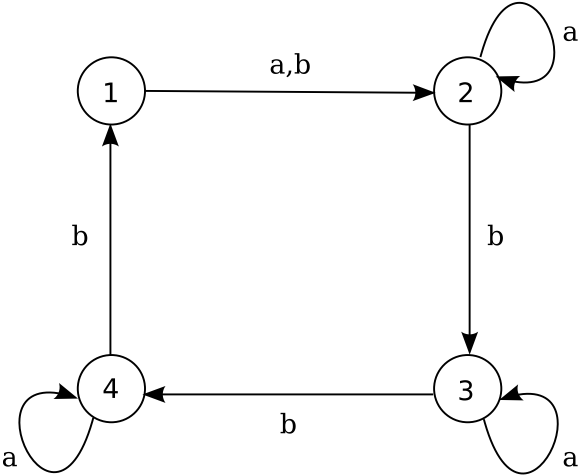
Synchronizing Automata and Černý's Conjecture

Narad Rampersad

Department of Mathematics
University of Liège

A finite automaton



Formal definition

- Here a finite automaton is a directed multigraph where
  - every vertex has constant out-degree k, and
  - the outgoing arcs of each vertex are labeled by distinct elements of a fixed k-element set.

Terminology

- We call the vertices states and denote the set of states by Q.
- We call the arcs transitions.
- Arcs are labeled by letters.
- A sequence of letters is called a word.

The transition function

- The transition function $\delta(p, a) = q$ denotes a transition from p to q labeled by a.
- If $w = w_1 w_2 \cdots w_n$ is a word then $\delta(q, w)$ is the state reached by starting at q and following the sequence of arcs labeled $w_1, w_2, \ldots, w_n$.
- If $A \subseteq Q$ then

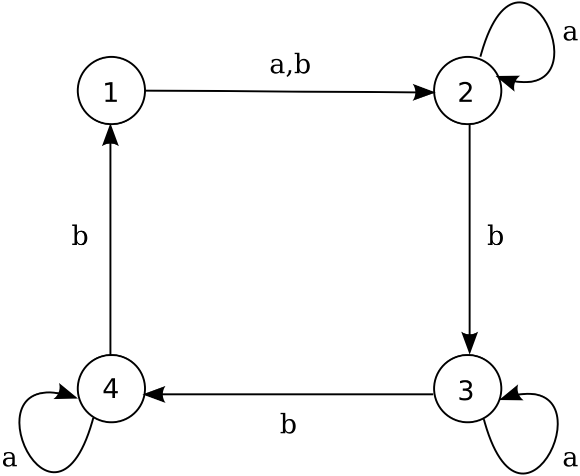
$$\delta(A, w) = \bigcup_{q \in A} \delta(q, w).$$

Synchronizing automata

- A word w such that $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$ is a reset word.
- An automaton with a reset word is synchronizing.
- Equivalently, there exists a state p and a word w such that $\delta(Q, w) = \{p\}$.
- Given an automaton, can we decide if it is synchronizing?
- If so, can we find the shortest reset word?

A synchronizing automaton

Reset word: *abbbabbba*.



Applications

- Moore's Gedanken-experiments (1950's):
- Imagine a satellite orbiting the moon.
- Its behaviour while on the dark side of the moon cannot be observed.
- When control is reestablished, we wish to reset the system to a particular configuration.

Applications

- Robotics (Natarajan 1980's):
- Imagine parts arriving on an assembly line with arbitrary orientations.
- The parts must be manipulated into a fixed orientation before proceeding with assembly.

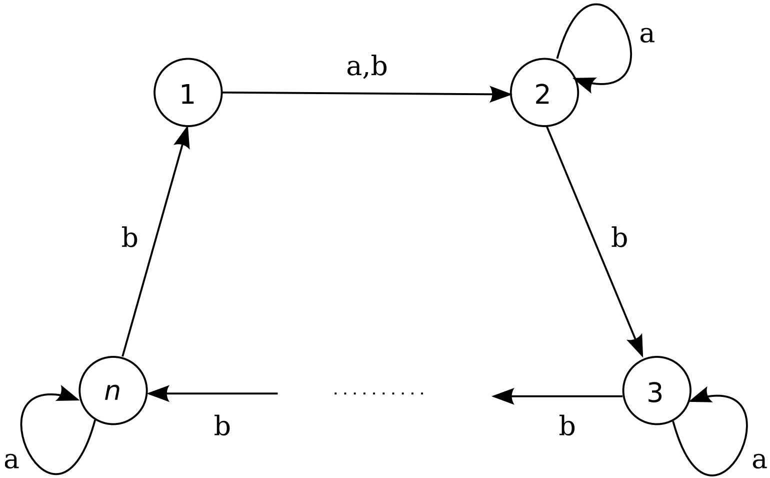
Černý's Conjecture

Černý's Conjecture (1964)

The shortest reset word of any synchronizing automaton with n states has length at most $(n-1)^2$.

Černý's construction

Reset word: $(ab^{n-1})^{n-2}a$ (length $(n-1)^2$).



Partial results

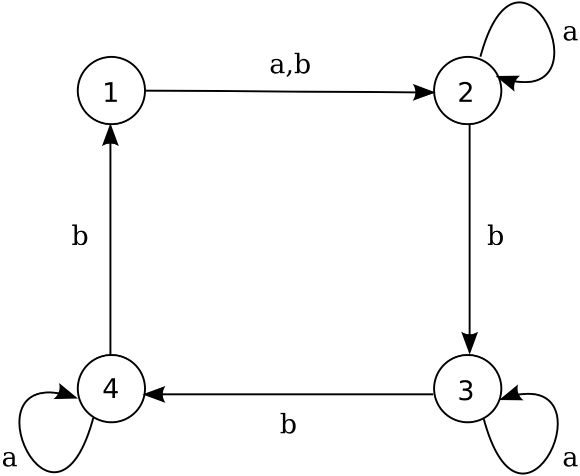
- E.g., Kari (2003) verified the conjecture for synchronizing automata whose underlying digraphs are Eulerian.
- Conjecture verified for several other classes of synchronizing automata.
- Steinberg (preprint) unified and simplified many of these proofs.

Best known upper bound

- M is a synchronizing automaton:
- There are sets $Q = P_1, P_2, \ldots, P_t$, and words $w_1, w_2, \ldots, w_{t-1}$, such that
  - $\delta(P_i, w_i) = P_{i+1}$, for $i = 1, \ldots, t-1$;
  - $|P_i| > |P_{i+1}|$, for $i = 1, \ldots, t-1$;
  - $|P_t| = 1$.
- $w = w_1 w_2 \cdots w_{t-1}$ is a reset word for M.

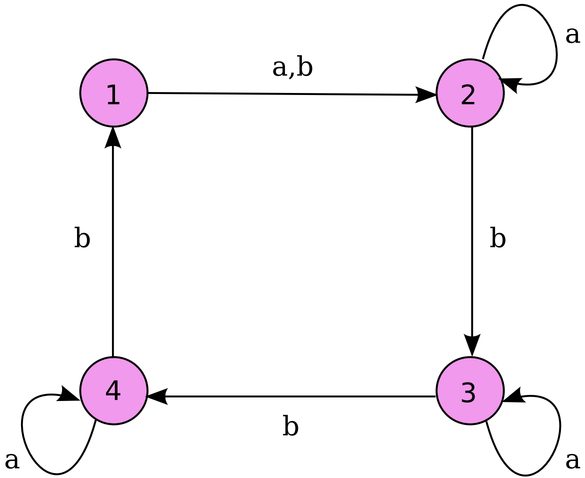
An example

Reset word: *a bbba bbba*.



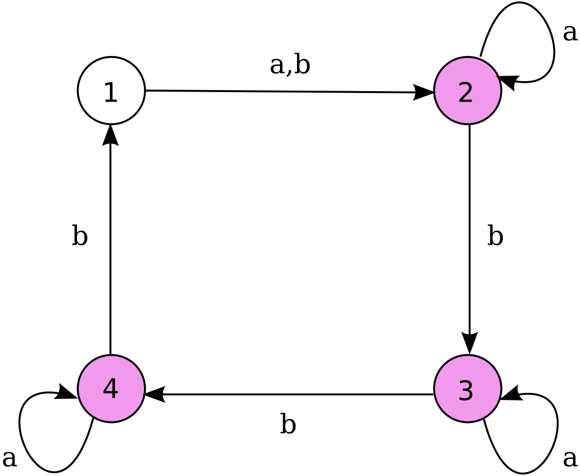
An example

Reset word: *a bbba bbba*.



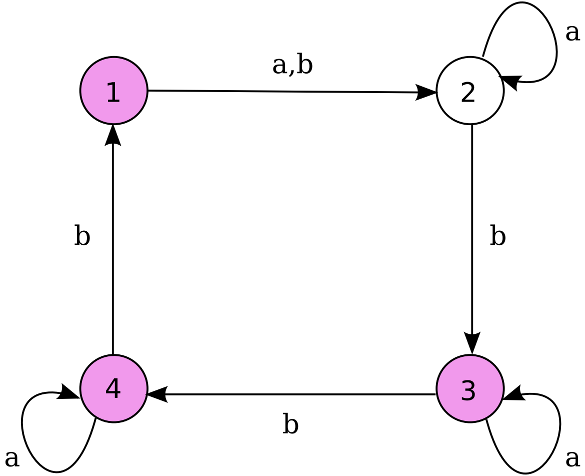
An example

Reset word: *a bbba bbba*.



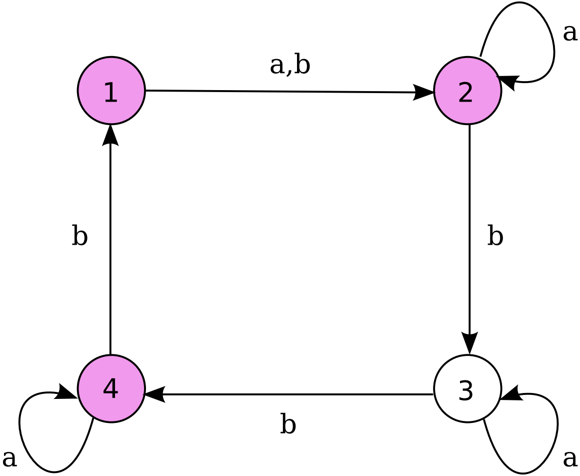
An example

Reset word: *a bbba bbba*.



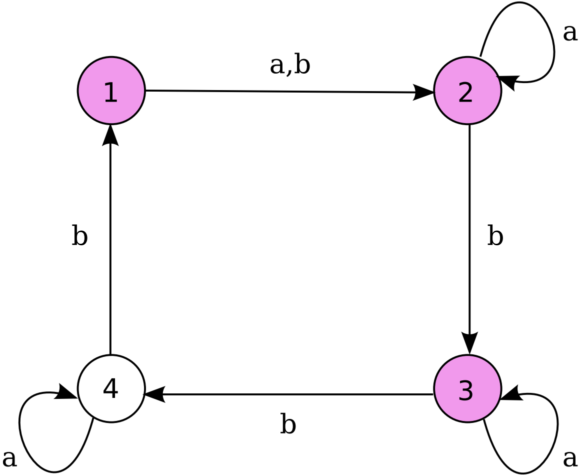
An example

Reset word: *a bbba bbba*.



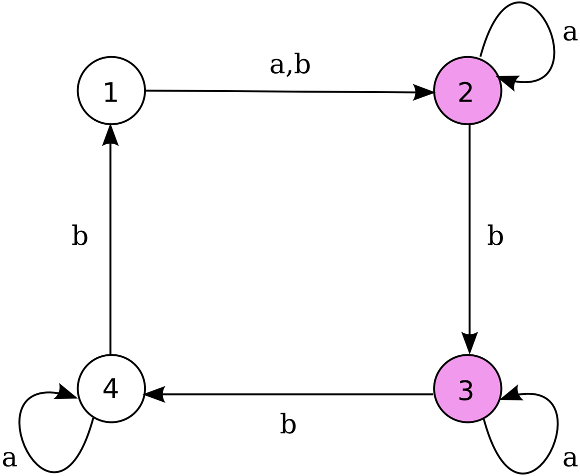
An example

Reset word: *a bbba bbba*.



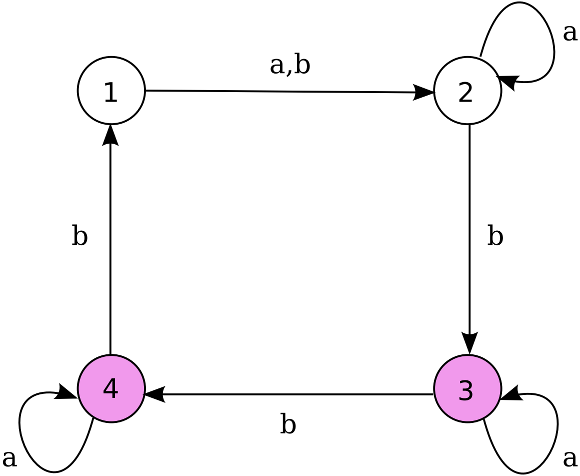
An example

Reset word: *a bbba bbba*.



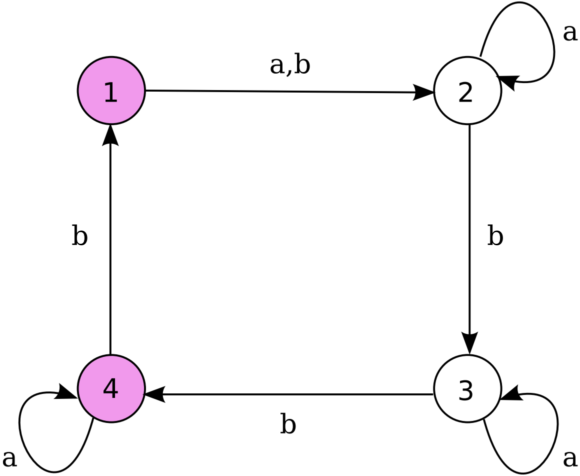
An example

Reset word: *a bbba* *b**bba*.



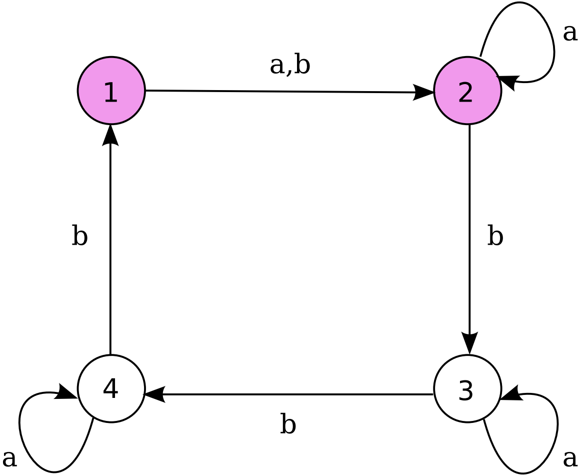
An example

Reset word: *a bbba bbba*.



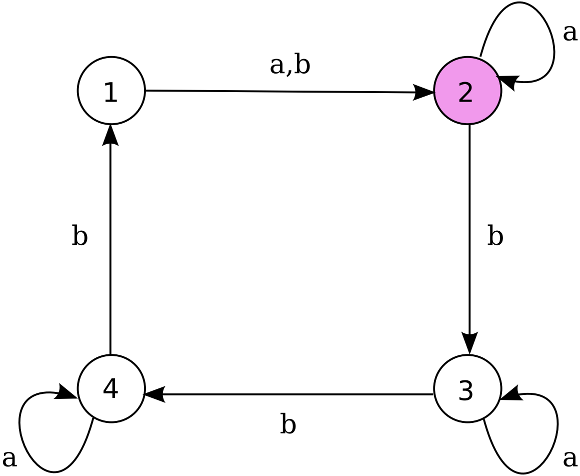
An example

Reset word: *a bbba bbba*.



An example

Reset word: *a bbba bbba*.



The greedy algorithm

Algorithm to find reset word w

Set $P_1 = Q$ and $t = 1$.

While $|P_t| > 1$:

Find a smallest word w_t such that $|\delta(P_t, w_t)| < |P_t|$.

Set $P_{t+1} = \delta(P_t, w_t)$ and increment t.

Return $w = w_1 w_2 \cdots w_{t-1}$.

Length of the reset word found

- What is the maximum length of w found by the greedy algorithm?
- In the worst case, $|P_i| - |P_{i+1}| = 1$, so that $t = n$.
- Consider a generic step k: i.e., P_k and w_k such that $|\delta(P_k, w_k)| < |P_k|$.
- What is the longest that w_k can be?

Length of the reset word found

- Let $w_k = a_1 a_2 \cdots a_{m+1}$ (the a's letters).
- There are sets $P_k = A_1, A_2, \ldots, A_{m+2}$ such that
  - $\delta(A_i, a_1) = A_{i+1}$ for $i = 1, \ldots, m+1$;
  - $|A_i| = |A_{i+1}|$ for $i = 1, \ldots, m$;
  - $|A_{m+1}| > |A_{m+2}|$.

Length of the reset word found

- For $i = 1, \ldots, m+1$,

$$|\delta(A_i, a_i \cdots a_{m+1})| < |A_i|.$$

- Thus there exists $q_i, q_i' \in A_i$ such that

$$\delta(q_i, a_i \cdots a_{m+1}) = \delta(q_i', a_i \cdots a_{m+1}).$$

- To each A_i, associate the set $B_i = \{q_i, q_i'\}$.

Length of the reset word found

- Note that $B_i \subseteq A_i$.
- Furthermore, for $i < j$, $B_j \nsubseteq A_i$.
- Otherwise, we would have a shorter word $w'_k = a_1 \cdots a_{i-1} a_j \cdots a_{m+1}$ such that $|\delta(P_k, w'_k)| < |P_k|$.

Length of the reset word found

- Let $\overline{A_i}$ denote the complement of A_i, i.e., the set $Q \setminus A_i$.
- We thus have
  - $B_i \cap \overline{A_i} = \emptyset$ for $i = 1, \ldots, m$;
  - $B_j \cap \overline{A_i} \neq \emptyset$ for $i < j$.
- What is the largest that m can be subject to these constraints?

A result from extremal set theory

Theorem (Frankl 1982)

Let $A_1, \ldots, A_m$ be sets of size r and let $B_1, \ldots, B_m$ be sets of size s such that

(a) $A_i \cap B_i = \emptyset$ for $i = 1, \ldots, m$;

(b) $A_i \cap B_j \neq \emptyset$ if $i < j$.

Then $m \leq \binom{r+s}{s}$.

A bound on the length of the reset word

- Let $|Q| = n$. Then $|\overline{A_i}| = n - k$ (since $|A_i| = k$) and $|B_i| = 2$ for $i = 1, \ldots, m$.
- By Frankl's result, $m \leq \binom{n-k+2}{2}$.
- Total length of the reset word at most

$$\sum_{k=2}^{n} \binom{n-k+2}{2} = \frac{n^3 - n}{6}.$$

Running time of the algorithm

- Originally conjectured by Fischler and Tannenbaum (1970) and (independently) by Pin (1981).
- After hearing Pin's 1981 talk, Frankl proved the necessary combinatorial result (independently rediscovered by Klyachko, Rystsov, and Spivak (1987)).
- Eppstein (1990) showed how to implement the greedy algorithm in $O(n^3 + kn^2)$ time.
- Greedy algorithm does not find a shortest reset word.

Finding a reset word of a given length

SYNCWORD

Given an automaton A and a positive integer k, does A have a reset word of length at most k?

- Clearly in NP since it suffices to "guess" a reset word of length at most $\min\{(n^3 - n)/6, k\}$.
- Eppstein showed it is NP-complete.

Finding a shortest reset word

MIN–SYNCWORD

Given an automaton A and a positive integer k, does A have a shortest reset word of length k?

- Olschewski and Ummels (preprint) showed it is DP-complete.

The class DP

- DP consists of all languages L such that $L = L_1 \setminus L_2$ for some languages L_1, L_2 in NP.
- A DP-complete problem is both NP-hard and coNP-hard.
- The canonical DP-complete problem is:

SAT–UNSAT

Given CNF formulae φ and ψ, is φ satisfiable and ψ unsatisfiable?

DP-completeness

- MIN-SYNCWORD clearly in DP, since it is the difference of SYNCWORD and

$$\{(A, k) : k > 0 \text{ and } (A, k-1) \in \text{ SYNCWORD}\}.$$

- To show DP-hardness, reduce from SAT-UNSAT.

Approximating the shortest reset word

Thereom (Berlinkov (preprint))

Unless $P = NP$, there is no polynomial-time algorithm to approximate the minimum length of a reset word for a given automaton within a constant factor.

Synchronizing colouring

- Start with a *strongly connected* directed multigraph G where every vertex has constant out-degree k.
- Is it possible to assign labels to the arcs so that G becomes synchronizing?
- If so, then G has a synchronizing colouring.

The road colouring problem

- Can graphs with synchronizing colourings be characterized?
- A graph is aperiodic if the gcd of the lengths of all of its cycles is 1.
- It is not hard to show that aperiodicity is a necessary condition.
- Adler and Weiss (1970) conjectured that it is also a sufficient condition.

The resolution of the problem

Theorem (Trahtman 2007)

Let G be a strongly connected directed multigraph where every vertex has constant out-degree k. Then G has a synchronizing coloring if and only if the the gcd of the lengths of all of its cycles is 1.

For further reading

- The literature on synchronizing automata is huge. For more information, see:
- Volkov's 2008 survey:

  http://csseminar.kadm.usu.ru/tarragona_volkov2008.pdf
- Jean-Eric Pin's webpage:

  http://www.liafa.jussieu.fr/~jep/Problemes/Cerny.html
- Avraham Trahtman's webpage:

  http://u.cs.biu.ac.il/~trakht/syn.html

The End