Math Physics Assignment

(Nov 4/05)

Due: Nov 14/05

60/60

From the book

① 5,87 –

② 5,89 –

③ 5.90 –

④ 5,96 –

⑤ 5,97 –

+ see attached sheets.

⑥ Show that the following three forms of $\nabla^2 \psi(r)$ in spherical polar coordinates are equivalent:

(a) $\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d\psi(r)}{dr}\right]$ (b) $\frac{1}{r}\frac{d^2}{dr^2}\left[r\psi(r)\right]$

(c) $\frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr}$

⑦ Consider the function : $f(r) = r^n$
Calculate ∇f, $\nabla\cdot(\hat{r} f(r))$, $\nabla^2 f$, $\nabla\times(\hat{r} f)$

⑧ A rigid body is rotating about a fixed axis with a constant angular velocity $\vec{\omega}$. Take $\vec{\omega}$ to be along $\hat{z}$. Express $\vec{r}$ in cylindrical coordinates and find

(a) $\vec{v} = \vec{\omega}\times\vec{r}$

(b) $\nabla\times\vec{v}$

(9) A particle m moves in response to a central force according to Newton's second law

$$m \ddot{\vec{r}} = \hat{r}\, f(r)$$

— (a) Show that this implies $\vec{r} \times \dot{\vec{r}} = \vec{C} = \text{constant}$

— (b) Argue coherently that this leads to Kepler's 2nd Law (look it up) for area of an orbit being swept out in time.

— (c) Deduce that the particle involved must move in a plane.

(10)

— (a) Express $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ in spherical polar coordinates. Hint: Think about equating ∇_{xyz} and $\nabla_{r\theta\phi}$.

— (b) Show that

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial \phi}$$

In Q.M. this is the z-component of angular momentum of a particle.

— (c) In Quantum Mechanics the angular momentum operator is $\vec{L} = -i\,(\vec{r} \times \nabla)$. Show that this implies

$$L_x + i L_y = e^{i\phi}\left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi}\right)$$

and

$$L_x - i L_y = -e^{-i\phi}\left(\frac{\partial}{\partial \theta} - i \cot\theta \frac{\partial}{\partial \phi}\right)$$

(11)

Problem on angles (2-D and 3-D)

From the time that the lower limb of the Sun touches the horizon, it takes approximately 2 minutes for the Sun to disappear beneath the horizon.

(a) Approximately what angle (expressed in <u>degrees</u> and <u>radians</u>) does the diameter of the Sun subtend at the Earth? Use a 2-D polar coordinates-style diagram and use the symbol $d\theta$ for the required angle. Is it consistent to treat $d\theta$ like an infinitesimal angle here?

(b) At what distance from your eye does a coin of about 3/4 inch diameter (e.g. dime or nickel) just block out the disk of the Sun?

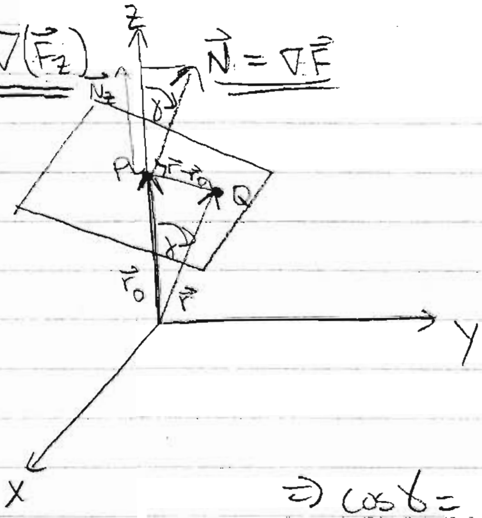
(c) What solid angle (in steradians) does the Sun subtend at the EARTH?

[All approximations must be justified in a quantitative style]

① 5.87 Prove acute angle γ between z axis and normal to surface $F(x,y,z)=0$ is $\sec\gamma = \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|}$

$\vec{F} = F(x,y,z) = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$ $\quad \vec{F}_z = F_z\hat{z}$

$\vec{N}_z = \nabla(\vec{F}_z)$ $\quad \vec{N} = \nabla\vec{F}$



$$\cos\gamma = \frac{|\vec{N}_z|}{|\vec{N}|} = \frac{|\nabla\vec{F}_z|}{|\nabla\vec{F}|}$$

$$\cos\gamma = \frac{\cancel{\nabla}|\vec{F}_z|}{\cancel{\nabla}|\vec{F}|}$$

$$|\vec{F}| = \sqrt{F_x^2+F_y^2+F_z^2}$$

$$\Rightarrow \cos\gamma = \frac{|\vec{F}_z|}{\sqrt{F_x^2+F_y^2+F_z^2}}$$

$$\Rightarrow \frac{1}{\cos\gamma} = \boxed{\sec\gamma = \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|\vec{F}_z|}}$$

2/2

② 5.89

Given $\vec{E}$ and $\vec{H}$ (vectors with continuous partial derivatives w.r.t. to position and time.

and $\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{H} = 0 \quad \nabla \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{H}}{\partial t} \quad \nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

Prove $\vec{E}$ and $\vec{H}$ satisfy $\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$

$$\nabla \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{H}}{\partial t} \Rightarrow \nabla \times (\nabla \times \vec{E}) = \nabla \times \left(\frac{-1}{c}\frac{\partial \vec{H}}{\partial t}\right)$$

$$= \frac{-1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{H})$$

but $\nabla \times \vec{H} = \frac{1}{c}\frac{\partial E}{\partial t} \Rightarrow \nabla \times (\nabla \times \vec{E}) = \frac{-1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$

also $\nabla \times (\nabla \times \vec{E}) = \nabla(\underbrace{\nabla \cdot \vec{E}}_{=0 \text{ since } \nabla \cdot \vec{E} = 0}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$

$$\therefore \boxed{\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}}$$

2/2

Similarly, $\nabla \times (\nabla \times \vec{H}) = \nabla \times \left(\frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right) = \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{E})$

$$= \frac{-1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2}$$

also $\nabla \times (\nabla \times \vec{H}) = \nabla(\underbrace{\nabla \cdot \vec{H}}_{=0 \text{ since } \nabla \cdot \vec{H} = 0}) - \nabla^2 \vec{H} = -\nabla^2 \vec{H}$

$$\therefore \boxed{\nabla^2 \vec{H} = \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2}}$$

2/2

$\therefore$ $\vec{E}$ and $\vec{H}$ satisfy the wave equation $\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$ where

$\vec{E}$ and $\vec{H}$ are electric and magnetic field vectors corresponding to a wave ψ propagating in free space at speed c.

③ 5.90

Using $\nabla\cdot\vec{E}=0$ $\quad \nabla\cdot\vec{H}=0$ $\quad \nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t}$ $\quad \nabla\times\vec{H}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

Show $\frac{\partial}{\partial t}\left\{\frac{1}{2}(E^2+H^2)\right\}+c\nabla\cdot(\vec{E}\times\vec{H})=0$

$$c\nabla\cdot(\vec{E}\times\vec{H})=c\vec{H}\cdot(\nabla\times\vec{E})-c\vec{E}\cdot(\nabla\times\vec{H})$$

$$=c\vec{H}\cdot\left(-\frac{1}{c}\frac{\partial\vec{H}}{\partial t}\right)-c\vec{E}\cdot\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$c\nabla\cdot(\vec{E}\times\vec{H})=-\left(\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)$$

$$\frac{\partial(\vec{H}\cdot\vec{H})}{\partial t}=\frac{\partial H^2}{\partial t}=\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}+\frac{\partial\vec{H}}{\partial t}\cdot\vec{H}=2\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}\Rightarrow\boxed{\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}=\frac{1}{2}\frac{\partial H^2}{\partial t}}$$

$$\frac{\partial(\vec{E}\cdot\vec{E})}{\partial t}=\frac{\partial E^2}{\partial t}=\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}=2\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\Rightarrow\boxed{\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}=\frac{1}{2}\frac{\partial E^2}{\partial t}}$$

$$\Rightarrow c\nabla\cdot(\vec{E}\times\vec{H})=-\frac{\partial}{\partial t}\left\{\frac{1}{2}(E^2+H^2)\right\}$$

$$\Rightarrow\frac{\partial}{\partial t}\left\{\frac{1}{2}(E^2+H^2)\right\}-\frac{\partial}{\partial t}\left\{\frac{1}{2}(E^2+H^2)\right\}=0$$

$$\therefore\boxed{\frac{\partial}{\partial t}\left\{\frac{1}{2}(E^2+H^2)\right\}+c\nabla\cdot(\vec{E}\times\vec{H})=0}$$

4/4

(4) 5.96

In cartesian coords: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

(a)

Transformation Eq'ns for Cylindrical Co-ords

$$\boxed{\begin{array}{ll} \hat{x} = \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi} & x = \rho\cos\phi \\ \hat{y} = \sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi} & y = \rho\sin\phi \\ \hat{z} = \hat{z} & z = z \end{array}}$$ plug into $\vec{r}$

$$\vec{r} = \rho\cos\phi(\cos\phi\hat{\rho} - \sin\phi\hat{\phi}) + \rho\sin\phi(\sin\phi\hat{\rho} + \cos\phi\hat{\phi}) + z\hat{z}$$

$$= \rho\hat{\rho}\underbrace{(\cos^2\phi + \sin^2\phi)}_{=1} + \rho\hat{\phi}\underbrace{(\sin\phi\cos\phi - \sin\phi\cos\phi)}_{=0} + z\hat{z}$$

$$\vec{r} = \rho\hat{\rho} + z\hat{z} \Rightarrow \frac{d\vec{r}}{dt} = \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \frac{dz}{dt}\hat{z} + z\cancel{\frac{d\hat{z}}{dt}}^{0}$$

$$\hat{\rho} = \cos\phi\hat{x} + \sin\phi\hat{y} \Rightarrow \frac{d\hat{\rho}}{dt} = \hat{x}\frac{d\cos\phi}{dt} + \cos\phi\cancel{\frac{d\hat{x}}{dt}}^{0} + \hat{y}\frac{d\sin\phi}{dt} + \sin\phi\cancel{\frac{d\hat{y}}{dt}}^{0}$$

$$\frac{d\hat{\rho}}{dt} = -\hat{x}\sin\phi\dot{\phi} + \hat{y}\cos\phi\dot{\phi} = \underline{\underline{\dot{\phi}\hat{\phi}}}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \Rightarrow \frac{d\hat{\phi}}{dt} = -\hat{x}\cos\phi\dot{\phi} - \hat{y}\sin\phi\dot{\phi} = \underline{\underline{-\dot{\phi}\hat{\rho}}}$$

Plugging in $\dot{\hat{\rho}} = \dot{\phi}\hat{\phi} \Rightarrow \frac{d\vec{r}}{dt} = \dot{\rho}\hat{\rho} + (\rho\dot{\phi})\hat{\phi} + \dot{z}\hat{z}$

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\hat{\rho}} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\dot{\hat{\phi}} + \ddot{z}\hat{z} + \cancel{\dot{z}\dot{\hat{z}}}^{0}$$

plug in $\dot{\hat{\rho}} = \dot{\phi}\hat{\phi}$ and $\dot{\hat{\phi}} = -\dot{\phi}\hat{\rho}$

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} - \rho\dot{\phi}\dot{\phi}\hat{\rho} + \ddot{z}\hat{z}$$

$$= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

(or, using notation $\hat{\rho} = \hat{e}_\rho$, $\hat{\phi} = \hat{e}_\phi$, $\hat{z} = \hat{e}_z$)

$$\boxed{\frac{d^2\vec{r}}{dt^2} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{e}_\rho + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z}$$

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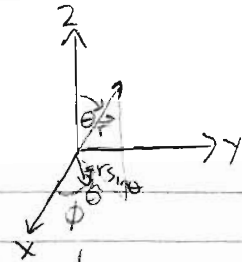

(b) In Spherical Co-ordinates:

$$\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$
$$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$
$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\vec{r} = r\hat{r} \Rightarrow \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\dot{\hat{r}} = \frac{d\sin\theta}{dt}\cos\phi\,\hat{x} + \sin\theta\frac{d\cos\phi}{dt}\hat{x} + \cancel{\sin\theta\cos\phi\frac{d\hat{x}}{dt}}^{\,0}$$
$$+ \frac{d\sin\theta}{dt}\sin\phi\,\hat{y} + \sin\theta\frac{d\sin\phi}{dt}\hat{y} + \cancel{\sin\theta\sin\phi\frac{d\hat{y}}{dt}}^{\,0}$$
$$+ \frac{d\cos\theta}{dt}\hat{z} + \cancel{\cos\theta\frac{d\hat{z}}{dt}}^{\,0}$$
$$= \hat{x}(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \hat{y}(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) + \hat{z}(-\dot{\theta}\sin\theta)$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$
$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

$$\Rightarrow \dot{\hat{r}} = \dot{\theta}(\hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta) + \dot{\phi}\sin\theta(-\hat{x}\sin\phi + \hat{y}\cos\phi)$$

(1) $$\Rightarrow \boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\hat{\phi}} \Rightarrow \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\,\hat{\phi}$$

(*) $$\boxed{\begin{aligned}\ddot{\vec{r}} = {} & \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} \\ & + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} + r\dot{\phi}\sin\theta\dot{\hat{\phi}}\end{aligned}}$$

$$\dot{\hat{\theta}} = \cancel{\dot{\hat{x}}\cos\theta\cos\phi}^{\,0} - \hat{x}\dot{\theta}\sin\theta\cos\phi - \hat{x}\dot{\phi}\cos\theta\sin\phi$$
$$+ \cancel{\dot{\hat{y}}\cos\theta\sin\phi} - \hat{y}\dot{\theta}\sin\theta\sin\phi + \hat{y}\dot{\phi}\cos\theta\cos\phi$$
$$- \cancel{\dot{\hat{z}}\sin\theta} - \hat{z}\dot{\theta}\cos\theta$$

$$= -\dot{\theta}\underbrace{(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)}_{\hat{r}} + \dot{\phi}\cos\theta\underbrace{(-\hat{x}\sin\phi + \hat{y}\cos\phi)}_{\hat{\phi}}$$

(2) $$\boxed{\dot{\hat{\theta}} = -\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\hat{\phi}}$$

$$\dot{\hat{\phi}} = -\cancel{\dot{\hat{x}}}^{0}\sin\phi - \hat{x}\dot{\phi}\cos\phi + \cancel{\dot{\hat{y}}}^{0}\cos\phi - \hat{y}\dot{\phi}\sin\phi$$

$$\dot{\hat{\phi}} = -\dot{\phi}\left(\hat{x}\cos\phi + \hat{y}\sin\phi\right)$$

$$= -\dot{\phi}\Big(\hat{r}(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi) + \hat{\theta}(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \hat{\phi}\underbrace{(-\sin\phi\cos\phi + \cos\phi\sin\phi)}_{0}\Big)$$

(3) $\Rightarrow \boxed{\dot{\hat{\phi}} = -\dot{\phi}\sin\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}}$

Plugging (1), (2) and (3) into (*)

$$\Rightarrow \ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} + r\dot{\theta}\dot{\phi}\cos\theta\hat{\phi} + \dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\hat{\phi} - r\dot{\phi}^2\sin^2\theta\,\hat{r} - r\dot{\phi}^2\cos\theta\sin\theta\,\hat{\theta}$$

$$\Rightarrow \boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\cos\theta\sin\theta)\hat{\theta} + (2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\hat{\phi}}$$

for the book's notation, $\hat{r} = \hat{e}_r$, $\hat{\theta} = \hat{e}_\theta$, $\hat{\phi} = \hat{e}_\phi$

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⑤ 5.97

If $F = x + 3y^2 - z^3$, $G = 2x^2yz$, $H = 2z^2 - xy$

evaluate $\dfrac{\partial(F,G,H)}{\partial(x,y,z)}$ at $(1,-1,0)$ $x=1,\ y=-1,\ z=0$

$$\frac{\partial(F,G,H)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix} \quad \left\{ \begin{array}{lll} \frac{\partial F}{\partial x} = 1 & \frac{\partial F}{\partial y} = 6y & \frac{\partial F}{\partial z} = -3z^2 \\ \frac{\partial G}{\partial x} = 4xyz & \frac{\partial G}{\partial y} = 2x^2z & \frac{\partial G}{\partial z} = 2x^2 \\ \frac{\partial H}{\partial x} = -y & \frac{\partial H}{\partial y} = -x & \frac{\partial H}{\partial z} = 4z \end{array} \right.$$

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ 4xyz & 2x^2z & 2x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= 1(2x^2z^2 + 2x^3y) - 6y(4xyz^2 + 2x^2y^2) - 3z^2(-4x^2yz + 2x^2yz)$$

$$= 2x^2z^2 + 2x^3y - 24xy^2z^2 - 12x^2y^3 + 12x^2yz^3 - 6x^2yz^3$$

$$= \underbrace{2(1)^2(0)^2}_{0} + 2(1)^3(-1) - \underbrace{24(1)(-1)^2(0)^2}_{0} - 12(1)^2(-1)^3 + \underbrace{12(1)^2(-1)(0)^3}_{0} + \underbrace{6(1)^2(-1)(0}_{0}$$

$$\boxed{\frac{\partial(F,G,H)}{\partial(x,y,z)} = 10 \quad \text{at } (1,-1,0)}$$

2/2

6

Show the following three forms of $\nabla^2 \psi(r)$ in spherical polar coordinates are equivalent.

(a) $\dfrac{1}{r^2}\dfrac{d}{dr}\left[r^2 \dfrac{d\psi(r)}{dr}\right]$ (b) $\dfrac{1}{r}\dfrac{d^2}{dr^2}\left[r\psi(r)\right]$

(c) $\dfrac{d^2\psi(r)}{dr^2} + \dfrac{2}{r}\dfrac{d\psi(r)}{dr}$

(a) $$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi(r)}{dr}\right] = \frac{1}{r^2}\left(r^2\frac{d^2\psi(r)}{dr^2} + 2r\frac{d\psi(r)}{dr}\right)$$

$$= \frac{d^2\psi(r)}{dr^2} + \frac{2}{r}\frac{d\psi(r)}{dr} = (c)$$

$\therefore$ (a) = (c)

(b) $$\frac{1}{r}\frac{d^2}{dr^2}\left[r\psi(r)\right] = = \frac{1}{r}\frac{d}{dr}\left[\psi(r) + r\frac{d\psi(r)}{dr}\right]$$

$$= \frac{1}{r}\left[\frac{d\psi(r)}{dr} + \left(\frac{d\psi(r)}{dr} + r\frac{d^2\psi(r)}{dr^2}\right)\right]$$

$$= \frac{2}{r}\frac{d\psi(r)}{dr} + \frac{d^2\psi(r)}{dr^2} = (c)$$

$\therefore$ (b) = (c) 4/4

$\therefore$ $\boxed{\text{since (a)=(c) and (b)=(c)} \Rightarrow \text{(a)=(b)=(c)}}$

(7) Consider $f(r) = r^n$
(using Spherical coordinates)

(a) $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\cancel{\frac{\partial f}{\partial \theta}}\hat{\theta} + \frac{1}{r\sin\theta}\cancel{\frac{\partial f}{\partial \phi}}\hat{\phi}$

$\boxed{\nabla f = n r^{n-1}\hat{r}}$ 2/2

(b) $\nabla\cdot(\hat{r}f(r)) = \frac{1}{r^2}\frac{\partial(r^2 f(r))}{\partial r} + 0 + 0$

$= \frac{1}{r^2}\frac{\partial(r^{n+2})}{\partial r} = \frac{(n+2)(r^{n+1})}{r^2} = \boxed{(n+2)r^{n-1}}$ 2/2

(c) $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cancel{\frac{\partial f}{\partial\theta}}\right) + \frac{1}{r^2\sin^2\theta}\cancel{\frac{\partial^2 f}{\partial\phi^2}}$

0 0

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(n r^2 r^{n-1}\right) = \frac{n}{r^2}\frac{\partial}{\partial r}\left(r^{n+1}\right) = \boxed{n(n+1)r^{n-2}}$ 2/2

(d) $\nabla\times(\hat{r}f) = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ r^n & 0 & 0\end{vmatrix} = \left[-r\hat{\theta}\left(-\cancel{\frac{\partial}{\partial\phi}} r^n\right) + r\sin\theta\,\hat{\phi}\left(-\cancel{\frac{\partial r^n}{\partial\theta}}\right)\right]\cdot\left(\frac{1}{r^2\sin\theta}\right)$

0 0

$\Rightarrow \boxed{\nabla\times(\hat{r}f) = 0}$ 2/2

(8) rigid body rotating on fixed axis with a constant angular velocity, $\vec{\omega}$ along $\hat{z}$ $\Rightarrow$ $\vec{\omega} = \omega\hat{z}$

$\vec{r} = \rho\hat{\rho} + z\hat{z}$ in cylindrical coordinates where $\rho \equiv$ radius of cylinder

$z \equiv$ point along z-axis

(a) $\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 0 & 0 & \omega \\ \rho & 0 & z \end{vmatrix} = \boxed{\omega\rho\hat{\phi}}$ 2/2

(b) $\nabla \times \vec{v} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\phi} & \frac{1}{\rho}\hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \rho(\omega\rho) & 0 \end{vmatrix} = \cancel{\frac{1}{\rho}\hat{\rho}\left(-\frac{\partial}{\partial z}\omega\rho^2\right)}^{\to 0} + \frac{1}{\rho}\hat{z}\left(\frac{\partial}{\partial\rho}\omega\rho^2\right)$

$= \boxed{2\omega\hat{z} = 2\vec{\omega}}$ 2/2

(9)

A particle m moves in response to a central force according to Newton's second law:

$$m\ddot{\vec{r}} = \hat{r} f(r) \quad \text{where} \quad f(r) = -\frac{Gm_1 m_2}{r^2}$$

(a) in 2-D polar coords, $\vec{r} = r\hat{r}$

$$\dot{\vec{r}} = \dot{r}\hat{r} + (r\dot{\phi})\hat{\phi}$$

$$m\ddot{\vec{r}} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

$$\Rightarrow m(\ddot{r} - r\dot{\phi}^2)\hat{r} + \underbrace{m(r\ddot{\phi} + 2\dot{r}\dot{\phi})}_{=0}\hat{\phi} = \left(-\frac{Gm_1 m_2}{r^2}\right)\hat{r}$$

$$\Rightarrow r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

$$\Rightarrow \frac{d}{dt}(r^2\dot{\phi}) = 0 \quad \text{but} \quad \frac{d}{dt}(\text{const}) = 0 \Rightarrow \boxed{r^2\dot{\phi} = C = \text{constant}}$$

$$\vec{r}\times\dot{\vec{r}} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ r & 0 & 0 \\ \dot{r} & r\dot{\phi} & 0 \end{vmatrix} = r^2\dot{\phi}\hat{z}$$

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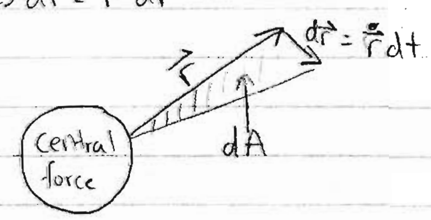
but it was already shown $r^2\dot{\phi} = \text{constant}$,

$$\therefore \boxed{\vec{r}\times\dot{\vec{r}} = r^2\dot{\phi}\hat{z} = C = \text{constant}}$$

(b) We have shown that $\vec{r}\times\dot{\vec{r}} = C = \text{constant}$

$\Rightarrow$ angular momentum, $\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times m\dot{\vec{r}} = m(\vec{r}\times\dot{\vec{r}}) = \text{constant}$

$\frac{d\vec{r}}{dt} = \dot{\vec{r}} \Rightarrow d\vec{r} = \dot{\vec{r}}\,dt$



because $\vec{L} = \text{const}$, the particle's motion is restricted to the plane formed by $\vec{r}$ and $\vec{v}\,(\dot{\vec{r}})$

$$dA = \frac{1}{2}|\vec{r}\times d\vec{r}| = \frac{1}{2}|\vec{r}\times\dot{\vec{r}}\,dt| \Rightarrow \frac{dA}{dt} = \frac{1}{2}|\vec{r}\times\dot{\vec{r}}| = \frac{1}{2}C = \text{constant}$$

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but $\frac{dA}{dt} = \text{const} \Rightarrow$ the radius vector from the central force to a particle sweeps out equal areas in equal time intervals

$\Rightarrow$ Kepler's 2nd Law

(c) angular momentum, $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\dot{\vec{r}} = m(\vec{r} \times \dot{\vec{r}}) = mC = \text{constant}$

but $\vec{L} = mC = mr^2\dot{\phi}\,\hat{z} = \text{constant} \Rightarrow$ there is no torque on the particle in the $\hat{z}$ direction caused by the central force.
This, in turn, implies the particle must move in a plane and is restricted to the $\hat{r}$ and $\hat{\phi}$ directions.

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(10)

(a) Express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical polar coordinates.

$$\nabla_{xyz} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla_{r\theta\phi} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Spherical unit vectors :

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

plugging in...

$$\nabla_{r\theta\phi} = (\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta)\frac{\partial}{\partial r}$$
$$+ (\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta} + (-\hat{i}\sin\phi + \hat{j}\cos\phi)\frac{1}{r\sin\theta}$$

$$\nabla_{r\theta\phi} = \hat{i}\left[\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$
$$+ \hat{j}\left[\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right]$$
$$+ \hat{k}\left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]$$

identify with ∇_{xyz} $\Rightarrow$

$$\boxed{\begin{aligned}\frac{\partial}{\partial x} &= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial y} &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\end{aligned}}$$

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(10) (b)

In spherical coordinates: $x = r\cos\phi\sin\theta$ $\quad y = r\sin\phi\sin\theta$

see (a) for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

$$\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = r\cos\phi\sin\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$- r\sin\phi\sin\theta\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \cancelto{0}{\left(r\cos\phi\sin^2\theta\sin\phi - r\cos\phi\sin^2\theta\sin\phi\right)}\frac{\partial}{\partial r} + \cancelto{0}{\left(\cos\phi\cos\theta\sin\theta\sin\phi - \cos\phi\cos\theta\sin\theta\sin\phi\right)}\frac{\partial}{\partial\theta}$$

$$+ \left(\cos^2\phi + \sin^2\phi\right)\frac{\partial}{\partial\phi} \quad \left(\text{but } \cos^2\phi + \sin^2\phi = 1\right)$$

$$= \frac{\partial}{\partial\phi} \Rightarrow \boxed{-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial\phi}}$$

M/M

(10) (c)

In Q.M. $\vec{L} = -i(\vec{r}\times\nabla)$ $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{r}\times\nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \hat{j}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \hat{k}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\Rightarrow L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

Spherical Coords:

$$\begin{bmatrix} x = r\cos\phi\sin\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\theta \end{bmatrix}$$

see (a) for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

$$L_x = -i\left[r\sin\phi\sin\theta\left(\cancel{\cos\theta\frac{\partial}{\partial r}} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + r\cos\theta\left(\cancel{\sin\theta\sin\phi\frac{\partial}{\partial r}} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right]$$

$$L_x = -i\left[\left(-\sin\phi\sin^2\theta - \sin\phi\cos^2\theta\right)\frac{\partial}{\partial\theta} - \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right] \qquad \left\{\frac{\cos\theta}{\sin\theta} = \cot\theta \quad \sin^2\theta + \cos^2\theta = 1\right\}$$

$$\boxed{L_x = i\left[\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right]}$$

$$L_y = -i\left[r\cos\theta\left(\cancel{\sin\theta\cos\phi\frac{\partial}{\partial r}} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - r\cos\phi\sin\theta\left(\cancel{\cos\theta\frac{\partial}{\partial r}} - \frac{\sin\theta}{r}\frac{\partial}{\theta}\right)\right]$$

$$L_y = -i\left[\left(\cos^2\theta\cos\phi + \sin^2\theta\cos\phi\right)\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$\boxed{L_y = -i\left[\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right]}$$

$$\Rightarrow L_x + iL_y = \left(i\sin\phi + \cos\phi\right)\frac{\partial}{\partial\theta} + \left(i\cos\phi\cot\theta - \sin\phi\cot\theta\right)\frac{\partial}{\partial\phi}$$

$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \qquad \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

$$\Rightarrow L_x + iL_y = \left(\frac{e^{i\phi} - e^{-i\phi}}{2} + \frac{e^{i\phi} + e^{-i\phi}}{2}\right)\frac{\partial}{\partial\theta} + \left[i\left(\frac{e^{i\phi} + e^{-i\phi}}{2}\right) - \left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)\right]\cot\theta\frac{\partial}{\partial\phi}$$

$$= e^{i\phi}\frac{\partial}{\partial\theta} + \frac{-e^{i\phi} - \cancel{e^{-i\phi}} - e^{i\phi} + \cancel{e^{-i\phi}}}{2i}\cot\theta\frac{\partial}{\partial\phi}$$

This will leave a -ve sgn. The trick is to multiply by $i^4 = 1$.

$$= e^{i\phi}\frac{\partial}{\partial\theta} + \frac{-e^{i\phi}}{i}\left(\frac{i}{i}\right)\cot\theta\frac{\partial}{\partial\phi}$$

$\frac{1}{i}(i^4) = i^3 = -i$

$$\Rightarrow \boxed{L_x + iL_y = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)}$$

$$L_x - iL_y = (i\sin\phi - \cos\phi)\frac{\partial}{\partial\theta} + (i\cos\phi\cot\theta + \sin\phi\cot\theta)\frac{\partial}{\partial\phi}$$

$$= -e^{-i\phi}\frac{\partial}{\partial\theta} + \left(\frac{-\cancel{e^{i\phi}} - e^{-i\phi} + \cancel{e^{i\phi}} - e^{-i\phi}}{2i}\right)\cot\theta\frac{\partial}{\partial\phi}$$

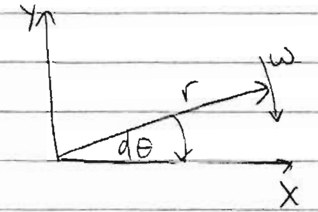
$$= -e^{-i\phi}\frac{\partial}{\partial\theta} + \frac{-e^{-i\phi}}{i}\left(\frac{i}{i}\right)\cot\theta\frac{\partial}{\partial\phi}$$

$$\Rightarrow \boxed{L_x - iL_y = -e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)}$$

3/3

(11)

(a)



given $dt = 2\,\text{min}$

angular velocity of Earth, $\omega = \dfrac{2\pi}{24\,\text{hr}} = \dfrac{2\pi}{1440\,\text{min}}$

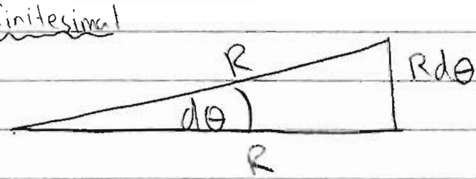
$$d\theta = \omega\,dt = \left(\frac{2\pi}{1440}\right)(2) = \frac{4\pi}{1440} = \frac{\pi}{360} = \boxed{8.73\times10^{-3}\ \text{radians}}$$

or $\boxed{0.500\ \text{degrees}}$

2/2

$\therefore$ Since $d\theta = 8.73\times10^{-3}$ is v. small, we can treat it like an infinitesimal angle.

(b) infinitesimal



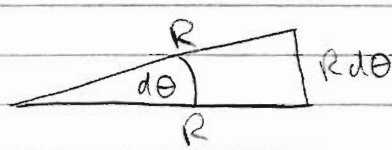
we found: $d\theta = 8.73\times10^{-3}$ rad

diameter of coin, $D = R\,d\theta = \frac{3}{4}$ inch

$$R\,d\theta = D \Rightarrow R = \frac{D}{d\theta} = \frac{(3/4)}{(8.73\times10^{-3})} = 85.9\ \text{in} = \boxed{2.1\ \text{m}}$$

2/2

(c)



diameter of sun, $D = R\,d\theta$

radius of sun, $r = \dfrac{D}{2} = \dfrac{R\,d\theta}{2}$

Steradian, $\Omega = \dfrac{A}{R^2} \Rightarrow d\Omega = \dfrac{dA}{R^2}$

$A = \pi r^2$

$\Rightarrow dA = \pi\left(\dfrac{R\,d\theta}{2}\right)^2 = \dfrac{\pi R^2 d\theta^2}{4}$

$$d\Omega = \frac{dA}{R^2} = \frac{\pi \cancel{R^2}\,d\theta^2}{4\cancel{R^2}} = \frac{\pi}{4}\,d\theta^2$$

but $d\theta = 8.73\times10^{-3}$ rad from (a)

$$d\Omega = \frac{\pi}{4}(8.73\times10^{-3})^2 = \boxed{5.99\times10^{-5}\ \text{steradians}}$$

2/2