Critical exponents of balanced words

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- This talk is about two things: repetitions in words and the balance property of words.
- Much (everything?) is known about these things over a binary alphabet.
- So we consider larger alphabets.
- But we need to understand the binary case first.

- let u be a finite word
- the alphabet is usually a finite subset of $\{0, 1, 2, \cdots\}$
- the length of u is $|u|$
- the number of times the letter a appears in u is $|u|_a$
- a word w (finite or infinite) over an alphabet A is balanced if for every $a \in A$ and every pair u, v of factors of w with $|u| = |v|$ we have

$$\big|\, |u|_a - |v|_a \,\big| \leq 1.$$

- the word 0020010201 is not balanced since $|00200|_0 = 4$ and $|10201|_0 = 2$
- the word 01201210210 is balanced

Over the binary alphabet, the class of infinite aperiodic balanced words coincides with the class of Sturmian words (Morse and Hedlund 1940). Sturmian words are first differences of irrational Beatty sequences.

Let α and ρ be real numbers with $0 < \alpha < 1$. A (slow) Beatty sequence is a sequence of the form

$$(\lfloor n\alpha + \rho \rfloor)_{n \geq 1} \text{ or } (\lceil n\alpha + \rho \rceil)_{n \geq 1}.$$

We consider only the case where α is irrational.

For $\alpha = \sqrt{2} - 1$ and $\rho = 0$ the Beatty sequence $(\lfloor n\alpha + \rho \rfloor)_{n \geq 1}$ is the sequence

$$0, 0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 7, 7, 7, 8, 8, 9, 9, 9, \ldots$$

The corresponding Sturmian sequence is the sequence of first differences:

$$0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, \ldots$$

Why is the Sturmian word balanced? An integer j appears in the Beatty sequence if and only if

$$\begin{aligned} & j = \lfloor i\alpha + \rho \rfloor \\ \Leftrightarrow \quad & \frac{\rho - j}{\alpha} - \left\lfloor \frac{\rho - j}{\alpha} \right\rfloor < \frac{1}{\alpha} \\ \Leftrightarrow \quad & \left(\frac{\rho - j - 1}{\alpha}, \frac{\rho - j}{\alpha} \right] \cap \mathbb{Z} \neq \emptyset. \end{aligned}$$

The number of distinct j's of the form $\lfloor i\alpha + \rho \rfloor$ for $a \le i < a + n$ is the number of integers in the interval

$$\left(\frac{\rho - a - n}{\alpha}, \frac{\rho - a}{\alpha} \right]$$

which is

$$\left\lfloor \frac{\rho - a}{\alpha} \right\rfloor - \left\lfloor \frac{\rho - a - n}{\alpha} \right\rfloor = \left\lfloor \frac{n}{\alpha} \right\rfloor \text{ or } \left\lceil \frac{n}{\alpha} \right\rceil .$$

- Any block of n consecutive terms in a Beatty sequence contains either $\lfloor n/\alpha \rfloor$ or $\lceil n/\alpha \rceil$ distinct terms.
- Any block of $n-1$ consecutive terms in the corresponding first difference sequence has either $\lfloor n/\alpha \rfloor - 1$ or $\lceil n/\alpha \rceil - 1$ 1's.
- Hence this binary sequence is balanced.

- We are interested in repetitions in balanced words.
- Let u be a finite word and write $u = u_0 u_1 \cdots u_{n-1}$, where the u_i are letters.
- A positive integer p is a period of u if $u_i = u_{i+p}$ for all i.
- Let $e = |u|/p$ and let z be the prefix of u of length p.
- z is a fractional root of u.
- We say that u has exponent e and write $u = z^e$.
- e.g., $01011010 = (01011)^{8/5}$
- A square (resp. cube) is a repetition with exponent 2 (resp. 3)

The critical exponent of an infinite word w is

$$E(w) = \sup\{r \in \mathbb{Q} : \text{there is a finite, non-empty factor of } w \text{ with exponent } r\}.$$

For example, the word

$$w = 012021012102012021\cdots$$

obtained by iterating the substitution

$$0 \to 012, \qquad 1 \to 02, \qquad 2 \to 1$$

contains no squares, but has repetitions with exponents arbitrarily close to 2, so $E(w) = 2$.

Dejean's Theorem

Given an alphabet A of size k, the least critical exponent among all infinite words over A is

$$\begin{cases} 7/4, & k = 3 \\ 7/5, & k = 4 \\ k/(k-1), & k = 2 \text{ or } k \geq 5. \end{cases}$$

However, imposing the balance property makes it much harder to avoid repetitions.

- To understand the repetitions in a Sturmian word, we need another equivalent definition of Sturmian words.
- Let α be an irrational real number between 0 and 1, called the slope.
- Suppose α has continued fraction expansion $\alpha = [d_0, d_1, d_2, d_3, \ldots]$.

The characteristic Sturmian word with slope α is the infinite word c_α obtained as the limit of the sequence of standard words s_n defined by

$$s_0 = 0, \quad s_1 = 0^{d_1 - 1}1, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad n \geq 2.$$

For $n \geq 2$, we also define the semi-standard words

$$s_{n,t} = s_{n-1}^{t} s_{n-2},$$

where $1 \leq t < d_n$.

One characteristic Sturmian word is of particular significance. Let $\phi = (1+\sqrt{5})/2$. The Fibonacci word is the characteristic Sturmian word

$$c_\theta = 010010100100101001010010010100\cdots$$

with slope $\theta := 1/\phi^2 = [0, 2, \overline{1}]$. We call the corresponding standard words the finite Fibonacci words:

$$f_0 = 0, \quad f_1 = 01, \quad f_2 = 010, \quad \ldots$$

- the word f_i has length F_{i+2} (the $(i+2)$-th Fibonacci number) and has F_{i+1} 0's and F_i 1's
- Mignosi and Pirillo (1992) showed that $E(c_\theta) = 2 + \phi$.
- The more general results of Damanik and Lenz (2002) and Justin and Pirillo (2001) show that this is minimal over all Sturmian words (i.e., all aperiodic balanced binary words).
- What about balanced words over larger alphabets?

- An infinite word y has the constant gap property if, for each letter a, there is some number d such that the distance between successive occurrences of a in y is always d.
- This is stronger than being periodic.
- $(0120)^\omega$ is periodic but is not a constant gap word (contains both 00 and 0120)
- $(0102)^\omega$ is a constant gap word

Theorem (Graham 1973; Hubert 2000)

A recurrent aperiodic word x is balanced if and only if x is obtained from a Sturmian word u over $\{0,1\}$ by the following procedure: replace the positions containing 0's in u by a periodic sequence y with constant gaps over some alphabet A and replace the positions containing 1's in u by a periodic sequence y' with constant gaps over some alphabet B, disjoint from A.

e.g., take the Sturmian word

$$u = 0101001010010101001010010101001010010100 \cdots ,$$

$y = (01)^\omega$ and $y' = (2324)^\omega$, then

$$x = 0213012041021302104120130214012031021401 \cdots$$

is balanced.

From the construction it is clear that to understand the repetitions in the word x one has to understand the repetitions in the underlying Sturmian word u.

A word x is a conjugate of a word y if we can write $x = uv$ and $y = vu$ for some words u and v.

Theorem (Damanik and Lenz 2002)

Let w be a primitive factor of a Sturmian word u.

- If w^2 is a factor of u then w is a conjugate of either a standard word s_k or a semi-standard word $s_{k,t}$.
- If w^3 is a factor of u then w is a conjugate of a standard word s_k.

- To create a balanced word with repetitions of small exponent, we try to choose the periodic words y and y' in such a way that the Hubert construction "breaks up" the repetitions in u.
- The analysis depends on understanding the number of 0's and 1's in the standard and semi-standard words modulo the periods of y and y'.

Proposition

There is an infinite ternary balanced word x_3 with critical exponent

$$E(x_3) = 2 + \frac{\sqrt{2}}{2} \approx 2.7071.$$

- Let $\alpha = \sqrt{2} - 1 = [0, \overline{2}]$.
- Let c_α be the characteristic Sturmian word with slope α.
- So c_α is the limit of the standard words s_k defined by

$$s_0 = 0, \quad s_1 = s_0 1, \quad s_k = s_{k-1}^2 s_{k-2}, \quad k \geq 2.$$

Define x_3 by replacing the 0's in c_α by $(01)^\omega$ and by replacing the 1's with 2's. We have

$$s_1 = 01, \qquad s_2 = 01010, \qquad s_3 = (01010)^2 01, \ldots$$

and

$$c_\alpha = 010100101001010100101001010100101001010010 \cdots$$

and

$$x_3 = 021201202102120210212012021201202102120 \cdots .$$

- The critical exponent of c_α is $3+\sqrt{2}$.
- Our goal is to show that the critical exponent of x_3 is $2+\frac{\sqrt{2}}{2}$.

- Let $(z')^e$ be a repetition of exponent $e \geq 2$ in x_3 ($e \in \mathbb{Q}$).
- Apply the morphism that sends $\{0, 1\} \to 0$ and $2 \to 1$ to x_3.
- We see that there is a corresponding repetition z^e of the same length in c_α.

$$0212012\,0212012 \to 0101001\,0101001$$

- Suppose that z is primitive.
- z is either a conjugate of one of the standard words s_k defined above or a conjugate of one of the semi-standard words

$$s_{k,1} = s_{k-1}s_{k-2}, \quad k \geq 2.$$

The lengths of the standard and semi-standard words are given in terms of q_n, defined by:

$$\frac{p_n}{q_n} = [d_0, d_1, d_2, d_3, \ldots, d_n],$$

where

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_n = d_n p_{n-1} + p_{n-2} \text{ for } n \geq 0;$$
$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_n = d_n q_{n-1} + q_{n-2} \text{ for } n \geq 0.$$

$|s_n| = q_n$ and $|s_{n,1}| = q_{n-1} + q_{n-2}$

The convergents have the following approximation property:

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \qquad (1)$$

The following fact is classical:

$$\frac{q_{n+1}}{q_n} = [d_{n+1}, d_n, \ldots, d_1]. \qquad (2)$$

- We return to a repetition $(z')^e$ in x_3 and the corresponding z^e in c_α.
- Suppose that z is a conjugate of a standard word s_k.
- Note that $|s_k|_0$ is odd for every $k \geq 1$.
- Hence $|z|_0$ is odd.
- Recall: z' is obtained by replacing each 0 in z with 0 or 1 alternately and the 1's with 2's.
- It follows that $z'z'$ cannot occur in x_3 so there is no repetition $(z')^e$ in x_3.

$$010100101001\,010100101001 \to 021201202102\,120210212012$$

- Now suppose that z is a conjugate of a semi-standard word.
- Then $|z| = q_{k-2} + q_{k-1}$ for some $k \geq 2$.
- Justin and Pirillo (2001) gave precise technical results about the exponents of certain kinds of repetitions in Sturmian words.
- One of these results is that the longest factor of c_α with this period has length $2(q_{k-2} + q_{k-1}) + q_{k-1} - 2$.

It follows that for a repetition z^e where z is a conjugate of a semi-standard word,

$$\begin{aligned} e &\leq \frac{2(q_{k-2} + q_{k-1}) + q_{k-1} - 2}{q_{k-2} + q_{k-1}} \\ &= 2 + \frac{q_{k-1} - 2}{q_{k-2} + q_{k-1}} \\ &= 2 + \frac{q_{k-1}/q_{k-2} - 2/q_{k-2}}{1 + q_{k-1}/q_{k-2}}. \end{aligned}$$

Now by (2) we have that q_{k-1}/q_{k-2} converges to $[2, \overline{2}] = \sqrt{2} + 1$, and by (1) we have

$$q_{k-1}/q_{k-2} < \sqrt{2} + 1 + 1/q_{k-2}^2.$$

Thus, we have

$$e < 2 + \frac{\sqrt{2} + 1 + 1/q_{k-2}^2 - 2/q_{k-2}}{\sqrt{2} + 2 - 1/q_{k-2}^2}.$$

The fraction on the right clearly tends to $(\sqrt{2}+1)/(\sqrt{2}+2) = \sqrt{2}/2$ as $k \to \infty$, and is increasing for $k \geq 3$, so the convergence is from below. Thus $e < 2 + \sqrt{2}/2$.

Indeed, for every $k \geq 2$, there are repetitions z^e in the Sturmian word c_α with exponent

$$e = 2 + \frac{(q_{k-1} - 2)/q_{k-2}}{1 + q_{k-1}/q_{k-2}} \xrightarrow{k\to\infty} 2 + \frac{\sqrt{2}}{2},$$

where the convergence is from below.

Now if z is the conjugate of a semi-standard word $s_{k,1}$, we note that $|s_{k,1}|_0$ is even for every $k \geq 2$ and so every such repetition z^e in c_α gives rise to a repetition $(z')^e$ in x_3, since $|z|_0$ in this case is even.

$$0101001\,0101001\,010 \to 0212012\,0212012\,021$$

- Finally, suppose that z^e is a repetition where z is not primitive.
- From existing results we can calculate that the critical exponent of c_α is $3+\sqrt{2}$.
- So z cannot have exponent ≥ 3.
- Thus z is a square and we have

$$e < \frac{3+\sqrt{2}}{2} < 2 + \frac{\sqrt{2}}{2}.$$

- We conclude that $E(x_3) = 2 + \frac{\sqrt{2}}{2}$.

For larger alphabets, we construct balanced words according to this table.

k	α	c.f.	y	y'
3	$\sqrt{2}-1$	$[0,\overline{2}]$	$(01)^\omega$	2^ω
4	$1/\phi^2$	$[0,2,\overline{1}]$	$(01)^\omega$	$(23)^\omega$
5	$\sqrt{2}-1$	$[0,\overline{2}]$	$(0102)^\omega$	$(34)^\omega$
6	$(78-2\sqrt{6})/101$	$[0,1,2,1,1,\overline{1,1,1,2}]$	0^ω	$(123415321435)^\omega$
7	$(63-\sqrt{10})/107$	$[0,1,1,3,\overline{1,2,1}]$	$(01)^\omega$	$(234526432546)^\omega$
8	$(23+\sqrt{2})/31$	$[0,1,3,1,\overline{2}]$	$(01)^\omega$	$(2345267325462375264325 76)^\omega$
9	$(23-\sqrt{2})/31$	$[0,1,2,3,\overline{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$
10	$(109+\sqrt{13})/138$	$[0,1,4,2,\overline{3}]$	$(01)^\omega$	$(2345672849632547682943652748 69)^\omega$

Table: Periodic words y and y' for the construction of x_k

We have shown

$$E(x_3) = 2 + \frac{\sqrt{2}}{2} \approx 2.7071$$

We can also prove that

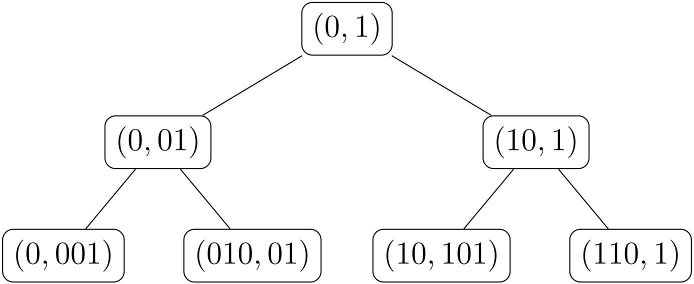
$$E(x_4) = 1 + \frac{\phi}{2} \approx 1.8090.$$

For $k \geq 5$, computer calculations suggest that

$$E(x_k) = \frac{k-2}{k-3}.$$

For values of k up to 9 (except for 4), we can show that these critical exponents are the smallest possible over k letters.

- We show optimality by a backtracking computer search over the tree of standard pairs.
- This is the tree of pairs with root $(0, 1)$ and each vertex (u, v) has children (u, uv) and (vu, v).


(0, 1)
(0, 01)
(10, 1)
(0, 001)
(010, 01)
(10, 101)
(110, 1)

- All finite balanced binary words appear in this tree.
- As we search, we take the longer of u, v and try all possible replacements of the 0's and 1's with constant gap sequences.
- If every replacement results in a word with too large a critical exponent, we backtrack.

- Why can't we prove the claimed critical exponents for $k \geq 5$?
- We have less information for repetitions z^e in Sturmian words that have exponent $e < 2$.
- We need to know the number of 0's and 1's in z modulo the periods of y and y' respectively.
- If $e \geq 2$, then z is an integer power of a conjugate of a standard or semi-standard word and we can count the number of 0's and 1's in such words.

- But if $e < 2$ then we cannot be sure that z is an integer power of a conjugate of a standard or semi-standard word.
- Saari showed that this is the case if z is the minimal fractional root of the repetition.
- But we have no information about non-minimal periods of factors of Sturmian words.
- So the main open problem is to prove the conjectured critical exponents for $k \geq 5$.

We can overcome this difficulty for the 4-letter alphabet with the following technical lemma, but this approach fails for larger alphabets.

Lemma

Let w be a factor of the Fibonacci word. Write $w = x^f$, where $f \in \mathbb{Q}$ and $|x|$ is the least period of w. Suppose that w has another representation $w = z^e$, where $e \in \mathbb{Q}$, z is primitive, and $|x| < |z|$. Then $e < 1 + \phi/2$.

So the main open problem is to prove the conjectured critical exponents for $k \geq 5$.

The End