Ceresa cycles of Fermat curves

(Based on joint work with Kumar Murty)

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Coarseness →

Three equivalence relations on alg. cycles: rational, algebraic, homological equivalence

rat. equivalent $\Longrightarrow$ alg. equivalent $\Longrightarrow$ hom. equivalent

$\not\Longleftarrow$ (trivial) $\not\Longleftarrow$ (Thm of Griffiths)

Thm (Griffiths '69): If X is a general quintic in $\mathbb{P}^4$, then

$$\frac{\mathcal{Z}_1(X)^{hom}}{\mathcal{Z}_1(X)^{alg}} \otimes \mathbb{Q} \neq 0.$$

Thm (Ceresa '83): If X is a general curve of genus >2, the Ceresa cycle of X is algebraically nontrivial.

X = (smooth projective) curve / $\mathbb{C}$, genus g>0

$Jac = Jac(X)$ = Jacobian of X = $\dfrac{Div^0(X)}{\text{princ. div's}}$

$e \in X$ (Base pt) $\rightsquigarrow$ $X \hookrightarrow Jac$, $x \mapsto x - e$

Image =: $X_e \in \mathcal{Z}_1(Jac)$ (1-dim'l alg. cycle on Jac)

Ceresa cycle of X with base pt e = $Cer_e(X) := X_e - (-1)_* X_e \in \mathcal{Z}_1^{hom}(Jac)$

$Cer_e(X)$ mod different equiv. relations:

Mod hom. ~	Trivial
Mod alg. ~	Independent of e
Mod rat. ~	Depends on e

Ceresa's thm ('83): If X = a general curve of genus >2, then $Cer(X)$ is alg. nontrivial.

Still at this point, no explicit example given!

Ceresa cycles of Fermat curves

Notation: F_n = Fermat curve of degree n, given by $x^n + y^n = z^n$

Thm* (B. Harris, '83): $Cer(F_4)$ is alg. nontrivial.

First explicit example of an alg. nontrivial hom. trivial cycle

Thm* (Bloch, '84): $Cer(F_4)$ is alg. of infinite order.

Further adaptations by Tadokoro, Otsubo, Kimura,... to other Fermat curves and quotients. In particular:

Thm* (Kimura, 2000): $Cer(F_7)$ is alg. of infinite order.

Adaptation of Bloch's

Thm* (Otsubo, 2012): $Cer(F_n)$ is alg. nonzero for all $4 < n \leq 1000$.

Thm* (Tadokoro, 2016): Ceresa cycle is alg. nonzero for some quotients of F_p for prime p<1000 and $\equiv 1 \mod 3$.

Adaptation of Harris'.

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* Uses Hodge theoretic (Griffiths) Abel-Jacobi map.

* Uses l-adic Abel-Jacobi map.

Both methods have their limitations. Bloch's gives stronger results, but much harder to implement.

Harris' argument in a nutshell

- $\forall$ variety Y: $\mathcal{Z}_d(Y)/\sim_{rat} =: CH_d(Y) \supset CH_d^{hom}(Y)$ (Hom. trivial subgroup)
- Griffiths Abel-Jacobi map

$$AJ : CH_d^{hom}(Y) \to JH_{2d+1}(Y) := H_{2d+1}(Y, \mathbb{C})\big/(F^{-d}H_{2d+1} + H_{2d+1}(Y, \mathbb{Z}))$$
$$\Gamma \mapsto \int_{\partial^{-1}\Gamma}$$

Griffiths intermediate Jacobian (a compact complex torus)

Γ alg. trivial $\Longrightarrow AJ(\Gamma)$ "vanishes" on $F^{d+2}H_{dR}^{2d+1}(Y)$

- With $Y = Jac(X), d = 1, Z = Cer(X)$ (X = a curve), get:

$Cer(X)$ alg. trivial $\Longrightarrow AJ(Cer(X))$ "vanishes" on $H^{3,0}(Jac(X)) = \bigwedge^3 H^{1,0}(X)$

- In his "long paper" in '83, Harris calculates $AJ(Cer(X))$ in terms of iterated integrals.

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1 dim'l for $X = F_4$

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In his "short paper" (also '83): Specializes to F_4, deduces → Long paper + above + Rohrlich on $H^{1,0}(F_n)$

$Cer(F_4)$ alg. trivial $\Longrightarrow$ Certain period integral is an integer.

— " — torsion $\Longrightarrow$ — " — rational.

$AJ(Cer(F_4))$ alg. trivial $\Longrightarrow$ Certain period integral is an integer.

—— " —— torsion $\Longrightarrow$ —— " —— rational.

- Can check nontriviality numerically.
- Adapted to other Fermat curves and some quotients by Tadokoro and Otsubo. There is a numerically verifiable sufficient condition for nontriviality mod alg. equiv. for a given F_n.

Limitations (due to it algorithmic nature and reliance on numerical approximations):

1) Can only check nontriviality (modulo algebraic equivalence)

2) Can only be used for finitely* many Fermat curves/quotients.

Modulo rational equivalence

Thm (E. - K. Murty, '21). For every prime p>7 and every choice of base point, $|AJ(Cer_e(F_p))| = \infty$. In particular, $Cer_e(F_p)$ is of infinite order modulo rational equivalence.

Proof was a combination of:

- Harris' and Pulte's works on Ceresa cycles and Hodge theory of π_1
- Works of Kaenders and Darmon-Rotger-Sols on alg. cycles and Hodge theory of π_1
- Rohrlich's analogue of Manin-Drinfeld
- Gross-Rohrlich's work on nontorsion points on Jacobians of the F_n

We'll sketch a simplified version of the proof that doesn't directly refer to Hodge theory of π_1.

Proof (simplified version)

Step 1) Reduction to the case where e is a cusp (i.e. satisfies xyz=0).

$$AJ(Cer_e(X)) = \underbrace{AJ(Cer_e(X))\Big|_{H^3_{prim}(Jac)}} \oplus \underbrace{AJ(Cer_e(X))\Big|_{H^1(X)\wedge cl(\Delta(X))}}$$

Harris: Independent of e.

Pulte ('88): linear comb. of e and the can. divisor (as a pt on Jac)

When $X = F_p$: By Rohrlich, if e is a cusp, this is torsion.

Step 2) Work with the modified diagonal cycle in X^3 instead

$$\Delta_{GKS,e}(X) := \{x,x,x\} - \{e,x,x\} - \{x,e,x\} - \{x,x,e\} + \{x,e,e\} + \{e,x,e\} + \{e,e,x\} \in CH_1^{hom}(X^3)$$

(Modified diagonal cycle of Gross, Kudla and Schoen)

Colombo and van Geeman ('93): $AJ(Cer_e(X)) \sim_{\mathbb{Q}^\times} AJ(\Delta_{GKS,e}(X))$

So we can instead show $|AJ(\Delta_{GKS,e}(X))| = \infty$.

Step 3) Relate to points on the Jacobian (Idea of Darmon-Rotger-Sols)

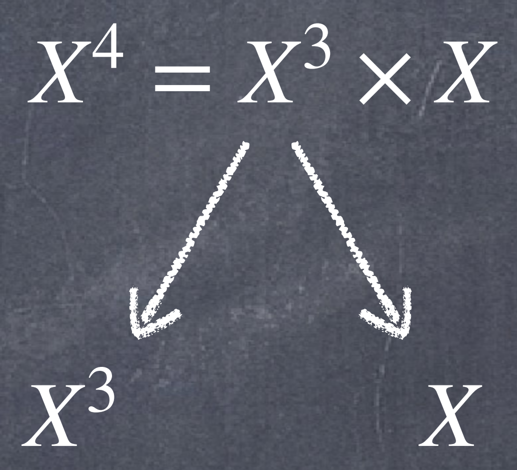
Let $\Gamma \in CH_1(X^2)$. The correspondence

$$\Gamma \times \Delta(X) \in CH_2(X^2 \times X^2) = CH_2(X^3 \times X)$$

gives a map

$$CH_1^{hom}(X^3) \to CH_0^{hom}(X) \overset{AJ}{=} Jac(X).$$

$$\Omega \mapsto (pr_4)_*(pr_{123}^*(\Omega) \cdot (\Gamma \times \Delta(X)))$$



Let $P_\Gamma :=$ the image of $\Delta_{GKS,e}(X)$.

If $|P_\Gamma| = \infty$ for some Γ, then $|AJ(\Delta_{GKS,e}(X))| = \infty$. (By functoriality of AJ.)

Step 4) Pick a suitable $\Gamma \in CH_1(F_p \times F_p)$.

Take Γ to be the graph of

$$\beta : F_p \to F_p$$
$$(x, y, z) \mapsto (-y, z, x)$$

Set $Q = (\zeta_6, \overline{\zeta_6}, 1) \in F_p$. Then

$P_\Gamma = (Q + \overline{Q} - 2e)$ + a pt supp. on the cusps

Torsion by Rohrlich

Gross-Rohrlich ('78): If p>7, then $|Q + \overline{Q} - 2e| = \infty$ (in $Jac(F_p)$).

□

Some questions

1) Fermat quotients

- Fix $p > 7$. For $1 \leq s \leq p-2$, consider

$$C_s : y^p = x(1-x)^s.$$

- The maps

$$F_p \to C_s \qquad (x, y, 1) \mapsto (x^p, xy^s, 1)$$

$$Jac(J_p) \sim \prod_{1 \leq s \leq p-2} Jac(C_s)$$

(Non-hyperelliptic cases)

- Gross-Rohrlich: If $s \notin \{1, (p-1)/2, p-2\}$, the image of $Q + \overline{Q} - 2e$ in $Jac(C_s)$ is of infinite order.

- Question: Can the argument be adapted to these non-hyperelliptic quotients?

- Partial answer: (Nemoto, 2024) Suppose $s^2+s+1 \equiv 0 \mod p$. Then $|AJ(Cer_e(C_s))| = \infty$ for every e.

"Pf": β descends to C_s. (In fact, this happens iff $s^2+s+1 \equiv 0 \mod p$.)

What about other quotients?

2) Algebraic equivalence and connection to Beilinson-Bloch conj.

Supp. p, s as above with $|AJ(Cer_e(X))| = \infty$.

$$AJ(Cer_e(C_s)) \in Ext^1(\mathbb{Q}(-1), \underbrace{\bigwedge^3 H^1(C_s)})$$

Decomposes into many components.

How does $Cer_e(C_s)$ (or its Abel-Jacobi image) decompose?

Beilinson-Bloch ⟿ $L(\bigwedge^3 H^1(X), s)$ should have a zero at 2.

Which factors have zeros?