Combinatorics on Words and Noncommutative Algebra

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Burnside's Problem

Bounded Burnside Problem

If G is a finitely generated group and there is an integer n such that $g^n = 1$ for every $g \in G$, then must G be finite?

General Burnside Problem

If G is a finitely generated group and every element of G has finite order, then must G be finite?

Counterexamples to Burnside's Problem

- Answer to both questions expected to be "yes"
- Counterexample to the General Burnside Problem given by Golod and Shafarevich (1964)
- Counterexample to the Bounded Burnside Problem given by Novikov and Adjan (1968)

Kurosh's Problem

Kurosh's Problem

If A is a finitely generated algebra over a field F and every element of A is nilpotent, then must A be finite dimensional over F?

- An algebra A is a vector space that is also a ring.
- A element $a \in A$ is nilpotent if $a^n = 0$ for some n.

The free noncommutative algebra

- F a field
- Let $T = F\langle x_1, x_2, \ldots x_d\rangle$ be the free noncommutative algebra over F generated by the variables $x_1, x_2, \ldots, x_d$.
- The monomials of T are words over the alphabet $x_1, x_2, \ldots, x_d$.
- T is the set of all F-linear combinations of such monomials: e.g.,

$$c_0x_3x_2x_1x_3 + c_1x_2x_2 + c_2x_3x_2x_1.$$

Homogeneous elements of T

- The degree of a monomial is its length as a word.
- An element of T is homogeneous if its monomials all have the same degree.
- Let S be a set of homogeneous elements, each of degree at least 2.
- Suppose S has at most r_i elements of degree i for $i \geq 2$.
- Let I be the two-sided ideal of T generated by S.

The Golod–Shafarevich construction

Golod–Shafarevich Theorem

If the coefficients in the power series expansion of

$$\left(1 - dz + \sum_{i \geq 2} r_i z^i\right)^{-1}$$

are nonnegative, then the quotient algebra T/I is infinite dimensional over F.

A particular case of the G–S theorem

- If S consists of monomials (i.e. words) we can rephrase the result in combinatorial terms.
- Let S be a set of words over an d-letter alphabet, each of length at least 2.
- Suppose S has at most r_i words of length i for $i \geq 2$.

A combinatorial reformulation

Theorem

If the power series expansion of

$$G(z) := \left(1 - dz + \sum_{i \geq 2} r_i z^i\right)^{-1}$$

has non-negative coefficients, then there are least $[z^n]G(z)$ words of length n over a d-letter alphabet that contain no word of S as a factor.

Squarefree words

- A square is a word of the form ww.
- A word is squarefree if it contains no square as a factor.

Squarefree words using 3 symbols (Thue 1906)

Iterate the substitution $0 \to 012$; $1 \to 02$; $2 \to 1$:

$$0 \to 012 \to 012021 \to 012021012102 \to \cdots$$

These words are squarefree.

Enumeration of squarefree words

Proposition

For $n \geq 0$ there are at least 5^n squarefree words of length n over an alphabet of size 7.

- Let S be the set of squares over an alphabet of size 7.
- For $n \geq 1$ the set S contains 7^n squares of length $2n$.

Applying the G–S theorem

- Define

$$\begin{aligned} G(z) &:= \left(1 - 7z + \sum_{i \geq 1} 7^i z^{2i}\right)^{-1} \\ &= \left(1 - 7z + \frac{7z^2}{1 - 7z^2}\right)^{-1} \\ &= 1 + 7z + 42z^2 + 245z^3 + 1372z^4 + 7546z^5 + \cdots . \end{aligned}$$

- One shows by induction that $[z^n]G(z) \geq 5^n$ for $n \geq 0$.

Counterexample to Kurosh's Problem

Goal

Construct an algebra A over a field F such that:

- A is finitely generated.
- Every element a of A is nilpotent (satisfies $a^n = 0$ for some n).
- A is infinite dimensional over F.

Constructing A as a quotient of the free algebra

- F a countable field
- Let $T = F\langle x_1, x_2, x_3\rangle$ be the free algebra over F.
- Let T' be the ideal of T consisting of all elements without a constant term.
- Want: an ideal I such that $A = T'/I$.
- Enumerate the elements of T' as $t_1, t_2, \ldots$.

Defining the ideal I

- Choose an integer $m_1 \geq 2$ and write

$$t_1^{m_1} = t_{1,2} + t_{1,3} + \cdots + t_{1,k_1},$$

  where each $t_{1,j}$ is homogeneous of degree j.

- Choose another positive integer m_2 so that

$$t_2^{m_2} = t_{2,k_1+1} + t_{2,k_1+2} + \cdots + t_{2,k_2}.$$

- Continue in this way for $t_3, t_4, \ldots$.
- Let I be the ideal generated by the $t_{i,j}$.

The quotient T'/I

- Each element of T'/I is nilpotent.
- An application of the G–S theorem ensures that T'/I is infinite dimensional over F.
- T'/I is a counterexample to Kurosh's Problem.

Counterexample to the General Burnside Problem

Goal

Construct a group G such that:

- G is finitely generated.
- Every element of G has finite order.
- G is infinite.

Constructing G from T/I

- Let F be the field with p elements.
- Let T and I be as defined above.
- Let a_1, a_2, a_3 be the elements $x_1 + I, x_2 + I, x_3 + I$ of T/I.
- Let G be the multiplicative semigroup in T/I generated by $1 + a_1$, $1 + a_2$, and $1 + a_3$.

Showing G is a group

- An element of G has the form $1 + a$ for some $a \in T'/I$.
- a is nilpotent, so for sufficiently large n, we have $a^{p^n} = 0$.
- In characteristic p we have $(1 + a)^{p^n} = 1 + a^{p^n} = 1$.
- Thus $1 + a$ has an inverse and G is a group.
- Every element $1 + a$ of G has finite order (a power of p).

Showing G is infinite

- Suppose G finite.
- F-linear combinations of elements of G form a finite dimensional algebra B.
- 1 and $1 + a_i$ are in G, so $(1 + a_i) - 1 = a_i$ is in B.
- $1, a_1, a_2, a_3$ generate T/I, which was previously shown to be infinite dimensional.
- B is thus infinite dimensional, a contradiction.
- We conclude G is infinite.

Growth of algebras

- A an algebra over a field F with generators $x_1, x_2, \ldots, x_d$
- V the vector space spanned by $x_1, x_2, \ldots, x_d$
- V^n the vector space spanned by monomials of degree n
- $A_n := F + V + V^2 + \cdots + V^n$
- $A = \bigcup_{n \geq 0} A_n$

Types of growth

- growth function of A: $d_V(n) := \dim_F(A_n)$.
- A has exponential growth: $d_V(n) \geq t^n$ for some $t > 1$.
- A has polynomial growth: $d_V(n) \leq cn^r$ for some non-negative integers c, r.

Growth of the free algebra

Example

- The free noncommutative algebra $F\langle x_1, \ldots, x_d \rangle$:

$$d_V(n) = \sum_{i=0}^{n} d^i = d^{n+1} - 1 \quad \text{(exponential).}$$

- The free commutative algebra $F[x_1, \ldots, x_d]$:

$$d_V(n) = \sum_{i=0}^{n} \binom{d+i-1}{i} = \binom{d+n}{n} \leq 2n^d \quad \text{(polynomial).}$$

Gelfand–Kirillov dimension

- Gelfand–Kirillov dimension of an algebra A:

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \log_n d_V(n)$$

- If $d_V(n)$ is exponential, then $\mathrm{GKdim}(A) = \infty$.
- If $d_V(n) \leq cn^r$, then $\mathrm{GKdim}(A) \leq r$.
- If A is finite dimensional, then $\mathrm{GKdim}(A) = 0$; otherwise, $\mathrm{GKdim}(A) \geq 1$.

Possible values for GK dimension

Bergman's Gap Theorem

There is no algebra A with $1 < \mathrm{GKdim}(A) < 2$.

Borho–Kraft; Warfield

For every real number $r \geq 2$, there is an algebra A with $\mathrm{GKdim}(A) = r$.

Monomial algebras

- I a two-sided ideal generated by monomials
- monomial algebra: an algebra $A := F\langle x_1, \ldots, x_d \rangle / I$
- The monomials of A of degree n are simply the words of length n that do not contain a generator of I as a factor.

Fact

For any finitely generated algebra A there is a monomial algebra B with the same growth function (hence the same GK dimension).

Complexity of sets of words

- A set L of words is factorial if whenever x is a word in L, every factor of x is also in L.
- The complexity function of L is the function $f(n)$ that counts the number of words of length n in L.

Theorem

Let L be a factorial set of words. If for some length n_0 we have $f(n_0) = n_0$, then there is a constant C such that $f(n) \leq C$ for all $n \geq 2n_0$. Moreover, $C \leq (n_0 + 1)^2/4$, and this bound is tight.

The complexity function

- Due independently to Kobayashi and Kobayashi (1993); Ellingsen and Farkas (1994); Balogh and Bollobás (2005).
- Either $f(n)$ bounded by a constant, or $f(n) \geq n + 1$ for all n.
- Bergman's gap theorem a consequence of this observation.

Complexity of infinite words

- $\mathbf{w}$ an infinite word
- L the set of finite factors of $\mathbf{w}$
- $f(n)$ the complexity function of L
- If $f(n) \leq C$, then $\mathbf{w}$ is eventually periodic.
- If $f(n) = n + 1$ for all n, the word $\mathbf{w}$ is called Sturmian.
- Sturmian words are aperiodic words of minimal complexity.
- First studied in depth by Morse and Hedlund (1940)

An example of a Sturmian word

The Fibonacci word

Iterate the substitution $0 \to 01$; $1 \to 0$:

$$0 \to 01 \to 010 \to 01001 \to 01001010 \to 0100101001001 \to \cdots$$

The infinite word obtained in the limit has $n+1$ factors of length n for all n.

Summary

We have seen applications of word combinatorics to:

- Burnside's Problem in group theory
- Kurosh's Problem in ring theory
- Growths of algebras

Other applications:

- PI-algebras (algebras satisfying a polynomial identity)
- Shirshov's Theorem
- etc.

The End