Some properties of a Rudin–Shapiro-like sequence

Narad Rampersad

(Joint work with Philip Lafrance and Randy Yee)

Department of Mathematics and Statistics
University of Winnipeg

Digital sequences

- there are several well-studied sequences whose n-th term is defined based on some property of the digits of n when written in some chosen base
- e.g., the sum-of-digits function $s_k(n) =$ the sum of the digits of the base-k representation of n
- e.g., $s_2(n)$ counts the number of 1's in the binary representation of n
- $(t_n)_{n\geq 0} = \left((-1)^{s_2(n)} \right)_{n\geq 0}$ is the Thue–Morse sequence

$$+1 \quad -1 \quad -1 \quad +1 \quad -1 \quad +1 \quad +1 \quad -1 \quad \cdots$$

The Rudin–Shapiro sequence

- let $e_{2;11}(n)$ denote the number of occurrences of 11 in the binary representation of n
- e.g., $e_{2;11}(235) = 3$ since $235 = [11101011]_2$
- $(r_n)_{n\geq 0} = \left((-1)^{e_{2;11}(n)} \right)_{n\geq 0}$ is the Rudin–Shapiro sequence

$$+1 \quad +1 \quad +1 \quad -1 \quad +1 \quad +1 \quad -1 \quad +1 \quad \cdots$$

Scattered subsequences

- instead of counting occurrences of a given block, we can count occurrences of a given pattern as a scattered subsequence in the digital representation of n
- e.g., $302 = [100101110]_2$ has 13 occurrences of 010 as scattered subsequences of its binary representation
- an occurrence of 10 as a scattered subsequence is called an inversion
- in general, an inversion in a word is an occurrence of ba as a scattered subsequence, where $b > a$

Counting inversions

- write $\mathrm{inv}_2(n)$ to denote the number of inversions in the binary representation of n
- define the sequence $(i_n)_{n\geq 0}$ by $i_n = (-1)^{\mathrm{inv}_2(n)}$, so

$$(i_n)_{n\geq 0} = \quad +1 \quad +1 \quad -1 \quad +1 \quad +1 \quad -1 \quad +1 \quad +1 \quad \cdots$$

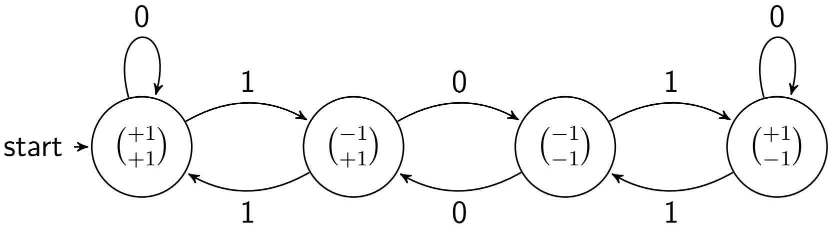
- this sequence has many similarities with the Rudin–Shapiro sequence

Generalized Rudin–Shapiro sequences

- Allouche and Liardet (1991) studied generalizations of the Rudin–Shapiro sequence
- fix $ab \neq 00$ and fix some positive integer d
- define $(u_n)_{n \geq 0}$ such that u_n equals the number of occurrences of ab as a scattered subsequence of the binary representation of n, where a and b occur at distance $d+1$ from each other
- taking $ab = 10$ counts the number of inversions where the inverted elements are separated by distance $d+1$

Representation as an automatic sequence

$(i_n)_{n\geq 0}$ is a 2-automatic sequence



Operation of the automaton

The automaton calculates i_n as follows: the binary digits of n are processed from most significant to least significant, and when the last digit is read, the automaton halts in the state

$$\begin{pmatrix} (-1)^{s_2(n)} \\ (-1)^{\mathrm{inv}_2(n)} \end{pmatrix}.$$

i_n is given by the lower component of the label of the state reached after reading the binary representation of n (the first component has the value t_n).

Generation by morphisms

$(i_n)_{n\geq 0}$ can be generated by iterating the morphism

$$A \to AB, \quad B \to CA, \quad C \to BD, \quad D \to DC,$$

to obtain the infinite sequence

$$ABCABDABCADCABCA\cdots$$

and then applying the recoding

$$A, B \to +1, \quad C, D \to -1.$$

Rudin–Shapiro morphism

cf. the Rudin–Shapiro sequence, which is obtained by iterating

$$A \to AB, \quad B \to AC, \quad C \to DB, \quad D \to DC,$$

and then applying the same recoding as above.

Recurrence relations

$$i_{2n} = i_n t_n \quad (1)$$

$$i_{2n+1} = i_n \quad (2)$$

- let w denote the binary representation of n
- the number of 10's in $w0$ equals the number of 10's in w plus the number of 1's in w

$$i_{2n} = (-1)^{\mathrm{inv}_2(n)+s_2(n)} = i_n t_n.$$

- $i_{2n+1} = i_n$ is clear, since appending a 1 to w does not change the number of 10's

Recurrence relations

$$\begin{aligned} i_{4n} &= i_n \\ i_{4n+1} &= i_{2n} \\ i_{4n+2} &= -i_{2n} \\ i_{4n+3} &= i_n. \end{aligned}$$

Proving the relations

The Thue–Morse sequence satisfies

$$t_{2n} = t_n \quad \text{and} \quad t_{2n+1} = -t_n.$$

Now we have

$$i_{4n} = i_{2n}t_{2n} = i_{2n}t_n = i_nt_nt_n = i_n,$$

where we have applied (1) twice. Similarly, we get

$$i_{4n+1} = i_{2(2n)+1} = i_{2n}$$

by applying (2).

Proving the relations

Next, we calculate

$$i_{4n+2} = i_{2(2n+1)} = i_{2n+1}t_{2n+1} = i_n(-t_n) = -i_{2n},$$

and finally,

$$i_{4n+3} = i_{2(2n+1)+1} = i_{2n+1} = i_n.$$

The summatory function of $(t_{3n})_{n\geq 0}$

Newman (1969) and Coquet (1983) studied the summatory function of the Thue–Morse sequence taken at multiples of 3. In particular,

$$\sum_{0\leq n<N} t_{3n} = N^{\log_4 3} G_0(\log_4 N) + \frac{1}{3}\eta(N),$$

where G_0 is a bounded, continuous, nowhere differentiable, periodic function with period 1, and

$$\eta(N) = \begin{cases} 0 & \text{if } N \text{ is even,} \\ (-1)^{s_2(3N-1)} & \text{if } N \text{ is odd.} \end{cases}$$

Summing Rudin–Shapiro

Brillhart, Erdős, and Morton (1983) and Dumont and Thomas (1989) studied the summatory function of the Rudin–Shapiro sequence. In this case,

$$\sum_{0 \le n < N} r_n = \sqrt{N} G_1(\log_4 N)$$

where again G_1 is a bounded, continuous, nowhere differentiable, periodic function with period 1.

Summing the inversions sequence

Define the summatory function $S(N)$ of $(i_n)_{n\geq 0}$ as

$$S(N) = \sum_{0\leq n\leq N} i_n.$$

The first few values of $S(N)$ are:

N	0	1	2	3	4	5	6	7
$S(N)$	1	2	1	2	3	2	3	4

A plot of the function $S(N)$

The smooth curves are plots of $\sqrt{2}\sqrt{N}$ and $(\sqrt{3}/3)\sqrt{N}$.

The growth of $S(N)$

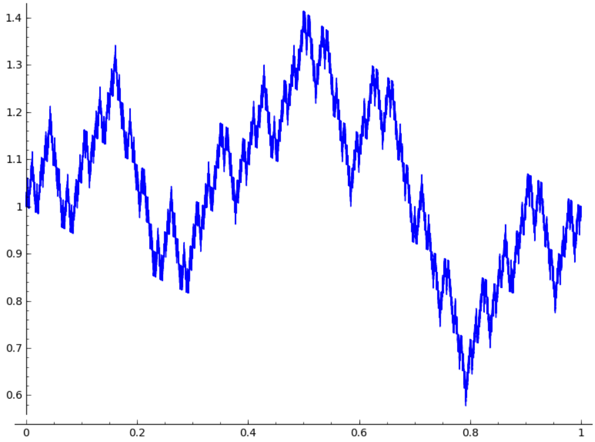
Theorem (Lafrance, R., Yee 2014)

There exists a bounded, continuous, nowhere differentiable, periodic function G with period 1 such that

$$S(N) = \sqrt{N} G(\log_4 N).$$

This can be obtained by a criterion from the book of Allouche and Shallit derived from techniques of Tenenbaum (1997).

A plot of the periodic function G



Upper and lower limits of oscillation

Theorem (Lafrance, R., Yee 2014)

We have

$$\liminf_{n\to\infty} \frac{S(n)}{\sqrt{n}} = \frac{\sqrt{3}}{3} \quad \text{and} \quad \limsup_{n\to\infty} \frac{S(n)}{\sqrt{n}} = \sqrt{2}.$$

Combinatorial properties of $(i_n)_{n\geq 0}$

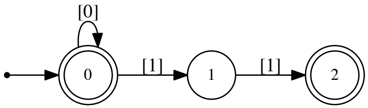
Theorem (Lafrance, R., Yee 2014)

The sequence $(i_n)_{n\geq 0}$ contains

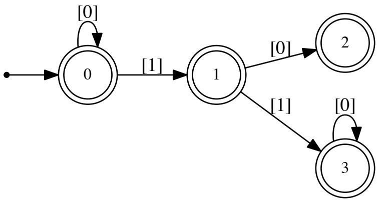
1. no 5-th powers,
2. cubes x^3 exactly when $|x| = 3$,
3. squares xx exactly when $|x| \in \{1, 2\} \cup \{3 \cdot 2^k : k \geq 0\}$.
4. arbitrarily long palindromes.

These results were verified by Jeffrey Shallit and Hamoon Mousavi using their automated prover for properties of automatic sequences.

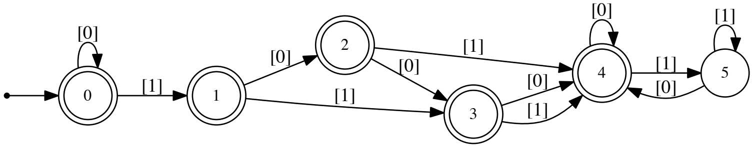
Automaton for period lengths of cubes in $(i_n)_{n\geq 0}$



Automaton for period lengths of squares in $(i_n)_{n\geq 0}$



Automaton for lengths of palindromes in $(i_n)_{n\geq 0}$



The "square root" property

The Rudin–Shapiro sequence satisfies the following:

There exists a constant C such that for all $N \geq 0$

$$\sup_{\theta \in \mathbb{R}} \left| \sum_{0 \leq n < N} r_n e^{2\pi i n \theta} \right| \leq C\sqrt{N}.$$

It would seem that (Allouche, personal communication) the sequence $(i_n)_{n \geq 0}$ does not have this property.

The End