Computer proofs of some combinatorial congruences

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(This is joint work with Jeffrey Shallit.)

We look at some computational methods for obtaining congruences for combinatorial sequences like

- the Catalan numbers

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \dots$$

  which count, among other things, the number of strings of properly nested parentheses of length $2n$, or the number of binary trees on n vertices;

- the Motzkin numbers

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots$$

  which count the number of lattice paths from $(0,0)$ to $(n,0)$ with steps $\nearrow$, $\searrow$, $\rightarrow$ and that don't dip below the x-axis;
- etc.

Let p be prime. Our main tool is the Freshman's Dream:

$$(1+x)^p \equiv_p 1 + x^p.$$

We also will work with base-p expansions. If

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r$$

we write

$$(n)_p = n_0 n_1 n_2 \cdots n_r$$

for the the base-p expansion of n written least-significant-digit first.

- Let's start with the central binomial coefficients:
- $\binom{2n}{n}$ is the coefficient of x^n in $(1+x)^{2n}$
- Suppose I would prefer $\binom{2n}{n}$ to be the constant term; i.e., the coefficient of x^0.
- Divide by x^n: $\binom{2n}{n}$ is the constant term of

$$\frac{(1+x)^{2n}}{x^n} = \left(\frac{(1+x)^2}{x}\right)^n = \left(\frac{1+2x+x^2}{x}\right)^n = \left(\frac{1}{x}+2+x\right)^n.$$

We can do the same for the Catalan numbers:

$$\begin{aligned} C_n &= \frac{1}{n+1}\binom{2n}{n} \\ &= \binom{2n}{n} - \binom{2n}{n-1} \\ &= \mathrm{ct}\left(\frac{1}{x}+2+x\right)^n - \mathrm{ct}\left(x\left(\frac{1}{x}+2+x\right)^n\right) \\ &= \mathrm{ct}\left(\left(\frac{1}{x}+2+x\right)^n(1-x)\right). \end{aligned}$$

The Motzkin numbers satisfy:

$$M_n = \sum_{k \geq 0} \binom{n}{2k} C_k = \mathrm{ct}\left(\left(\frac{1}{x} + 1 + x \right)^n (1 - x^2) \right).$$

Many similar combinatorial sequences involving sums of binomial coefficients can be written as the constant term of a Laurent polynomial with integer coefficients of the form $[P(x)]^n Q(x)$.

Let's evaluate the Catalan numbers modulo 2. Define

$$\begin{aligned} C_n &= \mathrm{ct}\left(\left(\frac{1}{x} + 2 + x\right)^n (1-x)\right) \\ &\equiv_2 \mathrm{ct}\left(\left(\frac{1}{x} + x\right)^n (1+x)\right) =: A_1(n). \end{aligned}$$

We compute $A_1(2n)$ and $A_1(2n+1)$.

$$\begin{aligned} A_1(2n) &= \mathrm{ct}\left((1/x + x)^{2n}(1+x)\right) \\ &\equiv_2 \mathrm{ct}\left((1/x^2 + x^2)^n(1+x)\right) \quad \text{(by Freshman's Dream)} \\ &= \mathrm{ct}\left((1/x^2 + x^2)^n\right) \\ &= \mathrm{ct}\left((1/x + x)^n\right) \\ &=: A_2(n) \end{aligned}$$

$$
\begin{aligned}
A_1(2n+1) &= \mathrm{ct}\left((1/x+x)^{2n+1}(1+x)\right) \\
&= \mathrm{ct}\left(((1/x+x)^2)^n(1/x+x)(1+x)\right) \\
&\equiv_2 \mathrm{ct}\left((1/x^2+x^2)^n(1/x+1+x+x^2)\right) \\
&= \mathrm{ct}\left((1/x^2+x^2)^n(1+x^2)\right) \\
&= \mathrm{ct}\left((1/x+x)^n(1+x)\right) \\
&= A_1(n)
\end{aligned}
$$

Now we compute $A_2(2n)$ and $A_2(2n+1)$.

$$\begin{aligned} A_2(2n) &= \mathrm{ct}\left((1/x + x)^{2n}\right) \\ &\equiv_2 \mathrm{ct}\left((1/x^2 + x^2)^n\right) \\ &= \mathrm{ct}\left((1/x + x)^n\right) \\ &= A_2(n) \end{aligned}$$

$$\begin{aligned} A_2(2n+1) &= \mathrm{ct}\left((1/x + x)^{2n+1}\right) \\ &\equiv_2 \mathrm{ct}\left((1/x^2 + x^2)^n (1/x + x)\right) \\ &= 0 \end{aligned}$$

So we get the recurrences

$$\begin{aligned} A_1(2n) &\equiv_2 A_2(n) \\ A_1(2n+1) &\equiv_2 A_1(n) \\ A_2(2n) &\equiv_2 A_2(n) \\ A_2(2n+1) &\equiv_2 0, \end{aligned}$$

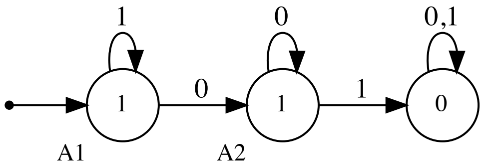
with initial conditions $A_1(0) = 1$, $A_2(0) = 1$.

Let $w = (n)_2$; i.e., w is the binary representation of n, written left-to-right. The previous recurrences can be written (with an abuse of notation) as

$$A_1(0w) \equiv_2 A_2(w)$$
$$A_1(1w) \equiv_2 A_1(w)$$
$$A_2(0w) \equiv_2 A_2(w)$$
$$A_2(1w) \equiv_2 0,$$

with initial conditions $A_1(0) = A_2(0) = 1$.

This can be represented graphically by the finite automaton



This is a machine that reads input $w = (n)_2$, digit-by-digit, and follows the arcs labeled by each digit read. If the machine ends in states labeled 1 (i.e., A_1 or A_2), then $C_n \equiv_2 1$ and if it ends in the 0 state, we have $C_n \equiv_2 0$.

We have thus proved the following folklore theorem:

Theorem

C_n is odd iff $(n)_2 = 1^k 0^j$; i.e., iff $n = 2^k - 1$.

(Here 1^k means a string of k 1's and 0^j means a string of j 0's.)

- This illustrates a general method due to Rowland and Zeilberger.
- We are given a sequence defined as $\mathrm{ct}([P(x)]^n Q(x))$, for some Laurent polynomials P and Q.
- Modulo any prime power, if we compute recurrence relations as we did above, this process will eventually terminate, giving a finite set of recurrence relations.
- We can then translate these recurrence relations into a finite automaton.

They implemented this in Maple, and were thus able to prove many hundreds of congruence results for the Catalan numbers and other sequences.

https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/meta.html

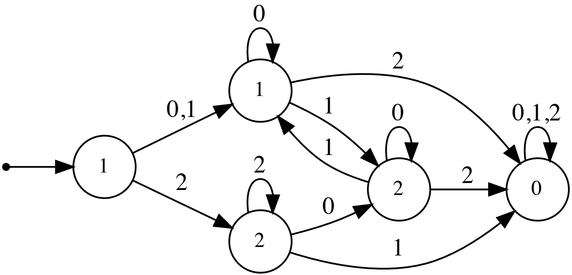
Rowland and Yassawi (2015) have also given a completely different method for computing finite automata for these kinds of combinatorial sequences.

Now let's look at the Catalan numbers C_n modulo 3. (Alter and Kubota (1973) studied the general case $C_n \bmod p$.)

Let

$$\mathbf{c}_3 = (C_n \bmod 3)_{n\geq 0}$$
$$= (1, 1, 2, 2, 2, 0, 0, 0, 2, 2, 2, 1, 1, 1, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 1, 1, 1, 0, 0, 0, 1,$$
$$1, 1, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$
$$0, 0, 0, 0, 0, 0, 2, 2, 2, 1, 1, 1, 0, 0, 0, 1, 1, 1, 2,$$
$$2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 2, 2,$$
$$2, 0, 0, 0, 2, 2, 2, 1, 1, 1, \ldots)$$

Applying the Rowland–Zeilberger method gives the automaton



which is rather more complicated than the modulo 2 automaton.

Theorem (Deutsch and Sagan 2006)

The runs of 0's in $\mathbf{c}_3$ begin at positions n where either

$$(n)_3 = 21^i \text{ or } (n)_3 = 21^i 0\{0,1\}^j,\ i \geq 1,\ j \geq 0,$$

and have length $(3^{i+2} - 3)/2$.

Theorem cont'd. (Deutsch and Sagan 2006)

The blocks of non-zero values in $\mathbf{c}_3$ are given by the following:

- The block 11222 occurs at position 0.
- The block 111222 occurs at all positions n where $(n)_3 = 2^i 0w$ for some $i \geq 2$ and some $w \in \{0,1\}^*$ that contains an odd number of 1's.
- The block 222111 occurs at all positions n where $(n)_3 = 2^i 0w$ for some $i \geq 2$ and some $w \in \{0,1\}^*$ that contains an even number of 1's.

We can obtain this result purely by computer using a program called Walnut (developed by Jeffrey Shallit's student Hamoon Mousavi). Suppose we are given

- A finite automaton reading input n in base-k and outputing the n-th term of a sequence $\mathbf{s}$; and,
- A formula φ in first-order-logic involving variables (over $\mathbb{N}$), constants, quantifiers, logical operations, ordering, addition and subtraction of natural numbers, and indexing into $\mathbf{s}$.
- We can also multiply by a constant (this is just repeated addition), but we can't multiply two variables.

- If φ has no free variables, Walnut will output either that φ is TRUE or φ is FALSE.
- If φ has free variables, Walnut will produce an automaton that accepts the base-k representations of the values of the free variables that satisfy φ.
- We won't get into the theory of how it evaluates these logical formulas.

e.g., the formula

$$\varphi := \exists i \, \forall j \, ((j \geq 0 \land j < 4) \Rightarrow \mathbf{c}_3(i+j) = 1)$$

asserts that there is a "run" of at least four 1's in $\mathbf{c}_3$.

In Walnut's language, this is

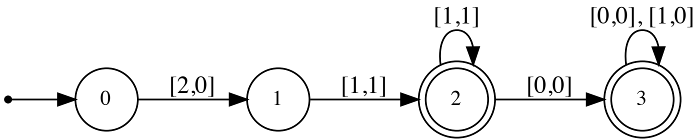
```
eval run4ones "?lsd_3 Ei Aj ((j>=0 & j<4) =>
    CAT3[i+j]=@1)":
```

and evaluates to "FALSE".

The Walnut command

```
eval cat3max0 "?lsd_3 n>=1 &
    (At t<n => CAT3[i+t]=@0) &
    CAT3[i+n]!=@0 & (i=0|CAT3[i-1]!=@0)":
```

produces the automaton



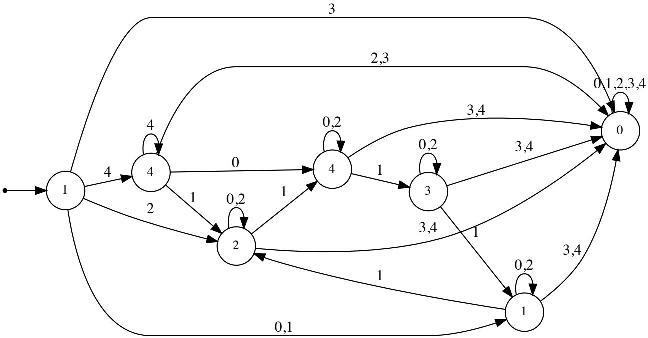
Examining the transition labels of the first component of the input gives the claimed representation for the starting positions of the runs of 0's

$$(i)_3 = 21^k \text{ or } (i)_3 = 21^k0\{0,1\}^j$$

and examining the transition labels of the second component gives the claimed length

$$(n)_3 = 01^k; \text{ i.e., } n = (3^{k+2} - 3)/2.$$

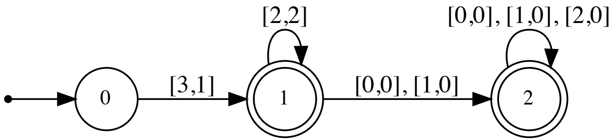
For $p = 5$, the Rowland–Zeilberger method gives the automaton



for

$\mathbf{c}_5 := (C_n \bmod 5)_{n \geq 0}$.

Using Walnut, one can obtain the following automaton for the runs of 0's in $\mathbf{c}_5$:



From this automaton we derive:

Theorem (R. and Shallit)

The runs of 0's in $\mathbf{c}_5$ begin at positions n where either

$$(n)_5 = 32^i \text{ or } (n)_5 = 32^i\{0,1\}\{0,1,2\}^j,\ i \geq 0,\ j \geq 0,$$

and have length $(5^{i+2} - 3)/2$.

We can easily characterize the non-zero blocks in $\mathbf{c}_5$ as well.

Let's examine the Motzkin numbers next. These are closely related to the central trinomial coefficients:

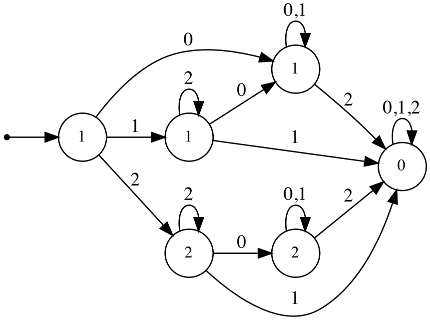
$$\begin{aligned}\sum_{k\geq 0}\binom{n}{2k}\binom{2k}{k} &= \sum_{k\geq 0}\binom{n}{2k}\,\mathrm{ct}\left(\frac{(1+x^2)^{2k}}{x^{2k}}\right)\\ &= \mathrm{ct}\left(\sum_{k\geq 0}\binom{n}{2k}\left(\frac{1+x^2}{x}\right)^{2k}\right)\\ &= \mathrm{ct}\left(1+\frac{1+x^2}{x}\right)^n\\ &= \mathrm{ct}\left(1+\frac{1}{x}+x\right)^n\end{aligned}$$

Hence

$$
\begin{aligned}
M_n &= \sum_{k\geq 0} \binom{n}{2k} C_k \\
&= \sum_{k\geq 0} \binom{n}{2k} \left[\binom{2k}{k} - \binom{2k}{k-1} \right] \\
&= \sum_{k\geq 0} \binom{n}{2k}\binom{2k}{k} - \sum_{k\geq 0} \binom{n}{2k}\binom{2k}{k-1} \\
&= \mathrm{ct}\left(1 + \frac{1}{x} + x\right)^n - \mathrm{ct}\left(x^2 \left(1 + \frac{1}{x} + x\right)^n\right) \\
&= \mathrm{ct}\left(\left(1 + \frac{1}{x} + x\right)^n (1 - x^2)\right)
\end{aligned}
$$

Now we can compute automata for $M_n \bmod p$ using the Rowland–Zeilberger algorithm.

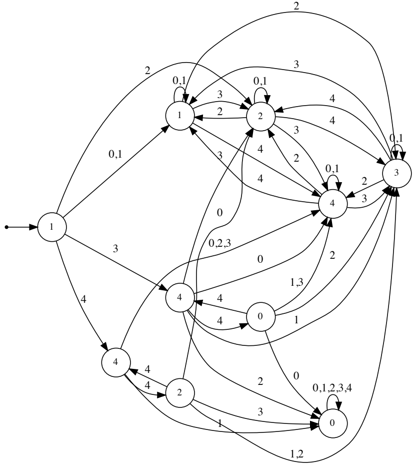
The automaton for $M_n \bmod 3$ is



- Let $\mathbf{m}_3 := (M_n \bmod 3)_{n\geq 0}$.
- Note that no matter where you are in the automaton, the input 02 takes you to the 0 state.
- So for any w, the input $w02$ results in output 0.
- Letting w run through all ternary strings of any fixed length, we find that $\mathbf{m}_3$ contains arbitrarily large runs of 0's.

$$
\begin{aligned}
\mathbf{m}_3 &= (M_n \bmod 3)_{n\geq 0} \\
&= (1, 1, 2, 1, 0, 0, 0, 1, 2, 1, 1, 2, 1, 0, 0, 0, 0, 0, 0, 0, \\
&0, 0, 0, 0, 0, 1, 2, 1, 1, 2, 1, 0, 0, 0, 1, 2, 1, 1, 2, \\
&1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\
&1, 2, 1, 1, 2, 1, 0, 0, 0, 1, 2, 1, 1, 2, 1, 0, 0, 0, 0, 0, 0, \\
&0, 0, 0, 0, 0, 0, 1, 2, 1, 1, 2, 1, 0, 0, 0, 1, 2, 1, 1, 2, \\
&1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)
\end{aligned}
$$

The automaton for $M_n \bmod 5$ is



$$\mathbf{m}_5 = (M_n \bmod 5)_{n\geq 0}$$

$$= (1, 1, 2, 4, 4, 1, 1, 2, 3, 0, 3, 3, 1, 0, 4, 2, 2, 4, 2, 4,$$

$$4, 4, 3, 0, 2, 1, 1, 2, 4, 4, 1, 1, 2, 3, 0, 3, 3, 1, 0, 4,$$

$$2, 2, 4, 2, 4, 4, 4, 3, 4, 3, 3, 3, 1, 2, 2, 3, 3, 1, 4, 0,$$

$$4, 4, 3, 0, 2, 1, 1, 2, 1, 2, 2, 2, 4, 3, 3, 2, 2, 4, 3, 3,$$

$$2, 2, 4, 1, 0, 1, 1, 2, \ldots)$$

- Unlike $\mathbf{m}_3$, the sequence does not contain arbitrarily long runs of 0's.
- With Walnut we can easily prove that the longest runs in $\mathbf{m}_5$ are $(1,1,1)$, $(2,2,2)$, $(3,3,3)$, and $(4,4,4)$.

- We have seen that $\mathbf{m}_3$ and $\mathbf{m}_5$ have very different behaviour.
- Burns (arxiv preprints) studied $\mathbf{m}_p$ for primes p between 7 and 29 using automata computed using the Rowland–Yassawi algorithm.
- His work suggests that depending on the value of p, the sequence $\mathbf{m}_p$:
- either behaves like $\mathbf{m}_3$, where 0 has density 1 (i.e., $p = 7, 17, 19$),
- or $\mathbf{m}_p$ behaves like $\mathbf{m}_5$, where 0 has density < 1 (i.e., $p = 11, 13, 23, 29$).

- $\mathbf{m}_5$ has another nice property: if a length-n pattern of residues occurs in $\mathbf{m}_5$, it is guaranteed to re-occur within the next $200n$ terms (in combinatorics on words, we call this phenomenon uniform recurrence).

The Walnut commands to prove this recurrence property are:

```
def mot5faceq "?lsd_5 At (t<n) =>
    (MOT5[i+t]=MOT5[j+t])":
eval tmp "?lsd_5 An (n>=1) => Ai Ej (j>i) &
    (j<i+200*n+1) & $mot5faceq(i,j,n)":
```

Problem

Characterize the primes p for which $\mathbf{m}_p$ is uniformly recurrent.

We guess that the answer to this problem is given by the sequence

$$2, 5, 11, 13, 23, 29, 31, 37, 53, \ldots$$

of primes that do not divide any central trinomial number.

This is sequence **A113305** of the OIES.

Theorem (Deutsch and Sagan)

The central trinomial coefficient T_n satisfies

$$T_n \equiv_3 \begin{cases} 1, & \text{if } (n)_3 \text{ does not contain a } 2; \\ 0, & \text{otherwise.} \end{cases}$$

Deutsch and Sagan proved this by an application of Lucas' Theorem; it is also immediate from the automaton produced by the Rowland–Zeilberger algorithm.

As with the Motzkin numbers, the behaviour of T_n modulo 5 is rather different from that modulo 3.

Theorem (R. and Shallit)

Let $\mathbf{t}_5 := (T_n \bmod 5)_{n\geq 0}$. Then

1. $\mathbf{t}_5$ does not contain 0 (i.e., T_n is never divisible by 5);
2. the only patterns that repeat three times in $\mathbf{t}_5$ are 111, 222, 333, and 444;

Theorem (cont'd.)

3. $\mathbf{t}_5$ is uniformly recurrent; Furthermore, if a length-n pattern w occurs at position i in $\mathbf{t}_5$, then there is another occurrence of w at some position j, where $i < j \leq i + 200n - 192$.
4. If w is a length-n pattern appearing in $\mathbf{t}_5$, then w appears at some position $i < 121n$.

Theorem (Deutsch and Sagan)

Let $(n)_p = n_0 n_1 \cdots n_r$. Then

$$T_n \equiv_p \prod_{i=0}^{r} T_{n_i}.$$

- An immediate consequence is that T_n is divisible by p if and only if one of the T_{n_i} is divisible by p.
- This criterion allows one to determine the primes p that do not divide any central trinomial coefficient; i.e., those in **A113305** of OEIS, which we conjectured in the previous section to be the ones for which $\mathbf{m}_p$ is uniformly recurrent.

Using the previous result of Deutsch and Sagan we can prove the following:

Theorem (R. and Shallit)

Let $\mathbf{t}_p$ be the sequence of central trinomial numbers modulo p. If the first p terms of $\mathbf{t}_p$ do not contain 0, but do contain a primitive root modulo p, then $\mathbf{t}_p$ is uniformly recurrent.

For $p = 5$, we have $(T_0, T_1, T_2, T_3, T_4) = (1, 1, 3, 7, 19)$, so $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4) = (1, 1, 3, 2, 4)$ contains the primitive root 2. The word

$$\mathbf{t}_5 = 1132411324334122214344231113 24 \cdots$$

is therefore uniformly recurrent.

- A computer calculation shows that for each prime p appearing in the list of initial values $2, 5, 11, 13, \ldots, 479$ of **A113305**, the first p terms of $\mathbf{t}_p$ always contain a primitive root modulo p.
- Hence, each of these $\mathbf{t}_p$'s are uniformly recurrent.

Problem

Prove this in general.

References

- Walnut can be downloaded here:

  https://cs.uwaterloo.ca/~shallit/walnut.html

- Rowland and Zeilberger's paper and accompanying material can be downloaded here:

  https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/meta.html

- Deutsch and Sagan's paper is: Congruences for Catalan and Motzkin numbers and related sequences, J. Number Theory 117 (2006), 191–215.

The End