Abelian Repetitions and Related Topics

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Abelian repetitions

Erdős 1961	abelian square: a word xx' such that x' is a permutation of x (like `reappear`)
Evdokimov 1968	abelian squares avoidable over 25 letters
Pleasants 1970	abelian squares avoidable over 5 letters
Justin 1972	abelian 5-powers avoidable over 2 letters
Dekking 1979	abelian 4-powers avoidable over 2 letters
	abelian cubes avoidable over 3 letters
Keränen 1992	abelian squares avoidable over 4 letters

Dekking's construction

- Define a map:

$$a \to aaab, \quad b \to abb.$$

- The limit of the sequence

$$a \to aaab \to aaabaaabaaababb \to \cdots$$

contains no abelian 4-power.

Avoiding patterns in the abelian sense

- Avoiding the pattern $xyyx$ in the abelian sense means avoiding all words $xyy'x'$ where x and x' (resp. y and y') are permutations of each other.
- What are the patterns that are avoidable in the abelian sense?

The Zimin patterns

- The Zimin patterns:

$$Z_1 = x, \quad Z_2 = xyx, \quad Z_3 = xyxzxyx, \quad \ldots$$

Conjecture (Currie and Linek 2001)

A pattern p with n variables is avoidable in the abelian sense if and only if Z_n avoids the pattern p in the abelian sense.

Avoiding long binary patterns

Theorem (Currie and Visentin 2008)

Any pattern over $\{x, y\}$ of length greater than 118 is avoidable in the abelian sense over a 2-letter alphabet.

The proof technique

- Let $\mathbf{w}$ be the infinite word

$$aaaabaaaabaaaabaaaababbbaaaab\cdots$$

generated by the map $a \to aaaab$ and $b \to abbb$.

- Define a weight function $f : \{a, b\}^* \to \mathbb{Z}_{11}$ by $f(a) = 3$, $f(b) = 10$, and $f(xy) = f(x) + f(y)$ for words x, y.
- Observe:

$$\{f(U) : U \text{ a prefix of } \mathbf{w}\} = \{0, 1, 2, 3, 6, 9\}.$$

An example (due to J. Currie)

- $\mathbf{w}$ contains the pattern $xyxyyxy$ in the abelian sense:

$$aaaabaaaabaaaabaaaababbbaaaab\cdots$$

- Write $\mathbf{w} = UX_1Y_1X_2Y_2Y_3X_3Y_4\mathbf{w}'$ (here $U = \epsilon$).
- $f(X_1) = f(X_2) = f(X_3) = 9$
- $f(Y_1) = f(Y_2) = f(Y_3) = f(Y_4) = 5$

Computing the prefix weights

Prefix weights:

$$f(U) = 0, \quad f(UX_1) = 9, \quad f(UX_1Y_1) = 3,$$

$$f(UX_1Y_1X_2) = 1, \quad f(UX_1Y_1X_2Y_2) = 6,$$

$$f(UX_1Y_1X_2Y_2Y_3) = 0, \quad f(X_1Y_1X_2Y_2Y_3X_3) = 9,$$

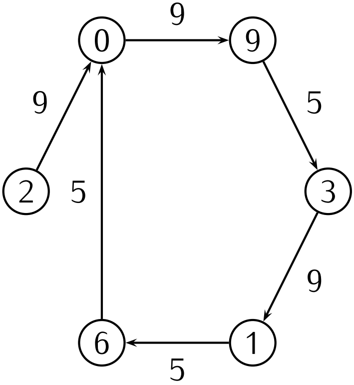
$$f(X_1Y_1X_2Y_2Y_3X_3Y_4) = 3$$

A graph-theoretic viewpoint

View as a walk on the graph $G(\{5,9\})$ with

- vertex set $\{0,1,2,3,6,9\}$, and
- edge set $\{(i,j) : j-i \pmod{11} \in \{5,9\}\}$.

The graph $G(\{5, 9\})$



Concluding that long patterns are avoidable

- If **w** contains p, then p can be walked on some $G(\{a, b\})$.
- If p contains $xxxx$, it is avoidable.
- Show: all long patterns walkable on some $G(\{a, b\})$ contain $xxxx$.
- Technical point: all long patterns should be walkable on $G(\{a, b\})$ with a, b non-zero.

Avoiding long abelian squares

Theorem (Entringer, Jackson, Schatz 1974)

Any binary word of length $k^2 + 6k$ contains an abelian square xx' with $|x| \geq k$.

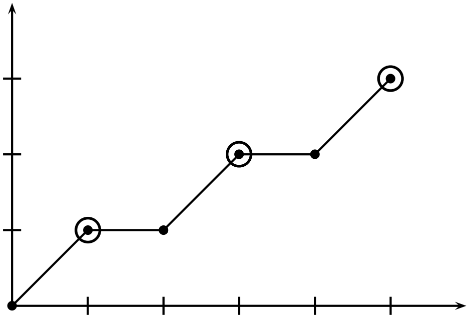
A lower bound

Theorem (Grant 2008)

Let q be a positive integer. There exists a binary word of length $q(q+1)$ which contains no abelian squares xx' with $|x| \geq \sqrt{2q(q+1)}$.

Geometrical interpretation

- Let w be a word. Let $S_i = \sum_{j=1}^{i} w[j]$. Plot (i, S_i).
- abelian square $\Rightarrow$ 3 equidistant collinear points.
- For $w = 10101$:



The construction

- Construct a sequence whose graph approximates a quadratic function.
- For $0 \leq i \leq q(q+1)$, define

$$a_i = \left\lfloor \frac{i^2}{2q(q+1)} \right\rfloor.$$

- Note: $a_i - a_{i-1} \in \{0, 1\}$ for $1 \leq i \leq q(q+1)$.
- Define w by $w_i = a_i - a_{i-1}$ for $1 \leq i \leq q(q+1)$.

An example for $q = 3$

$$
\begin{aligned}
(a_i) &= (\lfloor 1/24 \rfloor, \lfloor 4/24 \rfloor, \lfloor 9/24 \rfloor, \ldots, \lfloor 144/24 \rfloor) \\
&= (0, 0, 0, 0, 0, 1, 1, 2, 2, 3, 4, 5, 6)
\end{aligned}
$$

$$
(a_i - a_{i-1}) = (0, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1)
$$

$$
w = 000010101111
$$

Avoiding long abelian squares and cubes

Problem (Mäkelä 2002)

Is there a constant c such that the set of abelian cubes $xx'x''$ with $|x| \geq c$ is avoidable over a 2-letter alphabet?

Is there a constant d such that the set of abelian squares xx' with $|x| \geq d$ is avoidable over a 3-letter alphabet?

A variation

Theorem (Gerver and Ramsay 1979)

Let S be any set of non-collinear vectors in $\mathbb{Z}^2$. Any infinite S-walk contains arbitrarily large sets of collinear points.

A variation

Theorem (Gerver and Ramsay 1979)

Let S be any set of non-coplanar vectors in $\mathbb{Z}^3$. There is an infinite S-walk for which no $5^{11} + 1$ points are collinear.

Can this be improved?

Abelian complexity

- `roasting` and `organist` are abelian equivalent because they are anagrams.
- Let **w** be an infinite word.
- Define $\rho^{\mathrm{ab}}(n) =$ the number of abelian equivalence classes of words of length n in **w**.
- This is the abelian complexity function of **w**.

The binary case

Theorem (Coven and Hedlund 1973)

A recurrent infinite word has abelian complexity $\rho^{\mathrm{ab}}(n) = 2$ for all $n \geq 1$ if and only if it is Sturmian.

The ternary case

Problem (Rauzy 1982/83)

Does there exist a recurrent infinite word with abelian complexity $\rho^{\mathrm{ab}}(n) = 3$ for all $n \geq 1$?

Richomme, Saari, and Zamboni answered Rauzy's question in the affirmative.

The 4-letter case

Problem (Richomme, Saari, Zamboni 2009)

Does there exist a recurrent infinite word with abelian complexity $\rho^{\mathrm{ab}}(n) = 4$ for all $n \geq 1$?

Larger alphabets

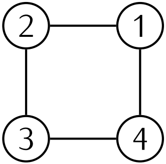
Theorem (Currie and R. 2009)

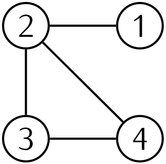
Let $k \geq 4$. There is no recurrent infinite word with abelian complexity $\rho^{\text{ab}}(n) = k$ for all $n \geq 1$.

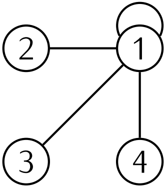
The idea of the proof

- Suppose w is a recurrent word over a 4-letter alphabet with constant abelian complexity 4.
- Define a graph G with:
  - vertex set $\{1, 2, 3, 4\}$, and
  - edges ij if either ij or ji is a factor of w.
- G is a spanning tree with one extra edge.

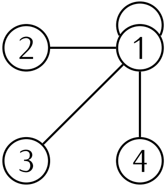
Examples of the graph G







A particular case



- Suppose 111 a factor of **w**.
- Note: $213, 214, 312$, etc. not factors.
- Let m be minimal such that $d1^me$ is a factor (wlog 21^m3 is a factor).

A particular case

- Then

$$121^m3,\ 1^m21^2,\ 1^m31^2,\ 1^m41^2$$

  are the 4 factors of length $m+3$ (up to permutation).
- Note: 1^{m+2} not a factor.
- Consider a factor $b1^kc$ where $b \in \{2,3\}$ and $c = 4$.
- Either $k = m$ or $k = m+1$. Both cases give a 5th factor. Contradiction.
- There are many other cases.

Conclusion

- Many questions already resolved for ordinary repetitions remain open for abelian repetitions.
- Problems regarding abelian repetitions are much more difficult.
- Abelian complexity: a relatively recent concept. Probably much more can be done in this area.

The End