}[\pi_1(X-\{\infty\},e)]}{I^{n+1}}\right)^{\vee} \longrightarrow H^1(X-\{\infty\},\mathbb{Z})^{\otimes n} \longrightarrow 0.$$

Tensoring with $\mathbb{C}$ this gives a short exact sequence

$$0 \longrightarrow L_{n-1}(X-\{\infty\},e) \overset{\text{inclusion}}{\longrightarrow} L_n(X-\{\infty\},e) \overset{(*)}{\longrightarrow} H^1(X-\{\infty\},\mathbb{C})^{\otimes n} \longrightarrow 0. \tag{14}$$

Tracking the procedure we see that the map $(*)$ sends $f \in L_n(X - \{\infty\}, e)$ to

$$\Big(\ [\gamma_1] \otimes \cdots \otimes [\gamma_n] \mapsto f((\gamma_1 - 1)\dots(\gamma_n - 1))\ \Big) \in (H_1(X-\{\infty\},\mathbb{C})^{\otimes n})^{\vee} = (H^1(X-\{\infty\},\mathbb{C}))^{\otimes n}.$$

(Here $\gamma_i \in \pi_1(X - \{\infty\}, e)$.) Equivalently, $(*)$ sends the iterated integral

$$f = \int \omega_1 \cdots \omega_n + \text{lower length terms},$$

where the ω_i are closed smooth 1-forms on $X - \{\infty\}$, to $[\omega_1] \otimes \cdots \otimes [\omega_n] \in (H^1(X - \{\infty\}, \mathbb{C}))^{\otimes n}$.

It is well-known that the isomorphism

$$\frac{L_n}{L_{n-1}}(X-\{\infty\},e) \cong H^1(X-\{\infty\})^{\otimes n}$$

induced by $(*)$ preserves the weight and Hodge filtrations, and thus is in fact an isomorphism of (mixed) Hodge structures (see for instance, [**10**, Proposition (5.3)] and its proof). We shall identify these two Hodge structures via this isomorphism, and refer to the map $(*)$ simply as the quotient map.

5.5. Let X, ∞ and e be as before. For simplicity in what follows we write H^1 for $H^1(X,\mathbb{C}) = H^1(X-\{\infty\},\mathbb{C})$.

Let $\mathbb{E}_e^\infty$ be the extension of mixed Hodge structures given by the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{L_1}{L_0}(X-\{\infty\},e) & \stackrel{\text{inclusion}}{\longrightarrow} & \frac{L_2}{L_0}(X-\{\infty\},e) & \longrightarrow & \frac{L_2}{L_1}(X-\{\infty\},e) & \longrightarrow & 0, \\ & & \wr\| & & & & \wr\| & & \\ & & H^1 & & & & H^1\otimes H^1 & & \end{array}$$

considered as an element of

$$\mathrm{Ext}(H^1\otimes H^1, H^1) \stackrel{\text{Carlson}}{\cong} J\underline{\mathrm{Hom}}(H^1\otimes H^1, H^1) \stackrel{\text{Poincaré duality}}{\cong} J\underline{\mathrm{Hom}}((H^1)^{\otimes 2}\otimes H^1, \mathbb{Z}(0)). \tag{15}$$

For every Hodge class $\xi \in H^1\otimes H^1$, let $\xi^{-1} : J\underline{\mathrm{Hom}}((H^1)^{\otimes 3},\mathbb{Z}(0)) \to J\underline{\mathrm{Hom}}(H^1,\mathbb{Z}(0))$ be the map that sends the class of $f : (H^1)^{\otimes 3} \to \mathbb{C}$ to the class of the map $\alpha \mapsto f(\xi\otimes\alpha)$. It is easy to see that via the identifications (15) and

$$\mathrm{Ext}(\mathbb{Z}(-1), H^1) \cong J\underline{\mathrm{Hom}}(\mathbb{Z}(-1), H^1) \stackrel{\text{Poincaré Duality}}{\cong} J\underline{\mathrm{Hom}}(H^1,\mathbb{Z}(0)),$$

for any extension $\mathbb{E} \in \mathrm{Ext}(H^1\otimes H^1, H^1)$, the element $\xi^{-1}(\mathbb{E}) \in \mathrm{Ext}(\mathbb{Z}(-1), H^1)$ is the pullback of $\mathbb{E}$ along the morphism $\mathbb{Z}(-1) \to H^1\otimes H^1$ given by $1 \mapsto \xi$.

Denote the diagonal of X by $\Delta(X) \in CH_1(X^2)$. The following result is due to Kaenders [**14**, Theorem 1.2].

THEOREM 3. Let $\xi_{\Delta(X)}$ be the $H^1\otimes H^1$ component of the class of $\Delta(X)$. Then

$$\xi^{-1}_{\Delta(X)}(\mathbb{E}_e^\infty) = AJ(-2g\infty + 2e + K),$$

where K is the divisor of a meromorphic 1-form on X and AJ is the Abel-Jacobi map $CH_0^{\mathrm{hom}}(X) \to J\underline{\mathrm{Hom}}(H^1,\mathbb{Z}(0))$ (see Paragraph 5.2).

Darmon, Rotger, and Sols more generally describe $\xi^{-1}(\mathbb{E}_e^\infty)$ for arbitrary Hodge classes, as follows. Let h be the composition

$$CH_1^{\mathrm{hom}}(X^3) \stackrel{\text{Abel-Jacobi}}{\longrightarrow} J\underline{\mathrm{Hom}}(H^3(X^3),\mathbb{Z}(0)) \stackrel{\text{Kunneth}}{\longrightarrow} J\underline{\mathrm{Hom}}((H^1)^{\otimes 3},\mathbb{Z}(0)). \tag{16}$$

Let

$$\begin{aligned} \Delta_e \;:=\; & \{(x,x,x) : x\in X\} - \{(e,x,x) : x\in X\} - \{(x,e,x) : x\in X\} - \{(x,x,e) : x\in X\} \\ + \; & \{(e,e,x) : x\in X\} + \{(e,x,e) : x\in X\} + \{(x,e,e) : x\in X\} \in CH_1^{\mathrm{hom}}(X^3) \end{aligned} \tag{17}$$

be the modified diagonal cycle of Gross, Kudla and Schoen in X^3, introduced and shown to be homologically trivial in [**6**]. Let Z_e^∞ be the cycle

$$Z_e^\infty := \{(x,x,\infty) : x\in X\} - \{(x,x,e) : x\in X\} \in CH_1^{\mathrm{hom}}(X^3). \tag{18}$$

Darmon, Rotger, and Sols prove the following result in [**4**, Theorem 2.5].

THEOREM 4. (a) For every Hodge class ξ,

$$\xi^{-1}(\mathbb{E}_e^\infty) = \xi^{-1}(h(-\Delta_e + Z_e^\infty)). \tag{19}$$

(b) Suppose X, e, ∞ are defined over a subfield $F \subset \mathbb{C}$. Denote the Jacobian of X by Jac. If ξ is the $H^1\otimes H^1$ Kunneth component of the class of an algebraic cycle on X^2 defined over F, then the points

$$\xi^{-1}(\mathbb{E}_e^\infty),\ \xi^{-1}(h(\Delta_e)) \in J\underline{\mathrm{Hom}}(H^1,\mathbb{Z}(0)) \stackrel{\text{Abel-Jacobi}}{\cong} \mathrm{Jac}(\mathbb{C})$$

are F-rational.

REMARK. While we shall not need it in the present paper, the identity (19) is in fact valid before applying ξ^{-1} (see [**5**, Paragraph 3.6.2]).

5.6. In view of Carlson's theorem (see Paragraph 5.1), to describe the extension $\mathbb{E}_e^\infty$ and prove results such as Theorems 3 and 4 one needs

(i) a retraction r of the inclusion map $H^1 \to \frac{L_2}{L_0}(X-\{\infty\}, e)$ defined over $\mathbb{Z}$, and

(ii) a section of the quotient map $\frac{L_2}{L_0}(X-\{\infty\}, e) \to H^1 \otimes H^1$ compatible with the Hodge filtrations.

The former is easy: One simply chooses loops $\beta_i \in \pi_1(X-\{\infty\}, e)$ representing a basis of $H_1(X, \mathbb{Z})$, and then defines $r = r_{\{\beta_i\}}$ by

$$r([f])([\beta_i]) = f(\beta_i - 1) \qquad (f \in L_2(X-\{\infty\}, e),\ [f] \in \frac{L_2}{L_0}(X-\{\infty\}, e)).$$

Note that one evaluates f at $\beta_i - 1$ (1 = the constant loop) rather than β_i to kill the constant term of f.

Defining a section as described in (ii) is not as straightforward. Of course, the objective will be achieved if one defines a section s of the quotient map $L_2(X-\{\infty\}, e) \to H^1 \otimes H^1$ compatible with the Hodge filtrations; then $s \pmod{L_0}$ will be our desired section of the map $\frac{L_2}{L_0}(X-\{\infty\}, e) \to H^1 \otimes H^1$. It is easy to define s on the subspace $H^{1,0} \otimes H^{1,0} + H^{0,1} \otimes H^{0,1}$: Given elements $[\eta_1] \otimes [\eta_2] \in H^1 \otimes H^1$, where η_1, η_2 are both holomorphic or anti-holomorphic on X, the integral $\int \eta_1\eta_2$ is closed on X, and one can define the section s on the subspace $H^{1,0} \otimes H^{1,0} + H^{0,1} \otimes H^{0,1}$ by

$$[\eta_1] \otimes [\eta_2] \ \mapsto\ \int \eta_1\eta_2 \,.$$

Extending this to the kernel K of the cup product $H^1 \otimes H^1 \to H^2(X)$ goes back to B. Harris [**13**] and Pulte [**17**]. The key is that K is the image of

$$L_2(X, e) \subset L_2(X-\{\infty\}, e) \to H^1 \otimes H^1,$$

so that in fact given $\Omega = \sum [\eta_i] \otimes [\eta_i'] \in K$, where η_i and η_i' are harmonic forms, one can lift Ω to a closed iterated integral on X. More precisely, recall that the space $\mathcal{E}^1(X)$ of smooth complex-valued 1-forms on X decomposes as $\mathcal{H} \oplus \mathcal{H}^\perp$, where $\mathcal{H}$ is the space of harmonic forms on X, and orthogonality is with respect to the inner product defined using the Hodge $*$ operator. Since $\Omega = \sum [\eta_i] \otimes [\eta_i'] \in K$, the 2-form $\sum \eta_i \wedge \eta_i'$ is exact on X. Since the elements of $\mathcal{H}$ are closed, one can choose $\nu \in \mathcal{H}^\perp$ such that $\sum \eta_i \wedge \eta_i' + d\nu = 0$. The differential ν is unique modulo exact differentials (as a closed element of $\mathcal{H}^\perp$ is exact). It follows that the iterated integral

$$\int \Big(\sum \eta_i\eta_i'\Big) + \nu \,,$$

which is closed by Lemma 1, does not depend on the choice of ν. Now one extends s to K by sending

$$\Omega \ \mapsto\ \int \Big(\sum \eta_i\eta_i'\Big) + \nu \,.$$

Note that this is consistent with the earlier assignment on $H^{1,0} \otimes H^{1,0} + H^{0,1} \otimes H^{0,1}$, as for $\Omega \in H^{1,0} \otimes H^{1,0} + H^{0,1} \otimes H^{0,1}$ one can simply take ν to be zero. Also, it is compatible with the Hodge filtration as if Ω is of type (1,1) one can choose ν to be of type (1,0).†

It remains to extend the section s from K to $H^1 \otimes H^1$. Darmon, Rotger and Sols do this in [**4**] using the Green function attached to a volume form on a Riemann surface. Indeed, take real harmonic forms η_0 and η_0' such that $[\eta_0] \otimes [\eta_0'] \notin K$. It is enough to define s at $[\eta_0] \otimes [\eta_0']$ (since the dimension of K is one less than the dimension of $H^1 \otimes H^1$). Let $g_{\infty,\eta_0\wedge\eta_0'}$ be the Green function attached to the 2-form $\eta_0 \wedge \eta_0'$ and divisor ∞; recall that this means $g_{\infty,\eta_0\wedge\eta_0'} : X - \{\infty\} \to \mathbb{R}$ is a smooth function satisfying the following properties:

(i) In a small neighborhood U of the point ∞, with a chart taken so that the point ∞ corresponds to $z = 0$, the function $g_{\infty,\eta_0\wedge\eta_0'}$ is of the form

$$-(\int_X \eta_0 \wedge \eta_0') \log z\bar{z} \ + \ \text{a smooth function on } U.$$

(ii) One has $dd^c g_{\infty,\eta_0\wedge\eta_0'} = \eta_0 \wedge \eta_0'$ on $X - \{\infty\}$, where $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$ with $\partial = \frac{\partial}{\partial z} dz$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}$ the usual operators.

(See Chapter 2 of [**16**], for instance.) Then $\nu := -\frac{1}{2\pi i}\partial g_{\infty,\eta_0\wedge\eta_0'}$ is of type (1,0), has a logarithmic singularity at ∞, and satisfies $\eta_0 \wedge \eta_0' + d\nu = 0$ on $X - \{\infty\}$. Now one completes the definition of our section s by sending

$$[\eta_0] \otimes [\eta_0'] \ \mapsto \ \int \eta_0 \wedge \eta_0' + \nu.$$

Kaenders' approach in [**14**] to extend the section s from K to $H^1 \otimes H^1$ is similar, albeit less explicit. He uses strictness of the differential of the complex $\mathcal{E}^{\cdot}(X \log \infty)$ of smooth differential forms with at most logarithmic singularity at ∞ with respect to the Hodge filtration to conclude existence of a differential $\mu \in \mathcal{E}^1(X \log \infty)$ of type (1,0) satisfying $\eta_0 \wedge \eta_0' + d\mu = 0$ on $X - \{\infty\}$.

5.7. An alternate Hodge section of $\frac{L_2}{L_0}(X - \{\infty\}, e) \to H^1 \otimes H^1$ and the third proof of Theorem 1. In this paragraph we give an alternate section of the natural map $\frac{L_2}{L_0} \to H^1 \otimes H^1$ in the case that $X = E$ is an elliptic curve. Then we deduce Theorem 1 from Kaenders' formula.

LEMMA 3. *Let α_1 be a nonzero holomorphic 1-form on E, and α_2 be a meromorphic differential with a single pole of order 2 at ∞. Then $\sigma : H^1 \otimes H^1 \to L_2(E - \{\infty\}, e)$ defined by*

$$[\alpha_i] \otimes [\alpha_j] \mapsto \int \alpha_i \alpha_j \tag{20}$$

is a section of the quotient map $L_2(E - \{\infty\}, e) \to H^1 \otimes H^1$ that is compatible with the Hodge filtration.

PROOF. That σ is a section is clear. We must verify compatibility with the Hodge filtration. This is clear for F^0 and F^2. Thus it remains to show that for $i \neq j$,

$$\int \alpha_i \alpha_j \in F^1 L_2(E - \{\infty\}, e).$$

†This follows from the following two facts: (1) The operator d in the complex $\mathcal{E}^{\cdot}(X)$ of smooth differential forms on X is strict with respect to the Hodge filtration, and (2) the projection $\mathcal{E}^1(X) \to \mathcal{H}^{\perp}$ preserves Hodge type.

Let us consider the case $i = 1$; the other case is similar. Let η_2 be a harmonic form on E whose cohomology class in $E - \{\infty\}$ coincides with that of α_2. Write $\alpha_2 = \eta_2 + df$ on $E - \{\infty\}$, where f is smooth and $f(e) = 0$. Then by (5), as functions on $\pi_1(E - \{\infty\}, e)$ we have

$$\int \alpha_1\alpha_2 = \int \alpha_1\eta_2 - f\alpha_1.$$

Since the differential of the complex $\mathcal{E}^{\cdot}(E \log \infty)$ of smooth differential forms on E with at most logarithmic singularity at ∞ is strict with respect to the Hodge filtration and $\alpha_1 \wedge \eta_2 \in \mathcal{E}^2(E \log \infty)$ is exact (of type (1,1)), there is $\mu \in \mathcal{E}^1(E \log \infty)$ of type (1,0) satisfying $\alpha_1 \wedge \eta_2 + d\mu = 0$. Then $\int \alpha_1\eta_2 + \mu \in F^1L_2(E - \{\infty\}, e)$. Thus it is enough to have

$$\int \mu + f\alpha_1 \in F^1L_1(E - \{\infty\}, e).$$

Since α_2 has a pole of order only 2 at ∞, the differential $f\alpha_1$ has a logarithmic singularity at ∞. The 1-form $\mu + f\alpha_1$ is a smooth differential form of type (1,0) with at most logarithmic singularity at ∞. The result follows. □

Let $\overline{\sigma} : H^1 \otimes H^1 \to \frac{L_2}{L_0}(E - \{\infty\}, e)$ be the composition of σ (defined in Lemma 3) and the quotient map $L_2(E - \{\infty\}, e) \to \frac{L_2}{L_0}(E - \{\infty\}, e)$. Then $\overline{\sigma}$ is a section of the quotient map $\frac{L_2}{L_0}(E - \{\infty\}, e) \to H^1 \otimes H^1$ compatible with the Hodge filtrations. Using the retraction r of the inclusion $H^1 \to \frac{L_2}{L_0}(E - \{\infty\}, e)$ described in Paragraph 5.6 and the section $\overline{\sigma}$ of the quotient map $\frac{L_2}{L_0}(E - \{\infty\}, e) \to H^1 \otimes H^1$, the extension $\mathbb{E}_e^\infty \in J\underline{\mathrm{Hom}}((H^1)^{\otimes 3}, \mathbb{Z}(0))$ is represented by the map

$$\psi : (H^1)^{\otimes 3} \longrightarrow \mathbb{C} \qquad [\alpha_i] \otimes [\alpha_j] \otimes PD([\beta_k]) \mapsto \int_{\beta_k} \alpha_i\alpha_j,$$

where PD denotes the Poincaré dual: $PD([\beta])$ for a loop β is $[\omega]$ if $\int_\beta - = \int_E \omega \wedge -$ on closed forms.

We may assume $\int_E PD(\beta_1) \wedge PD(\beta_2) = 1$. Then

$$[\alpha_1] = (-\int_{\beta_2} \alpha_1) \cdot PD(\beta_1) + (\int_{\beta_1} \alpha_1) \cdot PD(\beta_2).$$

Writing $\xi_{\Delta(E)} = \sum c_{ij}[\alpha_i] \otimes [\alpha_j]$, the element

$$\xi^{-1}_{\Delta(E)}(\mathbb{E}_e^\infty) \in J\underline{\mathrm{Hom}}(H^1, \mathbb{Z}(0)) \cong \frac{(F^1H^1)^\vee}{H_1(E, \mathbb{Z})}$$

(the identification via $[f] \mapsto [f|_{F^1H^1}]$, see Paragraph 5.2) is represented by the map $F^1H^1 \to \mathbb{C}$ which sends

$$[\alpha_1] \mapsto \psi(\xi_{\Delta(E)} \otimes [\alpha_1]) = \sum c_{ij} \left(-\int_{\beta_2} \alpha_1 \int_{\beta_1} \alpha_i\alpha_j + \int_{\beta_1} \alpha_1 \int_{\beta_2} \alpha_i\alpha_j \right). \tag{21}$$

On the other hand, the element $AJ(-2\infty + 2e) \in \dfrac{(F^1H^1)^\vee}{H_1(E,\mathbb{Z})}$ is represented by the map $F^1H^1 \to \mathbb{C}$ defined by

$$[\alpha_1] \mapsto -2\int_e^\infty \alpha_1 \tag{22}$$

(where the integral is over any path from e to ∞). By Theorem 3, the two maps (21) and (22) defer by an element of $H_1(E,\mathbb{Z})$ (note that the canonical divisor K is zero, since our curve is elliptic). Thus

$$\sum c_{ij}\left(-\int_{\beta_2}\alpha_1\int_{\beta_1}\alpha_i\alpha_j + \int_{\beta_1}\alpha_1\int_{\beta_2}\alpha_i\alpha_j\right) \equiv -2\int_e^\infty \alpha_1 \qquad (\mathrm{mod}\ \mathrm{Per}_{\mathbb{Z}}(\alpha_1)).$$

A straightforward calculation (see [**5**, Lemma 5.3.1]) shows

$$\begin{aligned} c_{21} = -c_{12} &= \frac{1}{\int_{\beta_1}\alpha_1\int_{\beta_2}\alpha_2 - \int_{\beta_2}\alpha_1\int_{\beta_1}\alpha_2} \\ c_{11} = c_{22} &= 0, \end{aligned}$$

and one gets the relation given in Theorem 1.

6. Generalization to arbitrary curves

In view of Theorem 4 the last method can be applied to produce explicit relations involving periods and quadratic periods of meromorphic differentials on an arbitrary punctured Riemann surface $X - \{\infty\}$, as long as one has an explicit description of the Hodge filtration on $L_2(X - \{\infty\}, e)$ in terms of such forms, analogous to the one given by Lemma 3 in $g = 1$ case. To make this precise, let us give a definition: A *good* set of differential forms for the triple (X, ∞, e) is a set of meromorphic 1-forms $\{\alpha_i\}_{i\le 2g}$ on X such that

(i) $\alpha_1, \dots, \alpha_g$ are holomorphic on X,
(ii) $\alpha_{g+1}, \dots, \alpha_{2g}$ have poles only at ∞,
(iii) the α_i represent a basis of $H^1(X - \{\infty\}) = H^1(X)$, and
(iv) the iterated integrals

$$\int \alpha_i\alpha_j$$

belong to $F^1L_2(X - \{\infty\}, e)$ if either i or j is $\le g$.

For instance, when $g = 1$ if α_1 is any nonzero holomorphic form, and α_2 is meromorphic with a single pole of order 2 at ∞, then α_1, α_2 form a good set of differentials. Note that the requirements are asking for a description of the Hodge filtration on $L_2(X - \{\infty\}, e)$ in terms of meromorphic forms, much like the classical description of the Hodge filtration on $H^1(X)$ in terms of such forms. Unfortunately, if $g > 1$ we do not know if a good set of differentials exists.

Recall that h denotes the composition map (16) and Δ_e is the modified diagonal cycle of Gross, Kudla, and Schoen in X^3 (defined in (17)).

PROPOSITION 2. Suppose $\{\alpha_i\}_{i\le 2g}$ is a good set of differentials for the triple (X, ∞, e). Let $\{\beta_k\}_{k\le 2g}$ be a set of generators of $\pi_1(X - \{\infty\}, e)$ whose homology classes form a basis of $H_1(X, \mathbb{Z})$.

Write $[\alpha_l] = \sum_k p_{lk} PD([\beta_k])$. Let $\xi = \sum c_{ij}[\alpha_i] \otimes [\alpha_j]$ be a Hodge class. Then $\xi^{-1}(h(\Delta_e))$ (resp. $\xi^{-1}(\mathbb{E}_e^\infty)$) is a torsion point if and only if

$$\left(\sum_k p_{lk} \int_{\beta_k} \sum_{i,j} c_{ij} \alpha_i \alpha_j \right)_{l \le g} \equiv \int_{\Delta(X)} \xi \left(\int_e^\infty \alpha_l \right)_{l \le g}$$

(resp. $\equiv 0$) mod the $\mathbb{Q}$-span of $\{(\int_{\beta_k} \alpha_l)_{l \le g} : k \le 2g\}$ in $\mathbb{C}^g$.

PROOF. The map $\alpha_i \otimes \alpha_j \mapsto \int \alpha_i \alpha_j \pmod{L_0}$ is a section of the quotient map $\frac{L_2}{L_0}(X - \infty, e) \to H^1 \otimes H^1$ that is compatible with the Hodge filtrations. Thus the extension $\mathbb{E}_e^\infty \in J\underline{\mathrm{Hom}}((H^1)^{\otimes 3}, \mathbb{Z}(0))$ is represented by

$$[\alpha_i] \otimes [\alpha_j] \otimes PD([\beta_k]) \mapsto \int_{\beta_k} \alpha_i \alpha_j,$$

and the point $\xi^{-1}(\mathbb{E}_e^\infty) \in \frac{(F^1 H^1)^\vee}{H_1(X, \mathbb{Z})}$ is represented by

$$[\alpha_l] \mapsto \sum_k p_{lk} \int_{\beta_k} \sum_{i,j} c_{ij} \alpha_i \alpha_j \qquad (l \le g).$$

Take a path γ_e^∞ in X from e to ∞. Then Z_e^∞ (defined in (18)) is the boundary of $\Delta(X) \times \gamma_e^\infty$. Thus $AJ(Z_e^\infty) \in J\underline{\mathrm{Hom}}((H^3(X^3), \mathbb{Z}(0))$ is represented by any map $\phi : H^3(X^3) \to \mathbb{C}$ whose restriction to $F^2H^3(X^3)$ is $\int_{\Delta(X) \times \gamma_e^\infty}$ (see Paragraph 5.2). The element $h(Z_e^\infty) \in J\underline{\mathrm{Hom}}((H^1)^{\otimes 3}, \mathbb{Z}(0))$ is represented by the restriction of ϕ to the subspace $(H^1)^{\otimes 3} \subset H^3(X^3)$ (the inclusion via Kunneth). It follows that $\xi^{-1}(h(Z_e^\infty)) \in \frac{(F^1 H^1)^\vee}{H_1(X, \mathbb{Z})}$ is represented by the map

$$[\alpha_l] \mapsto \phi(\xi \otimes [\alpha_l]) \overset{(*)}{=} \int_{\Delta(X) \times \gamma_e^\infty} \xi \otimes [\alpha_l] = \int_{\Delta(X)} \xi \int_{\gamma_e^\infty} \alpha_l \qquad (l \le g)$$

(for $(*)$ note that $\xi \otimes [\alpha_l] \in F^2H^3(X^3)$).

The assertions now follow from Theorem 4 in view of the descriptions of $\xi^{-1}(\mathbb{E}_e^\infty)$ and $\xi^{-1}(h(Z_e^\infty))$ given above. □

Note that in the following situations $\xi^{-1}(h(\Delta_e))$ is guaranteed to be torsion: (i) if X is hyperelliptic and e is a ramification point (as then $6\Delta_e = 0$ in the Chow group [**6**, Proposition 4.8]), and (ii) if X, e and ξ are defined over a subfield $F \subset \mathbb{C}$ and Jac(F) is finite (see Theorem 4(b)).

We end the article with a few remarks.

REMARKS. (1) In the case of an elliptic curve (with or without complex multiplication), taking ξ to be the Hodge class of a nonzero endomorphism gives (1) again. (In particular, the procedure does not give a new relation in the CM case.)

(2) In [**5**] we generalize Theorem 4 as follows. Let $\mathbb{E}_{n,e}^\infty$ be the extension

$$0 \longrightarrow \underset{\substack{\wr|| \\ (H^1)^{\otimes n-1}}}{\frac{L_{n-1}}{L_{n-2}}(X - \{\infty\}, e)} \xrightarrow{\text{inclusion}} \frac{L_n}{L_{n-2}}(X - \{\infty\}, e) \longrightarrow \underset{\substack{\wr|| \\ (H^1)^{\otimes n}}}{\frac{L_n}{L_{n-1}}(X - \{\infty\}, e)} \longrightarrow 0$$

of mixed Hodge structures. We show that $\mathbb{E}^{\infty}_{n,e}$, considered as an element of

$$\mathrm{Ext}((H^1)^{\otimes n},(H^1)^{\otimes n-1}) \cong J\underline{\mathrm{Hom}}((H^1)^{\otimes n},(H^1)^{\otimes n-1}) \overset{\text{Poincaré duality}}{\cong} J\underline{\mathrm{Hom}}((H^1)^{\otimes n}\otimes(H^1)^{\otimes n-1},\mathbb{Z}(0)),$$

is the Abel-Jacobi image of a homologically trivial algebraic cycle (see [**5**, Theorem 3.5.1]). This algebraic cycle is defined over $F \subset \mathbb{C}$ if X, e and ∞ are defined over F. As a result, if X, e, ∞ are defined over F, pullback by Hodge classes in $(H^1)^{\otimes 2n-2}$ that are defined over F gives F-rational points in the Jacobian of X (see [**5**, Theorem 4.1.1]). Similar to above, if there is a good set of differentials, the point $\xi^{-1}(\mathbb{E}^{\infty}_{n,e})$ is torsion if and only if certain relations between classical and quadratic periods hold. We also explicitly find the relations that may arise from this procedure for the diagonal of X^2 ($\xi \in (H^1)^{\otimes 4}$, $n = 3$), and show that when $g = 2$ already one would get new relations that are not seen by the diagonal of X and the extension $E^{\infty}_{2,e}$ considered in this article (see [**5**, Proposition 5.3.3]).

(3) Hain has also previously asked for a description of the Hodge filtration on the space of iterated integrals on a punctured curve in terms of meromorphic differentials (see Subsection 13.1 of [**11**]).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE ST., ROOM 6290, TORONTO, ONTARIO, CANADA, M5S 2E4
E-mail address: payman@math.utoronto.ca