Avoiding repetitions in words II

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Unavoidable regularity

van der Waerden's Theorem

If the natural numbers are partitioned into finitely many sets, then one set contains arbitrarily large arithmetic progressions.

Subsequences

- $\mathbf{w} = w_0 w_1 w_2 \cdots$
- subsequence: a word $w_{i_0} w_{i_1} \cdots$, where $0 \leq i_0 < i_1 < \cdots$.
- arithmetic subsequence of difference j: a word $w_i w_{i+j} w_{i+2j} \cdots$, where $i \geq 0$ and $j \geq 1$.

Unavoidable repetitions

vdW rephrased

For any infinite word $\mathbf{w}$ over a finite alphabet A, there exists $a \in A$ such that for all $m \geq 1$, $\mathbf{w}$ contains a^m in a subsequence indexed by an arithmetic progression.

Repetitions in arithmetic progressions

Theorem (Carpi 1988)

Let p be a prime and let m be a non-negative integer. There exists an infinite word over a finite alphabet that avoids $(1 + 1/p^m)$-powers in arithmetic progressions of all differences, except those differences that are a multiple of p.

Squares in arithmetic progressions

Corollary

There exists an infinite word over a 4-letter alphabet that contains no squares in any arithmetic progression of odd difference.

The construction

Let $q = p^{m+1}$. We construct an infinite word with the desired properties over the alphabet

$$\Sigma = \{n : 0 < n < 2q^2 \text{ and } q \nmid n\}.$$

Define $\mathbf{w} = a_1 a_2 \cdots$ as follows. For $n \geq 1$, write $n = q^t n'$, where $q \nmid n'$, and define

$$a_n = \begin{cases} n' \bmod q^2, & \text{if } t = 0; \\ q^2 + (n' \bmod q^2), & \text{if } t > 0. \end{cases}$$

An example of the construction

Take $p = 2$ and $m = 0$ (so that $q = 2$). Then $\Sigma = \{1, 3, 5, 7\}$ and, writing $n = 2^t n'$,

$$a_n = \begin{cases} n' \bmod 4, & \text{if } n \text{ is odd;} \\ 4 + (n' \bmod 4), & \text{if } n \text{ is even.} \end{cases}$$

It follows that

$$\mathbf{w} = 153517351537173515 3\cdots$$

contains no squares in arithmetic progressions of odd difference.

Proof of the construction

Recall: $\mathbf{w} = a_1 a_2 \cdots$ and $q = p^{m+1}$. For $n \geq 1$, write $n = q^t n'$, where $q \nmid n'$. Then

$$a_n = \begin{cases} n' \bmod q^2, & \text{if } t = 0; \\ q^2 + (n' \bmod q^2), & \text{if } t > 0. \end{cases}$$

Suppose $\mathbf{w}$ contains a $(1 + 1/p^m)$-power in an arithmetic progression of difference k, where k is not a multiple of p:

$$a_i a_{i+k} \cdots a_{i+(s-1)k} = a_{i+rk} a_{i+(r+1)k} \cdots a_{i+(r+s-1)k}$$

for some integers i, r, s satisfying $s/r \geq 1/p^m$.

Proof of the construction

If $a_i = a_{i+rk} > q^2$ then q divides both i and $i + rk$ and hence divides rk.

If $a_i = a_{i+rk} < q^2$, then $i \bmod q^2 = (i + rk) \bmod q^2$, so that q^2 divides rk.

In either case, since p does not divide k, it must be the case that q divides r.

So we write $r = q^\ell r'$ for some positive integers ℓ, r' with r' not divisible by q.

Proof of the construction

Recall that $s/r \geq 1/p^m$ and $q = p^{m+1}$, so that

$$\begin{aligned} s &\geq q^\ell r'/p^m \\ &= pq^{\ell-1}r' \\ &\geq pq^{\ell-1}. \end{aligned}$$

Thus $\{i, i+k, \ldots, i+(s-1)k\}$ forms a complete set of residue classes modulo $pq^{\ell-1}$.

There exists $j \in \{i, i+k, \ldots, i+(s-1)k\}$ such that

$$j \equiv q^{\ell-1} \pmod{pq^{\ell-1}}.$$

Proof of the construction

Write

$$\begin{aligned} j &= apq^{\ell-1} + q^{\ell-1} \\ &= q^{\ell-1}(ap+1), \end{aligned}$$

for some non-negative integer a.

We also have

$$\begin{aligned} j + rk &= q^{\ell-1}(ap+1) + q^{\ell} r' k \\ &= q^{\ell-1}(ap+1+qr'k). \end{aligned}$$

Proof of the construction

Also $a_j = a_{j+rk}$, and so from the definition of $\mathbf{w}$ we have

$$ap + 1 \equiv ap + 1 + qr'k \pmod{q^2},$$

so that $qr'k \equiv 0 \pmod{q^2}$.

This implies $r'k \equiv 0 \pmod{q}$. However, p does not divide k, and q does not divide r', so this congruence cannot be satisfied. This contradiction completes the proof.

The paperfolding word

- again take $p = 2$ and $m = 0$
- then $\mathbf{w} = 15351735153717351 53\cdots$
- apply the map $1, 5 \to 0$, $3, 7 \to 1$ to get
- $\mathbf{f} = 0010011000110110001\cdots$
- this is the ordinary paperfolding word

The Toeplitz construction

- Start with an infinite sequence of gaps, denoted ?.

$$? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad ? \quad \cdots$$

- Fill every other gap with alternating 0's and 1's.

$$0 \quad ? \quad 1 \quad ? \quad 0 \quad ? \quad 1 \quad ? \quad 0 \quad ? \quad 1 \quad ? \quad 0 \quad ? \quad 1 \quad \cdots$$

- Repeat.

$$0 \quad 0 \quad 1 \quad ? \quad 0 \quad 1 \quad 1 \quad ? \quad 0 \quad 0 \quad 1 \quad ? \quad 0 \quad 1 \quad 1 \quad \cdots$$

$$0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad ? \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad \cdots$$

Paperfolding words

- In the limit one obtains the ordinary paperfolding word:

$$0010011000110110\cdots$$

- At each step, one may choose to fill in the gaps by either

$$0101010101\cdots$$

or

$$1010101010\cdots.$$

- Different choices result in different paperfolding words.

Repetitions in paperfolding words

Theorem (Allouche and Bousquet-Mélou 1994)

If $\mathbf{f}$ is a paperfolding word and ww is a non-empty factor of $\mathbf{f}$, then $|w| \in \{1, 3, 5\}$.

2-dimensional words

- A 2-dimensional word $\mathbf{w}$ is a 2D array of symbols.
- $w_{m,n}$: the symbol of $\mathbf{w}$ at position (m, n).
- A word $\mathbf{x}$ is a line of $\mathbf{w}$ if there exists i_1, i_2, j_1, j_2, such that
  - $\gcd(j_1, j_2) = 1$ and
  - for $t \geq 0$, we have $x_t = w_{i_1+j_1 t, i_2+j_2 t}$.

Avoiding repetitions in higher dimensions

Theorem (Carpi 1988)

There exists a 2-dimensional word $\mathbf{w}$ over a 16-letter alphabet such that every line of $\mathbf{w}$ is squarefree.

Constructing the 2D-word

- Let $\mathbf{u} = u_0u_1u_2\cdots$ and $\mathbf{v} = v_0v_1v_2\cdots$ be infinite words over a 4-letter alphabet A that avoid squares in all arithmetic progressions of odd difference.
- Define $\mathbf{w}$ over the alphabet $A \times A$ by $w_{m,n} = (u_m, v_n)$.

Lines through the 2D-word

- Consider an arbitrary line

$$\begin{aligned} \mathbf{x} &= (w_{i_1+j_1t,i_2+j_2t})_{t\geq 0}, \\ &= (\,(u_{i_1+j_1t}, v_{i_2+j_2t})\,)_{t\geq 0}, \end{aligned}$$

  for some i_1, i_2, j_1, j_2, with $\gcd(j_1, j_2) = 1$.
- Without loss of generality, we may assume j_1 is odd.
- Then $(u_{i_1+j_1t})_{t\geq 0}$ is an arithmetic subsequence of odd difference of $\mathbf{u}$ and hence is squarefree.
- $\mathbf{x}$ is therefore also squarefree.

Abelian repetitions

Erdős 1961	abelian square: a word xx' such that x' is a permutation of x (like `reappear`)
Evdokimov 1968	abelian squares avoidable over 25 letters
Pleasants 1970	abelian squares avoidable over 5 letters
Justin 1972	abelian 5-powers avoidable over 2 letters
Dekking 1979	abelian 4-powers avoidable over 2 letters
	abelian cubes avoidable over 3 letters
Keränen 1992	abelian squares avoidable over 4 letters

The adjacency matrix of a morphism

Given a morphism $\varphi : \Sigma^* \to \Sigma^*$ for some finite set $\Sigma = \{a_1, a_2, \ldots, a_d\}$, we define the adjacency matrix $M = M_\varphi$ as follows:

$$M = (m_{i,j})_{1 \le i,j \le d}$$

where $m_{i,j}$ is the number of occurrences of a_i in $\varphi(a_j)$, i.e., $m_{i,j} = |\varphi(a_j)|_{a_i}$.

An example

$$\varphi : a \to ab$$
$$b \to cc$$
$$c \to bb.$$

$$M_\varphi = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \end{array}$$

Properties of M

- Define $\psi : \Sigma^* \to \mathbb{Z}^d$ by

$$\psi(w) = [|w|_{a_1}, |w|_{a_2}, \ldots, |w|_{a_d}]^T.$$

- Then

$$\psi(\varphi(w)) = M_\varphi \psi(w).$$

- By induction $M_\varphi^n = M_{\varphi^n}$, and hence

$$\psi(\varphi^n(w)) = M_\varphi^n \psi(w).$$

Dekking's construction

- Define a map:

$$a \to aaab, \quad b \to abb.$$

- The limit of the sequence

$$a \to aaab \to aaabaaabaaababb \to \cdots$$

contains no abelian 4-power.

Dekking's method

- the idea is to map letters to elements of $\mathbb{Z}/n\mathbb{Z}$ for some n
- abelian repetitions correspond to certain arithmetic progressions in $\mathbb{Z}/n\mathbb{Z}$
- show no such arithmetic progressions exist

Some definitions

- Let $\varphi : \Sigma^* \to \Sigma^*$ be a morphism.
- Call the words $\varphi(a)$, for $a \in \Sigma$, blocks.
- If $\varphi(a) = vv'$, $v' \neq \epsilon$, then v is a left subblock and v' a right subblock.
- Let G be a finite abelian group (written additively).
- $A \subseteq G$ is progression-free of order n if for all $a \in A$, $a, a+g, a+2g, \ldots, a+(n-1)g \in A$ implies $g = 0$.

φ-injectivity

- Let $f : \Sigma^* \to G$ be a morphism: i.e., $f(\epsilon) = 0$ and $f(a_1 a_2 \cdots a_i) = \sum_{1 \le j \le i} f(a_j)$.
- Call f a weight function.
- Let $v_1 v'_1, v_2 v'_2, \ldots, v_n v'_n$ be blocks.
- f is φ-injective if

$$f(v_1) = f(v_2) = \cdots = f(v_n)$$

implies either $v_1 = v_2 = \cdots = v_n$ or $v'_1 = v'_2 = \cdots = v'_n$.

The main criterion

Suppose that

(a) The adjacency matrix of φ is invertible.

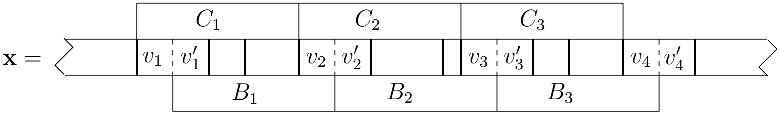
(b) $f(\varphi(a)) = 0$ for all $a \in \Sigma$;

(c) the set $A = \{f(v) : \ v$ a left subblock of $\varphi\}$ is progression-free of order $n+1$;

(d) f is φ-injective.

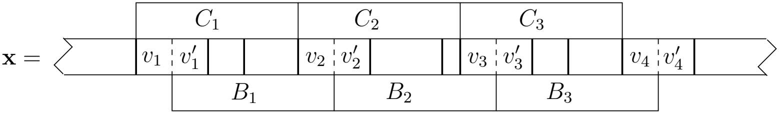
If φ is prolongable on a, and $\varphi^\omega(a)$ avoids "short" abelian n-powers, then $\varphi^\omega(a)$ is abelian n-power-free.

Proof

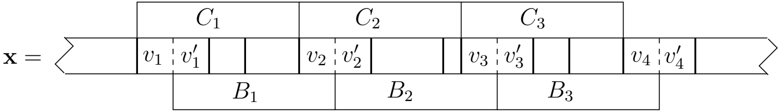


Let $\mathbf{x} = \varphi^\omega(a)$. Suppose $B_1 B_2 \cdots B_n$ is an abelian n-power in $\mathbf{x}$ with $|B_i|$ minimal.

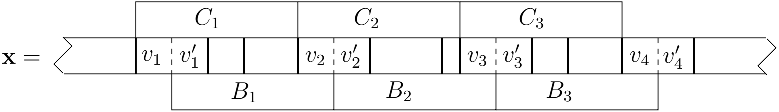
Suppose the B_i are not "short": i.e., $|B_i| > \max_{a\in\Sigma} |\varphi(a)|$.



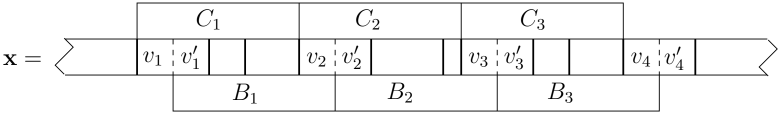
- Since the B_i's contain the same numbers of each letter, we have $f(B_1) = f(B_2) = \cdots = f(B_n)$.
- By hypothesis $f(\varphi(a)) = 0$ for every $a \in \Sigma$.
- Hence $f(B_i) = f(v_i') + f(v_{i+1})$.



- Since $f(v_i v_i') = 0$, we get $f(B_i) = -f(v_i) + f(v_{i+1})$.
- Thus the $f(v_i)$ form an $(n+1)$-term arithmetic progression with difference $f(B_i)$.
- This forces $f(v_1) = f(v_2) = \cdots = f(v_{n+1})$.
- φ-injectivity forces either $v_1 = v_2 = \cdots = v_{n+1}$ or $v_1' = v_2' = \cdots = v_{n+1}'$



In the first case, we "slide" the abelian n-power to the left by $|v_1|$ symbols to get another n-power $C_1 C_2 \cdots C_n$, which is aligned with blocks of φ. In the second case we slide to the right.



- Let D_i be such that $C_i = \varphi(D_i)$.
- Since $\mathbf{x} = \varphi(\mathbf{x})$, $D_1 D_2 \cdots D_n$ is a factor of $\mathbf{x}$.
- Now $\psi(C_i) = M\psi(D_i)$, where M is the matrix of φ.
- Since M is invertible and $\psi(C_1) = \psi(C_2) = \cdots = \psi(C_n)$, we have $\psi(D_1) = \psi(D_2) = \cdots = \psi(D_n)$.
- $D_1 \cdots D_n$ is a shorter abelian n-power, contradiction.

Avoiding abelian 4-powers

- We check that the morphism φ

$$a \to aaab, \quad b \to abb$$

  verifies the criterion we just proved.
- The matrix of φ is $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, which is invertible.
- Take $G = \mathbb{Z}/5\mathbb{Z}$.
- Define $f : \{a, b\}^* \to G$ by $f(a) = 1$ and $f(b) = 2$.
- $f(aaab) = f(abb) = 0$

Avoiding abelian 4-powers

- $A = \{0, 1, 2, 3\}$ is progression free of order 5
- f is φ-injective
- $\varphi^{\omega}(a)$ has no short abelian 4-powers
- by previous criterion, $\varphi^{\omega}(a)$ avoids abelian 4-powers

Avoiding abelian cubes

Define ϑ by $\vartheta(a) = aabc$, $\vartheta(b) = bbc$, and $\vartheta(c) = acc$. The same method shows that $\vartheta^\omega(a)$ avoids abelian cubes.

Avoiding abelian squares

- Finding a construction that avoids abelian squares over 4-letters is much harder.
- Keränen found one in 1992 by intensive computer search.
- The images of the letters under his morphism each have length 85.

Avoiding patterns in the abelian sense

- an abelian instance of the pattern $xxyxyx$ is a word

$$X_1X_2Y_1X_3Y_2X_4$$

  where the X_i are all abelian equivalent and the Y_i are all abelian equivalent; e.g.,

$$0012\ 0120\ 221\ 1020\ 122\ 2100$$

- relatively little is known about avoidability of patterns in the abelian sense

Avoiding long binary patterns in the abelian sense

Theorem (Currie and Visentin 2008)

Patterns over $\{x, y\}$ of length greater than 118 are avoidable in the abelian sense on a binary alphabet.

Avoiding sum-cubes

The following avoidability result is even stronger than avoiding abelian cubes:

Theorem (Cassaigne, Currie, Schaeffer, Shallit 2011)

There exists an infinite word over a finite subset of $\mathbb{N}$ that contains no three consecutive blocks of the same length and the same sum (sum-cube).

The word is the fixed point of the morphism

$$0 \to 03 \qquad 1 \to 43 \qquad 3 \to 1 \qquad 4 \to 01.$$

Avoiding sum-squares

Open problem

Does there exist an infinite word over a finite subset of $\mathbb{N}$ that contains no two consecutive blocks of the same length and the same sum (sum-square)?

The End