Dejean's Conjecture

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Repetitions in words

- A finite or infinite word

$$w = a_1 a_2 a_3 a_4 \cdots$$

has period p if $a_i = a_i + p$ for each i.

- If w has length ℓ and period p, then w is an k-power, where $k = \ell/p$.

Example

The word $aabaaba$ has periods 3 and 6 and is a $7/3$-power.

- In 1912 Thue constructed an infinite binary word

$$011010011001011010 01 \cdots$$

containing no k-powers with $k > 2$.

The repetition threshold

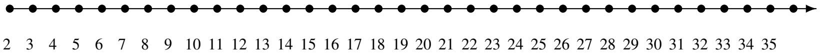
- A word containing no r-powers for any $r > k$ is called k^+-power-free.
- The repetition threshold over an n-letter alphabet is

$$RT(n) = \inf\{k : \text{some infinite word over an } n\text{-letter alphabet avoids } k\text{-powers.}\}$$

- In 1972 Françoise Dejean conjectured that for $n \geq 2$:

$$RT(n) = \begin{cases} 7/4, & n = 3 \\ 7/5, & n = 4 \\ n/(n-1), & n \neq 3, 4. \end{cases}$$

History of the conjecture



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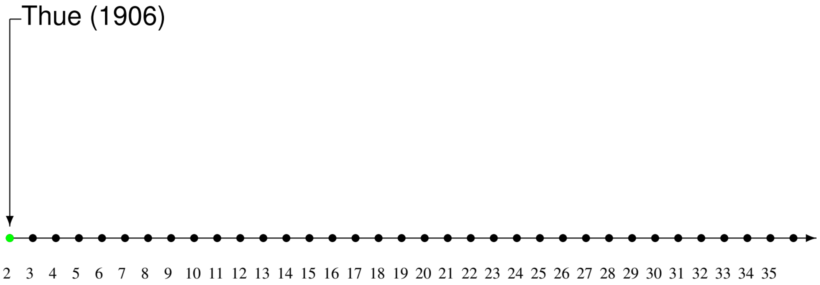

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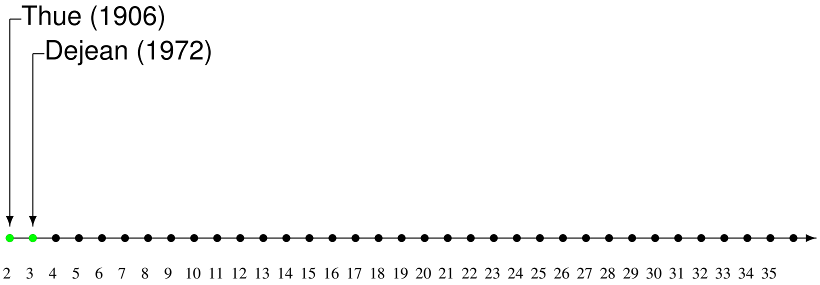

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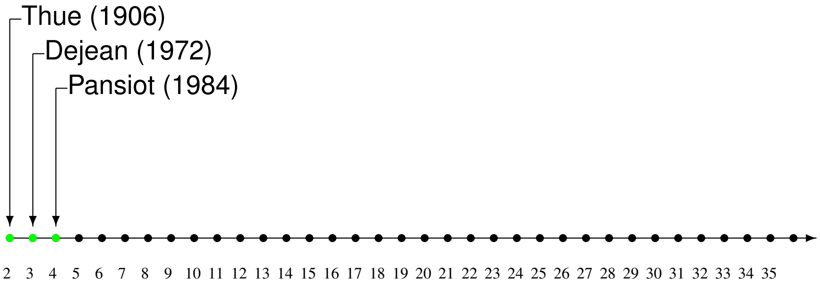

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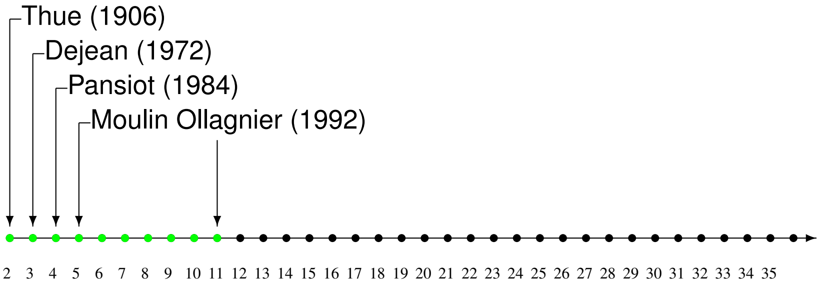

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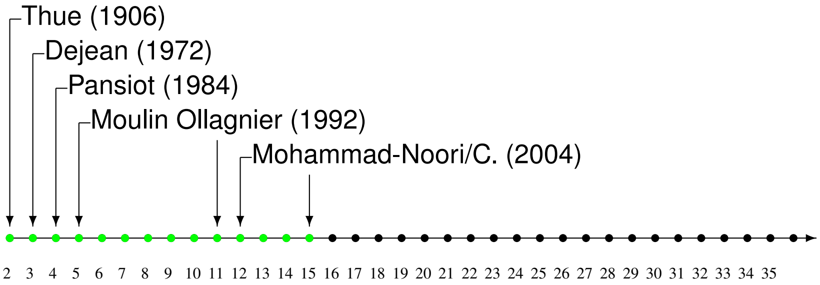

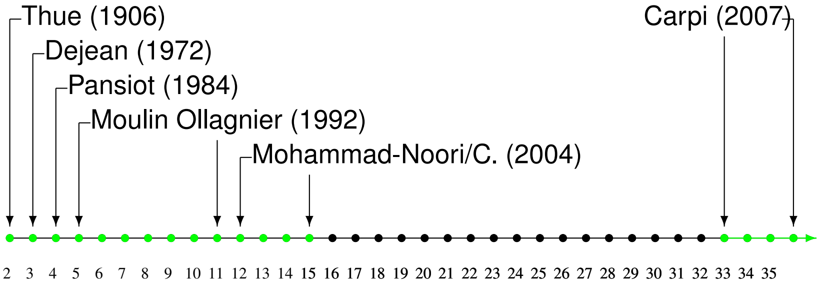
Thue (1906)
Dejean (1972)
Pansiot (1984)
Moulin Ollagnier (1992)
Mohammad-Noori/C. (2004)
Carpi (2007)
2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

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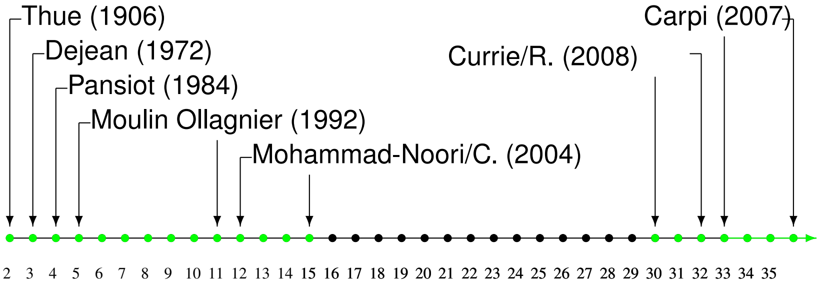

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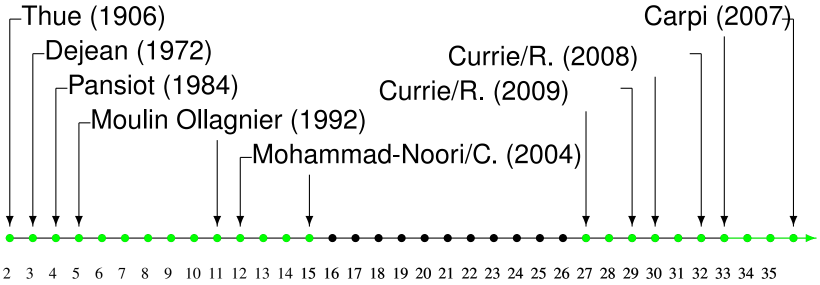

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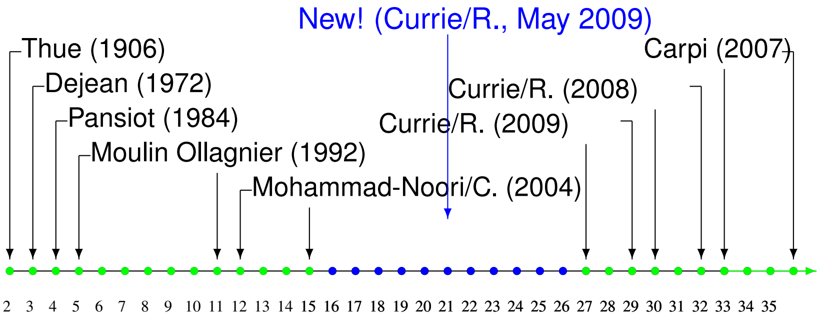

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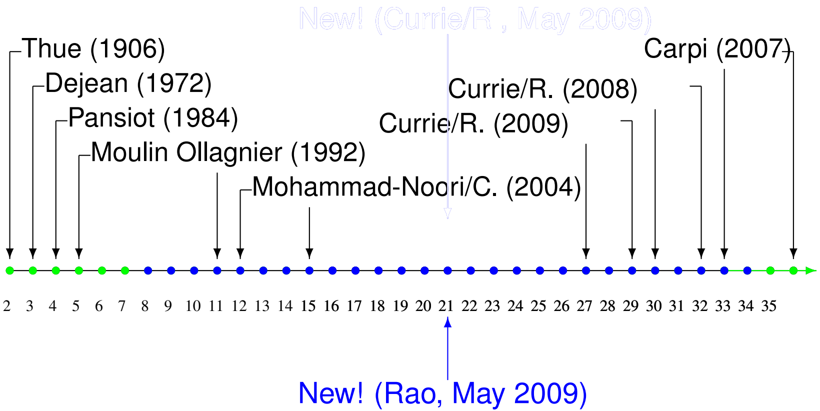

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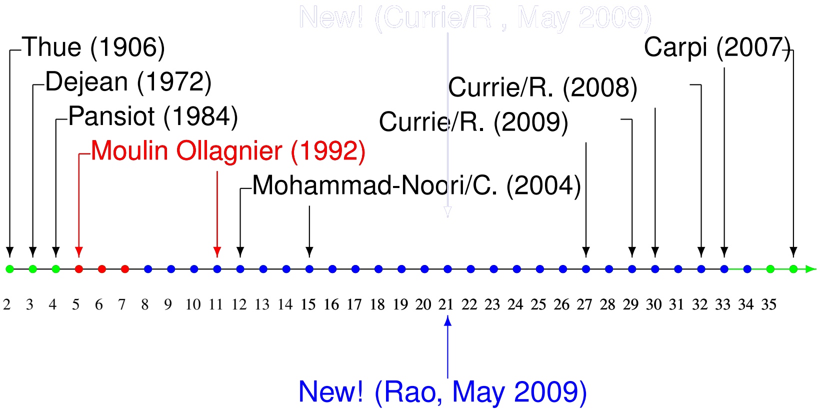

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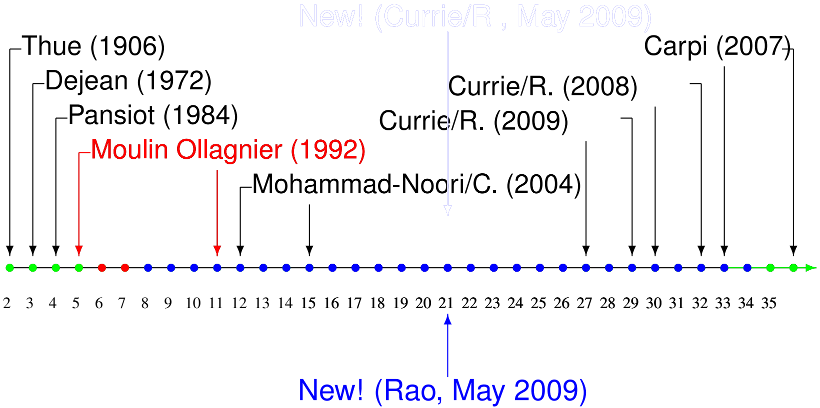

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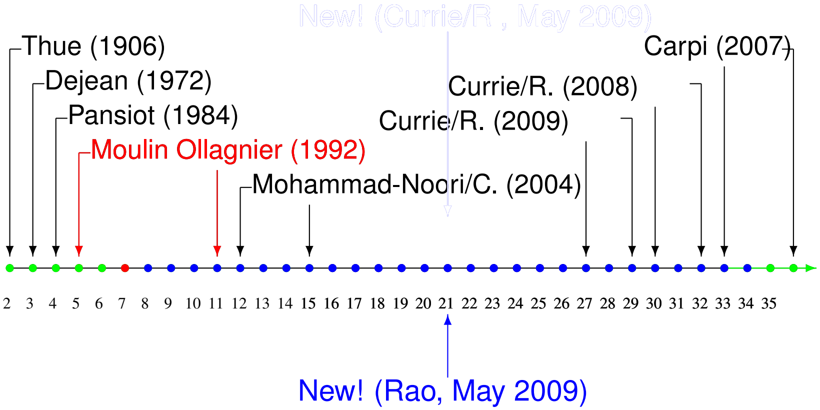

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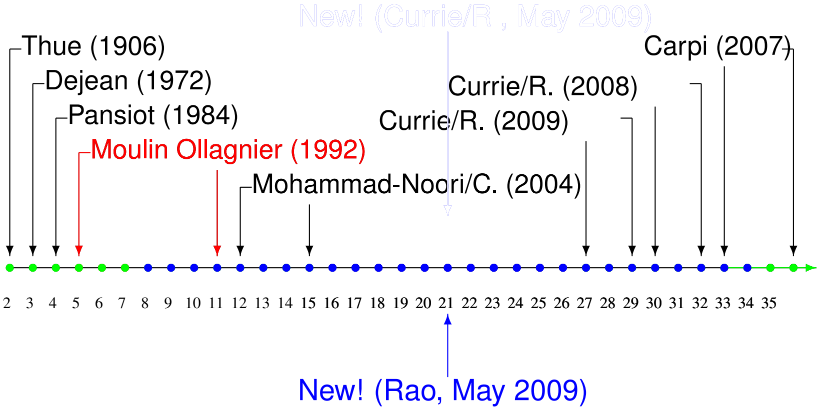

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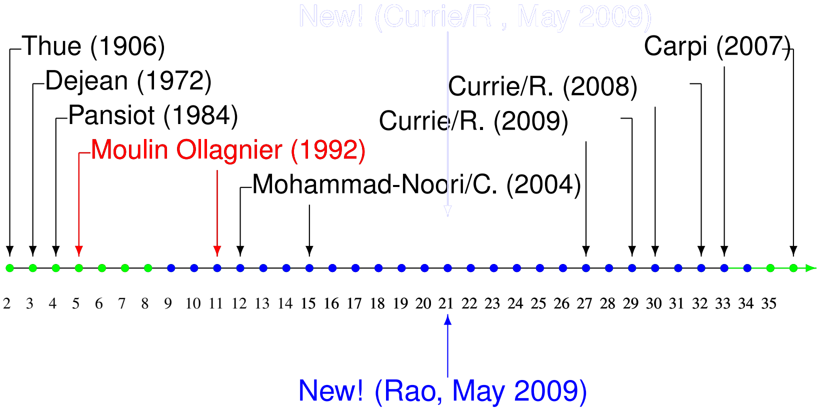

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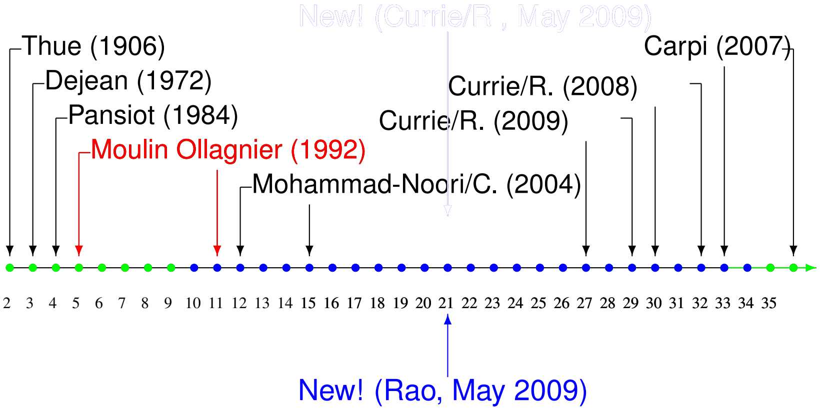

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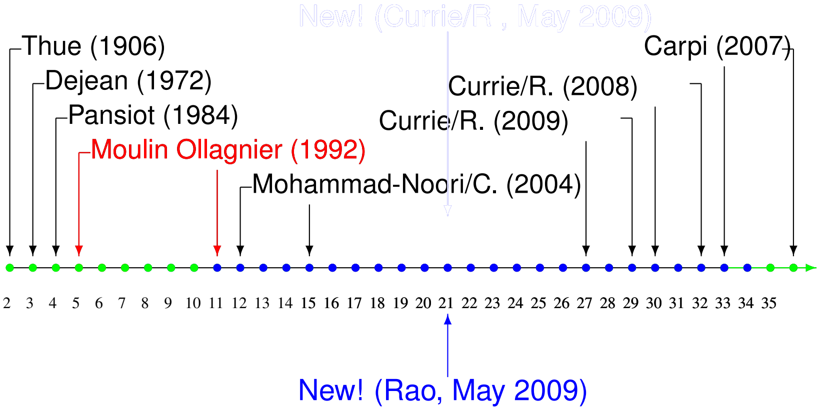

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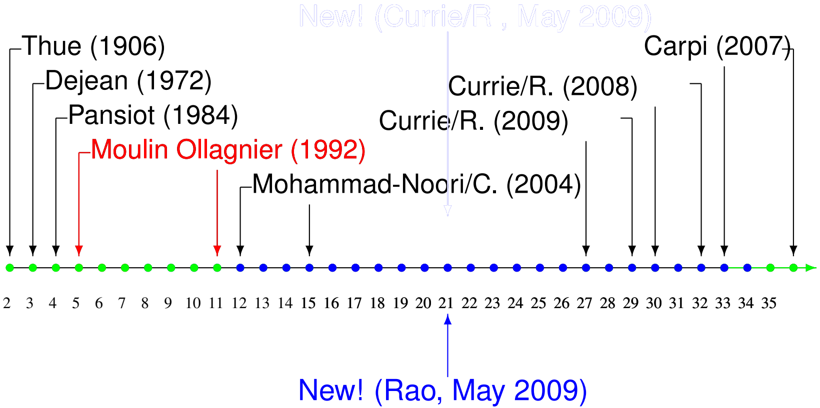
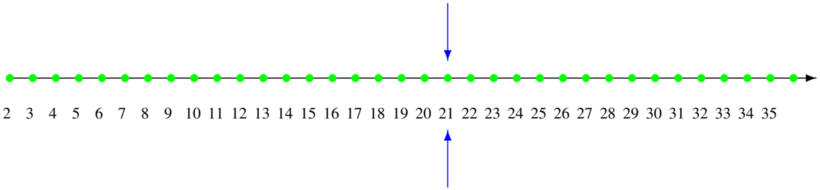


Dejean's conjecture proved

Dejean's conjecture proved! (Currie/R., May 2009)



Dejean's conjecture proved! (Rao, May 2009)

The Pansiot encoding

- Let the alphabet size n be fixed.
- A word of length at least $n+2$ must contain a $(n/(n-1))$-power.
- If a word avoids $(n/(n-1))^+$-powers, every block of length $n-1$ consists of $n-1$ different letters.
- The letter following a block y of length $n-1$ is either
  - the first letter of y; or
  - the unique letter that does not occur in y.
- We encode the first case with a 0 and the second case with a 1.
- This is the Pansiot encoding.
- Given the Pansiot encoding we can uniquely reconstruct the original word.

The Pansiot encoding

Example (n=6)

The word

$$123451632415$$

has Pansiot encoding

$$0101101.$$

We can reconstruct the original word from the prefix 12345 and the code 0101101.

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The word

$$12345163 2415$$

has Pansiot encoding

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Pansiot's construction

- To prove Dejean's conjecture for $n = 4$, Pansiot needed to show that there exists a $(7/5)^+$-power-free word $\mathbf{w}$.
- It suffices to find the binary Pansiot encoding of $\mathbf{w}$.
- Pansiot constructed such a binary word by iterating the morphism

$$0 \to 101101, \quad 1 \to 10$$

as follows

$$1 \to 10 \to 10101101 \to 10101101101011011010101110110 \to \cdots$$

yielding in the limit the infinite word

$$10101101101011011010101110110 \cdots .$$

A map into the symmetric group

- Moulin Ollagnier proved Dejean's conjecture for $5 \leq n \leq 11$.
- He made the following observation:
- A word $w = a_1 a_2 \cdots a_{n-1}$ containing no repeated letter can be associated with a permutation:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & b \end{pmatrix},$$

where b is the unique letter that does not occur in w.

A map into the symmetric group

- Moving from one $(n-1)$-letter block to the next $(n-1)$-letter block by a "0" in the Pansiot encoding corresponds to multiplication on the right by
$$\sigma_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-1 & k \\ 2 & 3 & 4 & \cdots & 1 & k \end{pmatrix}.$$
- Moving from one block to the next by a "1" corresponds to multiplication on the right by
$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-1 & k \\ 2 & 3 & 4 & \cdots & k & 1 \end{pmatrix}.$$

A map into the symmetric group

Example (n=6)

Word:

$$\color{red}{12345}\color{black}{1632415}$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

$$1\textcolor{red}{23451}632415$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\textcolor{red}{\sigma_0} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 & 6 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

12345163241 5

Pansiot encoding:

0101101

Permutation:

$$\sigma_0\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 1 & 6 & 2 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

$$123\color{red}{451632}\color{black}{2415}$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\sigma_0\sigma_1\color{red}{\sigma_0}\color{black}{} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 6 & 3 & 2 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

$$1234\color{red}{51632}\color{black}{415}$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\sigma_0\sigma_1\sigma_0\color{red}{\sigma_1}\color{black}{} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 3 & 2 & 4 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

$$123451632415$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 3 & 2 & 4 & 5 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

$$123451632415$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 4 & 1 & 5 \end{pmatrix}$$

A map into the symmetric group

Example (n=6)

Word:

$$123451632415$$

Pansiot encoding:

$$0101101$$

Permutation:

$$\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_0\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix}$$

A map into the symmetric group

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Word:

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Permutation:

$$\sigma_0\sigma_1\sigma_0\sigma_1\sigma_1\sigma_0\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix}$$

A map into the symmetric group

- We thus have a canonical map ψ from the binary Pansiot codewords to the symmetric group S_n defined by

$$\begin{aligned} 0 &\rightarrow \sigma_0 \\ 1 &\rightarrow \sigma_1, \end{aligned}$$

and if $y = y_0 y_1 \cdots y_\ell$ is a word over $\{0, 1\}$, then

$$y \rightarrow \sigma_{y_0} \sigma_{y_1} \cdots \sigma_{y_\ell}.$$

Kernel repetitions

- Let x be the Pansiot encoding of a word w over an n-letter alphabet.
- Suppose $x = pe$ with e a prefix of pe, p non-empty.
- We call p the period and e the excess.
- If $|e| \geq n - 1$ and $\psi(p)$ is the identity permutation, we call x a kernel repetition.
- In this case w is a repetition of exponent

$$(|pe| + n - 1)/|p|.$$

Kernel repetitions

Example (n=4)

Word:

$$w = 1234134123413$$

Pansiot encoding:

$$x = \underbrace{1100011}_{p}\underbrace{110}_{e}$$

Permutation:

$$\psi(p) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Thus, x is a kernel repetition and w is a repetition of exponent

$$(|pe| + n - 1)/|p| = (10 + 4 - 1)/7 = 13/7.$$

Moulin Ollagnier's approach

- Generate an infinite Pansiot encoding $\mathbf{x}$ by iterating a binary morphism f.
- The word $\mathbf{x}$ must not contain a kernel repetition $x = pe$ with $(|pe| + n - 1)/|p| > RT(n)$.
- Moulin Ollagnier imposes the following algebraic condition on the morphism f:

$$\psi(f(0)) = \tau^{-1}\psi(0)\tau, \quad \psi(f(1)) = \tau^{-1}\psi(1)\tau.$$

- The algebraic condition ensures that f maps kernel repetitions to kernel repetitions.
- All sufficiently long kernel repetitions are the images under f of a shorter kernel repetition (more or less).

Moulin Ollagnier's approach

- We only have to check finitely many kernel repetitions in $\mathbf{x}$ to verify that none have $(|pe| + n - 1)/|p| > RT(n)$.
- Recall: $\mathbf{x}$ encodes a word $\mathbf{w}$ over an n-letter alphabet.
- We must also check that $\mathbf{w}$ does not contain other forbidden repetitions that do not arise from kernel repetitions in $\mathbf{x}$.
- These other repetitions can have length at most $(n-1)^2$, so again there are only finite many words to check.
- Moulin Ollagnier found by computer search suitable binary morphisms to generate the words $\mathbf{x}$ for $5 \leq n \leq 11$.
- For example, his morphism for $n = 5$ is

$$\begin{aligned} 0 &\rightarrow 0101011011010101110110 \\ 1 &\rightarrow 1010101011011011101101. \end{aligned}$$

The final resolution of the conjecture

- The major breakthrough was Carpi's proof of the conjecture for $n \geq 33$.
- By strengthening one part of Carpi's construction, we improved this to $n \geq 27$.
- We resolved the remaining open cases by extending Moulin Ollagnier's computer calculations to find suitable morphisms.
- Our constructions can easily be verified by checking that they satisfy the criteria previously established by Moulin Ollagnier.
- Rao independently resolved the last open cases by a different method.
- He found morphisms, which, when applied to the Thue–Morse word, give the desired Pansiot encoding.

Our calculations

- We searched for candidate morphisms f.
- We looked for uniform morphisms ($f(0)$ and $f(1)$ have the same length).
- We "guessed" that $f(0)$ and $f(1)$ should have length $4n - 4$ or $4n$.
- We did a backtracking search to find candidates of length $4n - 4$ for $f(0)$ and $f(1)$ (for $n = 21$, we searched for words of length $4n$).
- The candidates also had to satisfy Moulin Ollagnier's algebraic condition.
- We determined the number of iterates of f we needed to examine for forbidden repetitions.
- If no forbidden repetitions were found, we concluded that f generates a word witnessing the correctness of Dejean's conjecture for alphabet size n.

Thank you!