Computing π

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- π is commonly defined as the ratio of a circle's circumference to its diameter
- not the only possible definition (e.g., π is twice the least positive x for which $\cos x = 0$)
- π is ubiquitous in mathematics, including many surprising contexts
- for instance: the probability that when one tosses a coin $2n$ times it comes up heads the same number of times that it comes up tails is approximately $1/\sqrt{\pi n}$

- It is well known (Lambert 1761) that the constant

$$\pi = 3.14159265358979323846264\,33 \cdots$$

  is irrational.
- Its decimal expansion is non-repeating and non-terminating.
- It seems to behave like a random sequence.
- So how can we compute its expansion?

Some pre-calculus era estimates of π:

- Babylonians (ca. 2000 BC):

$$25/8 = 3.125$$

- Egyptians (ca. 2000 BC):

$$256/81 \approx 3.1604$$

- Archimedes (ca. 250 BC):

$$223/71 < \pi < 22/7 \quad (3.1408 < \pi < 3.1429)$$

- Madhava (ca. 1400 AD):

$$3.14159265359$$

- In the pre-calculus era the principal techniques were geometrical.
- Archimedes approximated the circumference of a circle by using inscribed and circumscribed polygons with many sides.
- The post-calculus era used infinite series.

Archimedes' geometrical method can be boiled down to the following iterative procedure:

Set $a_0 = 2\sqrt{3}$ and $b_0 = 3$.

Define

$$\begin{aligned} a_{n+1} &= \frac{2a_n b_n}{a_n + b_n} \\ b_{n+1} &= \sqrt{a_{n+1} b_n} \end{aligned}$$

The b_n (or a_n) give approximations to π.

If Archimedes had a computer (and was able to accurately compute $\sqrt{3}$) he could have computed these approximations to π:

n	Approx. to π
1	3.10582854123025
2	3.13262861328124
3	3.13935020304687
4	3.14103195089051
5	3.14145247228546
6	3.14155760791186
7	3.14158389214832
8	3.14159046322805
9	3.14159210599927

The Gregory–Leibniz series found in the 1670's (known to Madhava ca. 1400) is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

This comes from the infinite series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots ,$$

valid for $|x| \leq 1$, and the fact that

$$\arctan 1 = \frac{\pi}{4}.$$

The series for arctan is derived as follows. In elementary calculus we show that

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt.$$

Replacing the integrand by the series

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots$$

and integrating term-by-term gives (for $-1 < x < 1$)

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots .$$

Validity for $x = \pm 1$ follows from results in advanced calculus.

We can approximate π by truncating the Gregory–Leibniz series but this does not give accurate estimates.

# of terms of sum	Approx. to π
1	4.00000000000000
2	2.66666666666667
3	3.46666666666667
4	2.89523809523810
5	3.33968253968254
6	2.97604617604618
7	3.28373848373848
8	3.01707181707182

- We do not even have the first place after the decimal point correct.
- We need to take several hundred terms of the sum to get the first two decimal places correct.
- To get the first six decimal places we would need tens of thousands of terms.

A much better way to approximate π comes from another arctangent identity attributed to Euler in 1738:

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

We can approximate each of these arctangent values by substituting $x = 1/2$ and $x = 1/3$ into the arctangent series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

and truncating after some number of terms.

This gives the following (much better) approximations to π:

# of terms of sum	Approx. to π
1	3.33333333333333
2	3.11728395061728
3	3.14557613168724
4	3.14085056176106
5	3.14174119743369
6	3.14156158787759
7	3.14159934096620
8	3.14159118436091
9	3.14159298133457
10	3.14159257960635

But how does one demonstrate the identity

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3} \ ?$$

We will derive the addition formula for arctangent in the following form:

$$\arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} = \arctan \frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2},$$

valid whenever the left-hand side is between $-\pi/2$ and $\pi/2$.

We start with the addition formulae for sine and cosine:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Divide the two equations to get

$$\tan(\alpha + \beta) = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Now divide numerator and denominator by $\cos\alpha\cos\beta$ to get the addition formula for tangent:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Now we set

$$\alpha = \arctan \frac{a_1}{b_1} \text{ and } \beta = \arctan \frac{a_2}{b_2}$$

and substitute into the addition formula for tangent.

We get

$$\tan\left(\arctan\frac{a_1}{b_1}+\arctan\frac{a_2}{b_2}\right)=\frac{\frac{a_1}{b_1}+\frac{a_2}{b_2}}{1-\left(\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}\right)}$$
$$=\frac{a_1b_2+a_2b_1}{b_1b_2-a_1a_2},$$

and so

$$\arctan\frac{a_1}{b_1}+\arctan\frac{a_2}{b_2}=\arctan\frac{a_1b_2+a_2b_1}{b_1b_2-a_1a_2}.$$

Now we derive Euler's arctangent identity as follows:

$$\begin{aligned}
\arctan\frac{1}{2} + \arctan\frac{1}{3} &= \arctan\left(\frac{1\cdot 3 + 1\cdot 2}{2\cdot 3 - 1\cdot 1}\right) \\
&= \arctan\frac{5}{5} \\
&= \arctan 1 \\
&= \frac{\pi}{4}.
\end{aligned}$$

An even better arctangent identity for computing π is due to Machin in 1706:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

This formula can also be proved using the same method.

First we compute

$$\begin{aligned}2\arctan\frac{1}{5} &= \arctan\frac{1}{5} + \arctan\frac{1}{5}\\ &= \arctan\left(\frac{5+5}{25-1}\right)\\ &= \arctan\frac{10}{24}\\ &= \arctan\frac{5}{12}.\end{aligned}$$

It follows that

$$\begin{aligned} 4\arctan\frac{1}{5} &= 2\arctan\frac{5}{12} \\ &= \arctan\frac{5}{12} + \arctan\frac{5}{12} \\ &= \arctan\left(\frac{5\cdot 12 + 5\cdot 12}{12^2 - 5^2}\right) \\ &= \arctan\frac{120}{119}. \end{aligned}$$

Finally, we have

$$\begin{aligned}
\arctan\frac{1}{239}+\frac{\pi}{4} &= \arctan\frac{1}{239}+\arctan\frac{1}{1} \\
&= \arctan\left(\frac{1+239}{239-1}\right) \\
&= \arctan\frac{240}{238} \\
&= \arctan\frac{120}{119}.
\end{aligned}$$

Consequently,

$$4 \arctan \frac{1}{5} = \arctan \frac{1}{239} + \frac{\pi}{4},$$

and so

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239},$$

as claimed.

This identity gives the following approximations to π:

# of terms of sum	Approx. to π
1	3.18326359832636
2	3.14059702932606
3	3.14162102932503
4	3.14159177218218
5	3.14159268240440
6	3.14159265261531
7	3.14159265362355
8	3.14159265358860
9	3.14159265358984
10	3.14159265358979

- Until the 1970's, most modern methods for computing π used a similar arctangent identity.
- In the mid-1800's, a Viennese calculator named Johann Zacharias Dase was able to compute π to 200 places upon being shown how to use the formula

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}.$$

Many other series representations (also products, continued fractions, etc.) of π are known. Newton used

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3 \cdot 8} - \frac{1}{5 \cdot 32} - \frac{1}{7 \cdot 128} - \frac{1}{9 \cdot 512} - \cdots\right).$$

He recorded 15 digits in his diary, but stated,

> *"I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."*

A remarkable formula for π was obtained by Borwein, Bailey, and Plouffe in 1996:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right).$$

The formula is special because it allows one to compute individual "digits" of the base-16 expansion of π directly, without computing any others.

- current record for most digits of π computed: 12.1 trillion digits (Yee and Kondo, 2013)
- For most practical purposes fewer than 20 digits accuracy is sufficient.
- Why do people do these calculations?

The End