Mathematical Physics

Lecture, Monday, October 1, 2001

finish up example — potential calculation

Recall: We were given

$$\vec{F} = ay\,(y^2-3z^2)\,\hat{e}_x + 3ax(y^2-z^2)\,\hat{e}_y - 6axyz\,\hat{e}_z$$

$\vec{\nabla}\times\vec{F} = 0 \Rightarrow V(x,y,z)$ exists, via method 1

we concluded that $\boxed{V(x,y,z) = 3axyz^2 - axy^3}$

which implies $\vec{F} = -\vec{\nabla}V$ which reproduced $\vec{F}$.

— — — — —

Method 2 $V(x,y,z)$ calculation:

Write:

$$\vec{F} = -\left\{\frac{\partial V}{\partial x}\hat{e}_x + \frac{\partial V}{\partial y}\hat{e}_y + \frac{\partial V}{\partial z}\hat{e}_z\right\} = ay\,(y^2-3z^2)\,\hat{e}_x + 3ax\,(y^2-z^2)\,\hat{e}_y - 6axyz\,\hat{e}_z$$

$\boxed{\text{pull out} - \text{sign}}$ — put it back in at end of discussion

Therefore $\dfrac{\partial V}{\partial x} = ay\,(y^2-3z^2)$

integrating (partially) wrt x, we may write

$$V = \int ay(y^2-3z^2)\,dx + f(y,z)$$

$$V = ayx\,(y^2-3z^2) + f(y,z)$$

$\longrightarrow$

over

Now $\frac{\partial V}{\partial y} = F_y - 3ax(y^2 - z^2)$

or
$$\frac{\partial}{\partial y}\left(ayx[y^2 - 3z^2] + f(x,y)\right) = 3ax(y^2 - z^2)$$

$$3ax(y^2 - z^2) + \frac{\partial f}{\partial y} = 3ax\,(y^2 - z^2) \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow f = f(z) \text{ only}$$

finally

$$\frac{\partial V}{\partial z} = F_z = -6axyz.$$

$$\Downarrow$$

$$-6axyz + \frac{\partial f}{\partial z} = -6axyz. \Rightarrow f(z) = \text{const at most.}$$

$\therefore$

$$V = axy\,(y^2 - 3z^2)$$

put in − sign

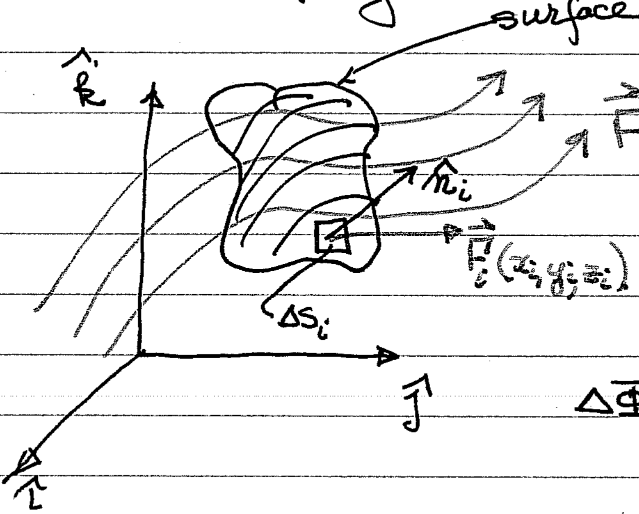
$$\boxed{V(x,y,z) = 3axyz^2 - axy^3 + \underline{\text{Const}} \quad \text{as before}}$$

as required

Lecture, Monday, October 1, 2001

The theorem of Gauss – Divergence Theorem

The Concept of flux.



the flux thru the surface ΔS_i

$$\Delta\Phi_i = |\vec{F}_i(x_i,y_i,z_i)|\,|\hat{n}_i|\,\Delta S_i \cos(\vec{F}_i, \hat{n}_i)$$

Note the above guarantees that $\vec{F}_i \perp$ to $\hat{n}_i \Delta S_i$

hence

$$\Phi \cong \sum_{i=1}^{N} \Delta\Phi_i = \sum_{i=1}^{N} |\vec{F}_i(x_i,y_i,z_i)|\,|\hat{n}_i|\,\Delta S_i \cos(\vec{F}_i, \hat{n}_i)$$

$$\Phi = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta\Phi_i = \lim_{N\to\infty} \sum_{i=1}^{N} |\vec{F}_i(x_i,y_i,z_i)|\,|\hat{n}_i|\,\Delta S_i \cos(\vec{F}_i, \hat{n}_i)$$

$$\text{flux} \equiv \Phi = \iint_{\text{surface}} \vec{F}\cdot\hat{n}\,ds$$ (this is the definition of flux)

over →

Divergence Theorem [the theorem of Gauss]

Statement:

$$\underset{\substack{\text{Closed}\\ \text{surface}}}{\iint} \vec{F}\cdot\hat{n}\,ds = \underset{\substack{\text{Volume}\\ \text{enclosed}}}{\iiint} \vec{\nabla}\cdot\vec{F}\; d(\text{vol})$$

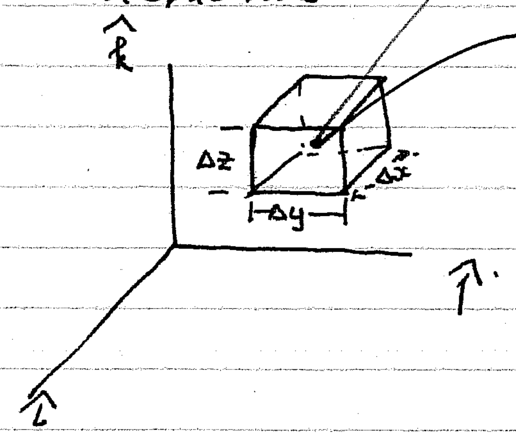
or if we back off from the integral (r, h.s), we conclude.

$$\vec{\nabla}\cdot\vec{F} = \lim_{\Delta vol \to 0}\left(\frac{\iint_{\text{closed}} \vec{F}\cdot\hat{n}\,ds}{\Delta(\text{vol})}\right) = \lim_{\Delta vol \to 0}\left\{\frac{\text{flux thru closed surface}}{\Delta vol}\right\}$$

flux/vol.
flux density

Let us look at $\vec{F}(x,y,z)$.

x,y,z in middle of our box [cube]



Recall: for a fn of one variable say $f(x)$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \simeq \frac{df}{dx}$$

or we may write

$$f(x+\Delta x) \simeq f(x) + \left.\frac{df}{dx}\right|_{x} \Delta x$$

(Taylor expansion essentially to first order in Δx.)

hence $g(x,y,z)$

$$\frac{g(x+\Delta x, y, z) - g(x,y,z)}{\Delta x} \simeq \frac{\partial g}{\partial x}$$

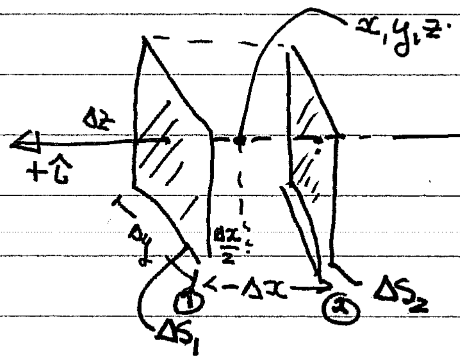
$$g(x+\Delta x, y, z) - g(x,y,z) \simeq \left.\frac{\partial f}{\partial x}\right|_{xyz} \Delta x$$

We wish to evaluate

$$\iint_{\text{Closed}} \vec{F}\cdot\hat{n}\,ds \simeq \sum_{\substack{i \\ \text{all} \\ \Delta S_i}} \vec{F}_i\cdot\hat{n}_i\,\Delta S_i$$

N.B. for our cube, we have six ΔS_i & six $\hat{n}_i$ to worry about.

let us look at pair of surfaces $\perp$ to $\hat{i}$ & $-\hat{i}$



Call pair of surfaces $\perp$ to $\hat{i}$ & $-\hat{i}$ directions

ΔS_1, ΔS_2, respectively.

Now along x, $\vec{F}$ has a component. $F_x(x,y,z)$

We now evaluate the flux passing thru surface (1) & thru. surface (2), Φ_1, Φ_2

$$\Phi_1 = \vec{F}\cdot\underbrace{\hat{n}_1\,\Delta S_1}_{\cdot\,\hat{i}\,\Delta S_1} = F_x\left(x+\tfrac{\Delta x}{2}, y, z\right)\Delta y\,\Delta z.$$

$$\Phi_2 = \vec{F}\cdot\underbrace{\hat{n}_1\,\Delta S_2}_{\cdot(-\hat{i})\,\Delta S_2} = -F_x\left(x-\tfrac{\Delta x}{2}, y, z\right)\Delta y\,\Delta z.$$

Now the total flux from this pair

$$\Phi_1+\Phi_2 = \left[F_x\left(x+\tfrac{\Delta x}{2}, \Delta y, \Delta z\right) - F_x\left(x-\tfrac{\Delta x}{2}, y, z\right)\right]\Delta y\,\Delta z$$

Now recall Taylor series expansion to first order in x.

$$f(x+\Delta x, y, z) \simeq f(x,y,z) + \left.\frac{\partial f}{\partial x}\right|_{x,y,z} \Delta x .$$

hence, we may write:

$$\Phi_1 + \Phi_2 = \left\{ \left[F_x(x,y,z) + \left.\frac{\partial F_x}{\partial x}\right|_{xyz} \frac{\Delta x}{2} \right] - \left[F_x(x,y,z) - \left.\frac{\partial F_x}{\partial x}\right|_{xyz} \frac{\Delta x}{2} \right] \right\} \Delta y \Delta z .$$

$$= \left\{ \left.\frac{\partial F_x}{\partial x}\right|_{xyz} \left[\frac{\Delta x}{2} + \frac{\Delta x}{2} \right] \right\} \Delta y \Delta z .$$

$$= \left.\frac{\partial F_x}{\partial x}\right|_{xyz} \Delta x \Delta y \Delta z .$$

Now do exactly the same for flux pairs $\Phi_3 + \Phi_4 \perp$ to $\hat{j}$ direction & for $\Phi_5 + \Phi_6 \perp$ to $\hat{k}$ direction, apply symmetry argument. hence, without any further ado!

$$\Phi_3 + \Phi_4 = \left[F_y(x, y+\frac{\Delta y}{2}, z) - F_y(x, y-\frac{\Delta y}{2}, z) \right] \Delta x \Delta z$$

$$= \left\{ \left[F_y(x,y,z) + \left.\frac{\partial F_y}{\partial y}\right|_{xyz} \frac{\Delta y}{2} \right] - \left[F_y(x,y,z) - \left.\frac{\partial F_y}{\partial y}\right|_{xyz} \frac{\Delta y}{2} \right] \right\} \Delta x \Delta z$$

$$= \partial F_y \Big| \Delta x \Delta y \Delta z$$

& $\Phi_5 + \Phi_6 = \left[F_z\left(x, y, z+\frac{\Delta z}{2}\right) - F_z\left(x, y, z-\frac{\Delta z}{2}\right)\right] \Delta x \Delta y.$

$$= \left\{\left[F_z(x,y,z) + \frac{\partial F_z}{\partial z}\bigg|_{xyz} \frac{\Delta z}{2} - \left\{F_z(x,y,z) - \frac{\partial F_z}{\partial z}\bigg|_{xyz} \frac{\Delta z}{2}\right\}\right\} \Delta x \Delta y.$$

$$= \frac{\partial F_z}{\partial z} \Delta x \Delta y \Delta z.$$

— — — — .

hence, we must conclude

$$\vec{\nabla}\cdot\vec{F} = \lim_{\Delta vol \to 0} \frac{\left\{\begin{matrix}\text{Flux} \\ \text{thru enclosed} \\ \text{surface}\end{matrix}\right\}}{\{\Delta vol\}} = \lim_{\Delta vol \to 0} \frac{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta x \Delta y \Delta z}{\{\Delta x \Delta y \Delta z\}}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \quad \{\text{definition of divergence}\}$$

OR

$$\iint_{\text{closed surface}} \vec{F}\cdot\hat{n}\,ds = \iiint_{\text{volume enclosed}} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz$$

$$= \iiint_{\text{volume enclosed}} \vec{\nabla}\cdot\vec{F}\, d(vol)$$

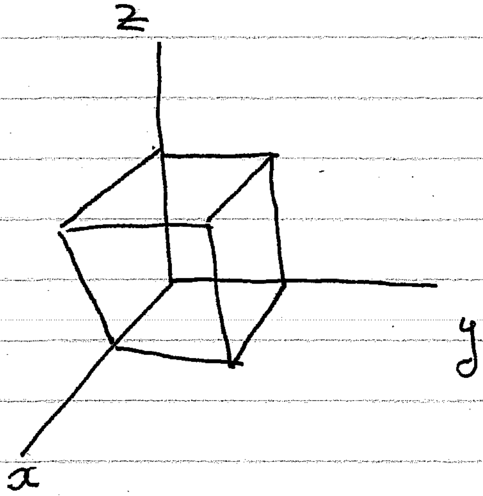
We have established the theorem of Gauss - divergence theorem

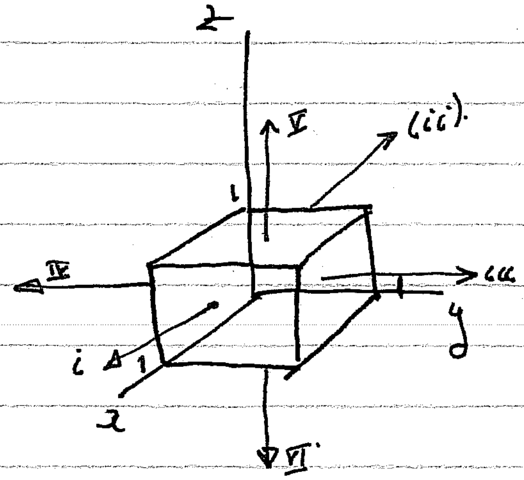
Example: Griffiths 1.10 pg32

Check out the divergence theorem.

$$\vec{F}(x,y,z) = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$$

on a unit cube





$$\underset{\substack{\text{Closed}\\\text{surface}}}{\iint} \vec{F}\cdot\hat{n}\, ds \quad = \quad \underset{\substack{\text{vol}\\\text{encl}}}{\iiint} \vec{\nabla}\cdot\vec{F}\, d\text{vol}$$

lhs rhs

rhs
$$\vec{\nabla}\cdot\vec{F} = 2(x+y)$$

&

$$\underset{\substack{\text{vol}\\\text{encl}}}{\iiint} \vec{\nabla}\cdot\vec{F}\, d\text{vol} = \underset{\substack{\text{vol}\\\text{encl}}}{\iiint} 2(x+y)\, dx\, dy\, dz$$

$$= 2\int_0^1\int_0^1\int_0^1 (x+y)\, dx\, dy\, dz.$$

$$\int_0^1 (x+y)\,dx = \left(\frac{x^2}{2}+yx\right)\Big|_0^1 = \tfrac{1}{2}+y\,; \qquad \int_0^1 \left(\frac{1}{2}+y\right)dy = \left(\tfrac{1}{2}y+\tfrac{y^2}{2}\right)\Big|_0^1$$

$$= \tfrac{1}{2}+\tfrac{1}{2} = 1$$

& $\displaystyle\int_0^1 1\,dz = 1$

Evidently, $\displaystyle\iiint_{\text{vol encl}} \vec{\nabla}\cdot\vec{F} = 2(1)(1) = 2$

Now, let us look at l.h.s, i.e. the surface integral.

To evaluate the surface integral, we <u>must</u> consider separately the six sides of the cube.

i) $\displaystyle\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y^2\,dy\,dz = 1/3$

ii) $\displaystyle\iint \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 y^2\,dy\,dz = -1/3$

iii) $\displaystyle\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (2x(1)+z^2)\,dx\,dz = \int_0^1 [x^2+z^2x]\Big|_0^1 dz = \int_0^1 [1+z^2]\,dz = \left(z+\frac{z^3}{3}\right)\Big|_0^1$

$y=1$ here

$$= 1+\tfrac{1}{3} = \underline{\underline{4/3}}$$

iv) $\displaystyle\iint \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 (2x(0)+z^2)\,dx\,dz = -\int_0^1 z^2x\Big|_0^1 dz = \frac{-z^3}{3}\Big|_0^1 = -1/3$

$y=0$ here

v) $\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1 \int_0^1 2y(1)\,dx\,dy = \int_0^1 2yx\Big|_0^1 dy = y^2\Big|_0^1 = 1.$

$z=1$ here

VI $\iint \vec{F}\cdot\hat{n}\,ds = -\int_0^1 \int_0^1 2y(0)\,dy\,dy = 0$

$z=0$ here

hence, the total flux

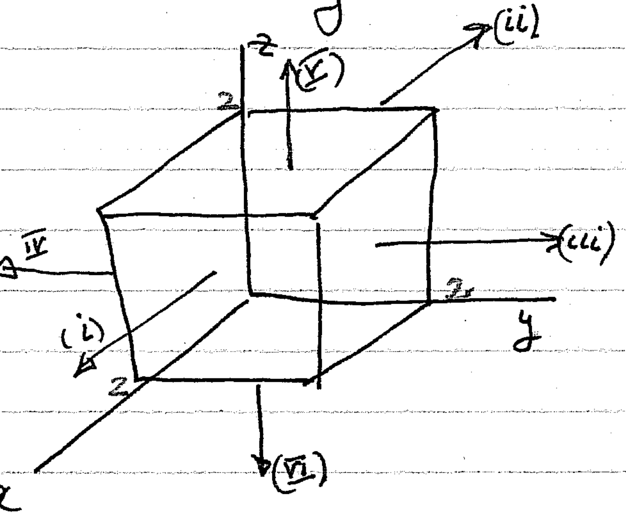
$$\oiint_{\text{Closed surface}} \vec{F}\cdot\hat{n}\,ds = 1/3 - 1/3 + 4/3 - 1/3 + 1 = 2$$

as expected — as required

Problem 1.32 – Griffiths. pg. 33

Given $\boxed{\vec{F} = xy\,\hat{i} + 2yz\,\hat{j} + 3zx\,\hat{k}}$ on a cube of side 2.

Check out divergence Theorem for This situation



r.h.s $\boxed{\vec{\nabla}\cdot\vec{F} = y + 2z + 3x}$

$$\iiint_{\text{vol encl}} \vec{\nabla}\cdot\vec{F}\,d\text{vol} = \int_0^2\int_0^2\int_0^2 (y + 2z + 3x)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 \left(yx + 2xz + \frac{3x^2}{2}\right)\Big|_0^2 dy\,dz$$

$$= \int_0^2\int_0^2 (2y + 4z + 6)\,dy\,dz$$

$$= \int_0^2 (y^2 + 4zy + 6y)\Big|_0^2 dz$$

→ over

$$= \int_0^2 (4+8z+12)\,dz = \int_0^2 (16+8z)\,dz$$

$$= \left[16z+4z^2\right]\Big|_0^2 = 32+16 = 48$$

N.B. this relatively easy to do!!

Now we evaluate $\displaystyle\iint_{\text{Closed}} \vec{F}\cdot\hat{n}\,ds$ on the six sides!!

(i) $$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 2y\,dy\,dz = \int_0^2 4\,dz = 8$$

$x=2$ here

ii) $$\iint \vec{F}\cdot\hat{n}\,ds = -\int_0^2\int_0^2 (0)y\,dy\,dz = 0$$

$x=0$ here.

iii) $$\cancel{\iint \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 (4x+z^2)\,dx\,dz = \int \left(2x^2+z^2x\right)\Big|_0^2 dz}$$

$\cancel{y=2 \text{ here}}$

$$\cancel{= \int_0^2 (8+2z^2)\,dz = \left(8z+\frac{2z^3}{3}\right)\Big|_0^2 = 16+}$$

iii) $$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 4z\,dx\,dz.$$

$y=2$

$$= \int_0^2 \left[4zx\right]\Big|_0^2 dz = \int_0^2 8z\,dz = 4z^2\Big|_0^2 = 16$$

(IV) $$\iint \vec{F}\cdot\hat{n}\,ds = -\int_0^2\int_0^2 z(0)z\,dx\,dz = 0$$
$y=0$ here

(V) $$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^2 3x(2)\,dx\,dy = \int_0^2\int_0^2 6x\,dx\,dy$$
$z=2$

$$= \int_0^2 3x^2\Big|_0^2 dy = \int_0^2 12\,dy = 12y\Big|_0^2$$

$$= 24$$

(VI) $$\iint \vec{F}\cdot\hat{n}\,ds = -\int_0^2\int_0^2 3x(0)\,dx\,dy = 0$$
$z=0$ here

∴ the total flux

$$\underset{\text{Closed surface}}{\oiint} \vec{F}\cdot\hat{n}\,ds = 8+0+16+0+24+0 = 48$$ as expected, as required

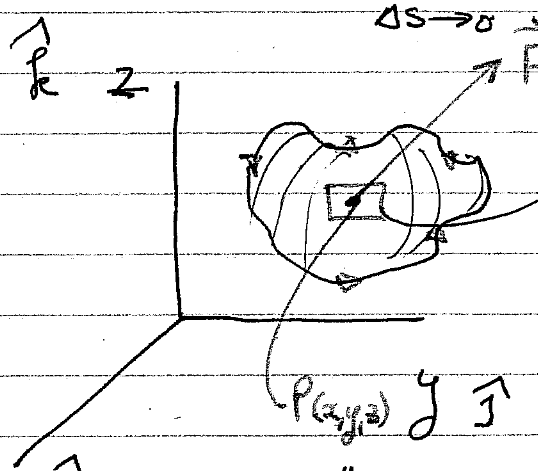
Lecture, Wednesday, October 3, 2001

The curl – Stoke's theorem

Claim: $\displaystyle\oint_{\substack{\text{Closed}\\\text{path}}} \vec{F}\cdot d\vec{r} = \iint_{\substack{\text{Cap}\\\text{surface}}} \vec{\nabla}\times\vec{F}\cdot\hat{n}\, ds$

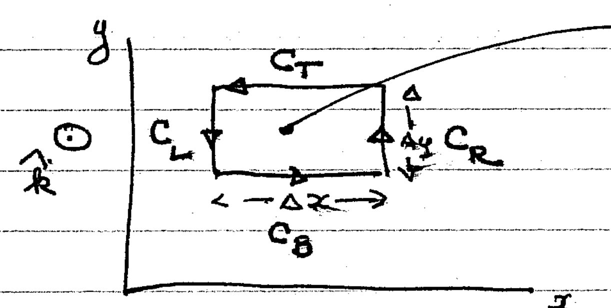
Now, back off integral, & write.

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \lim_{\Delta s\to 0}\left(\frac{1}{\Delta s}\oint_{\substack{\text{closed}\\\text{path}}} \vec{F}\cdot d\vec{r}\right) = \frac{\text{circulation}}{\text{area }\perp}$$



what to do?!

project this rectangular area into the (x-y) plane.



$P(x,y,z)$ at center $\sim z \simeq$ constant

here z = constant (i.e. zero)

Let us consider the $\vec{\text{Circulation of } F}$ over a small (infinitesimal) rectangle which is ∥ – to x-y plane (i.e. projected onto x-y plane), with sides Δx & Δy with the point $P(x,y,z)$ at center of rectangle

N.B $\boxed{\text{The sense of integration:}}$ we carry out the path of integration in a counter-clockwise direction (i.e: the area enclosed by the path is $\boxed{\text{"always on your left"}}$ looking down on the x-y plane.

The line integral is broken up into four parts:

C_B (bottom), C_R (right), C_T (top), C_L (left).

$\boxed{\text{N.B}}$ It is assumed that the rectangle is small and we shall eventually take the limit as it shrinks to zero; we will approximate the integral over each segment by $\vec{F}\cdot d\vec{r}$ evaluating $\vec{F}$ at the $\boxed{\text{center}}$ of the segment (i.e. essentially a Taylor expansion)

Let us begin:

$\boxed{\text{Along } C_B}$ $\displaystyle\int_{C_B} \vec{F}\cdot d\vec{r} = \int F_x\,dx \simeq F_x\left(x, y-\frac{\Delta y}{2}, z\right)\Delta x.$

[note $\hat{i}$ is positive in this direction]

$\boxed{\text{along } C_T}$ we write

$$\int_{C_T} \vec{F}\cdot d\vec{r} = -\int F_x\,dx \simeq -F_x\left(x, y+\frac{\Delta y}{2}, z\right)\Delta x$$

$\boxed{\text{note } \hat{i} \text{ is negative in this direction}}$ over and now let us add those contributions.

$$\int_{C_B+C_T} \vec{F}\cdot d\vec{r} \simeq -\left[F_x\left(x, y+\frac{\Delta y}{2}, z\right) - F_x\left(x, y-\frac{\Delta y}{2}, z\right)\right]\Delta x$$

$$\simeq -\frac{\left[F_x\left(x, y+\frac{\Delta y}{2}, z\right) - F_x\left(x, y-\frac{\Delta y}{2}, z\right)\right]}{\Delta y}\Delta x\Delta y.$$

but $F_x\left(x, y+\frac{\Delta y}{2}, z\right) \simeq F_x(x,y,z) + \left.\frac{\partial F_x}{\partial y}\right|_{xyz}\frac{\Delta y}{2}$.

& $F_x\left(x, y-\frac{\Delta y}{2}, z\right) \simeq F_x(x,y,z) - \left.\frac{\partial F_x}{\partial y}\right|_{xyz}\frac{\Delta y}{2}$.

or we may now conclude.

$$\int_{C_B+C_T} \vec{F}\cdot d\vec{r} = -\frac{\left\{\left[F_x(x,y,z) + \left.\frac{\partial F_x}{\partial y}\right|_{xyz}\frac{\Delta y}{2}\right] - \left[F_x(x,y,z) - \left.\frac{\partial F_x}{\partial y}\right|_{xyz}\frac{\Delta y}{2}\right]\right\}}{\Delta y}\Delta x\Delta y$$

but $\Delta S = \Delta x\Delta y$

(I) or $\displaystyle \frac{1}{\Delta S}\int_{C_B+C_T} \vec{F}\cdot d\vec{r} \simeq -\left.\frac{\partial F_x}{\partial y}\right|_{xyz}$.

$\left(\frac{1}{\Delta S} = \frac{1}{\Delta x\Delta y}\right)$

Now let us do exactly the same for C_L, C_R $\longrightarrow$ over here $\int_{C_R} \vec{F}\cdot d\vec{r} \simeq \int F_y\,dy \simeq F_y\left(x+\frac{\Delta x}{2}, y, z\right)\Delta y$ $\boxed{\text{Note: here } \hat{j} \text{ is +ve}}$

and $\int_{C_L} \vec{F}\cdot d\vec{r} \simeq \int F_y\,dy \simeq -F_y\left(x-\frac{\Delta x}{2}, y, z\right)\Delta y$ $\boxed{\text{note: here } \hat{j} \text{ is -ve}}$

Now, recall $F_y\left(x+\frac{\Delta x}{2}, y, z\right) \simeq F_y(x,y,z) + \left.\frac{\partial F_y}{\partial x}\right|_{xyz}\frac{\Delta x}{2}.$

& $F_y\left(x-\frac{\Delta x}{2}, y, z\right) \simeq F_y(x,y,z) - \left.\frac{\partial F_y}{\partial x}\right|_{xyz}\frac{\Delta x}{2}$

Now add & obtain.

$\boxed{II}$ $\dfrac{\int_{C_R} \vec{F}\cdot d\vec{r} + \int_{C_L} \vec{F}\cdot d\vec{r}}{\Delta S = \Delta x \Delta y.} = \dfrac{\left\{\left[F_y(x,y,z) + \left.\frac{\partial F_y}{\partial x}\right|_{xyz}\frac{\Delta x}{2}\right] - \left[F_y(x,y,z) - \left.\frac{\partial F_y}{\partial x}\right|_{xyz}\frac{\Delta x}{2}\right]\right\}\Delta y}{\Delta S = \Delta x \Delta y.}$

$=$

$= \left.\dfrac{\partial F_y}{\partial x}\right|_{xyz}.$

Now add (I) + (II) & obtain.

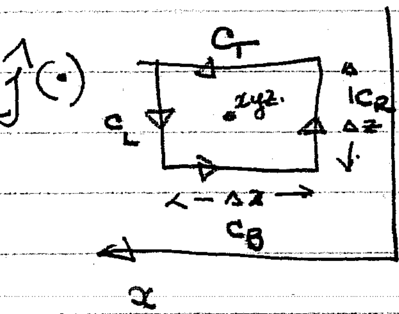
$$\lim_{\Delta S \to 0}\left\{\frac{1}{\Delta S}\oint_{C_B + C_T + C_R + C_L} \vec{F}\circ d\vec{r}\right\} = \boxed{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}}$$

& evidently

$$\boxed{(\vec{\nabla}\times\vec{F})\cdot\hat{k} = (\vec{\nabla}\times\vec{F})_z = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}$$

N.B.

Now project $\square$ into x-z plane — y = constant.



Along C_B

$$\int_{C_B} \vec{F}\cdot d\vec{r} = \int F_x\,dx \simeq -F_x\left(x,y,z-\frac{\Delta z}{2}\right)\Delta x$$

N.B. path in $-\hat{i}$ direction

along C_T

$$\int_{C_T} \vec{F}\cdot d\vec{r} = \int F_x\,dx \simeq F_x\left(x,y,z+\frac{\Delta z}{2}\right)\Delta x$$

N.B. path in $+\hat{i}$ direction

or using the Taylor expansion to first order in Δx

$$(\mathrm{I})\quad \frac{1}{\Delta S}\int_{C_B+C_T} \vec{F}\cdot d\vec{r} = \frac{\left\{\left[F_x(x,y,z) + \left.\frac{\partial F_x}{\partial z}\right|_{xyz}\frac{\Delta z}{2} - F_x(xyz) + \left.\frac{\partial F_x}{\partial z}\right|_{xyz}\frac{\Delta z}{2}\right\}\Delta x}{\Delta x\,\Delta z}$$

$$(\mathrm{I})\quad \frac{1}{\Delta S}\int_{C_B+C_T} \vec{F}\cdot d\vec{r} = \frac{\partial F_x}{\partial z}$$

— — — — — —

$\boxed{\text{Along } C_R}$: $\displaystyle\int_{C_R} \vec{F}\cdot d\vec{r} \simeq \int F_z\, dz \simeq F_z\left(x-\frac{\Delta x}{2}, y, z\right)\Delta z$ $\boxed{\text{in } +\text{ve } \hat{k} \text{ direction}}$

$\boxed{\text{Along } C_L}$: $\displaystyle\int_{C_L} \vec{F}\cdot d\vec{r} = \int F_z\, dz \simeq -F_z\left(x+\frac{\Delta x}{2}, y, z\right)\Delta z$ $\boxed{\text{in } -\text{ve } \hat{k} \text{ direction}}$

Now add & divide by ΔS.

$$\frac{1}{\Delta S}\int_{C_R+C_L} \vec{F}\cdot d\vec{r} \simeq -\left[\frac{F_z\left(x+\frac{\Delta x}{2}, y, z\right) - F_z\left(x-\frac{\Delta x}{2}, y, z\right)}{\Delta x\,\Delta z}\right]\Delta z$$

(II) $$= -\frac{\partial F_z}{\partial x}.$$

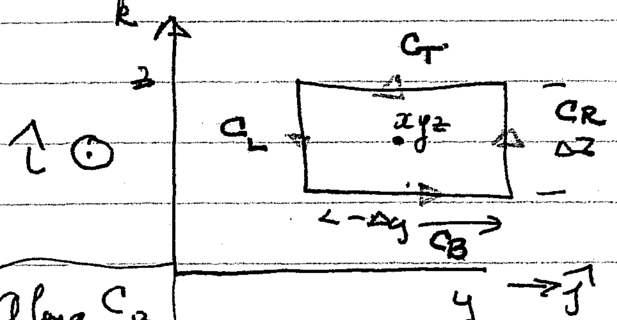
Now add (I) + (II).

$$\lim_{\Delta S\to 0}\left[\frac{1}{\Delta S}\oint \vec{F}\cdot d\vec{r}\right] = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}.$$

$$\text{or } (\vec{\nabla}\times\vec{F})\cdot\hat{j} = (\vec{\nabla}\times\vec{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$$

Finally, Project rectangle into z-y plane.

direction $\underline{x = \text{constant}}$



$\boxed{\text{along } C_B}$

$$\int_{C_B} \vec{F}\cdot d\vec{r} = \int F_y\, dy \simeq F_y\left(x, y, z - \frac{\Delta z}{2}\right)\Delta y$$

$\boxed{\hat{j} \text{ dr is positive}}$

$\boxed{\text{along } C_T}$

$$\int_{C_T} \vec{F}\cdot d\vec{r} = \int F_y\, dy \simeq -F_y\left(x, y, z + \frac{\Delta z}{2}\right)\Delta y$$

$\boxed{\hat{j} \text{ dr is negative}}$

$$\text{or } \frac{1}{\Delta S}\int_{C_B + C_T} \vec{F}\cdot d\vec{r} \simeq -\left\{\left[F_y(x,y,z) + \left.\frac{\partial F_y}{\partial z}\right|_{xyz}\frac{\Delta z}{2}\right] - \left\{F_y(xyz) - \left.\frac{\partial F_y}{\partial z}\right|_{xyz}\frac{\Delta z}{2}\right\}\Delta y\right\}$$

(I)
$$= -\frac{\partial F_y}{\partial z}$$

finally,

$\boxed{\text{Along } C_R}$

$$\int \vec{F}\cdot d\vec{r} = \int F_z\, dz \simeq F_z\left(x, y + \frac{\Delta y}{2}, z\right)\Delta z$$

$\boxed{\hat{k} \text{ is positive here}}$

$\boxed{\text{along } C_L}$

$$\int \vec{F}\cdot d\vec{r} = \int F_z\, dz \simeq -F_z\left(x, y - \Delta y/2, z\right)\Delta z$$

$\boxed{\hat{k} \text{ is negative here}}$

adding & dividing by Δs

$$\frac{1}{\Delta S}\int_{C_R+C_L} \vec{F}\cdot d\vec{r} = \left\{\left[F_z(xyz) + \left.\frac{\partial F_z}{\partial y}\right|_{xyz}\frac{\Delta y}{2} - \left\{F_z(x,yz) - \left.\frac{\partial F_z}{\partial y}\right|_{xyz}\frac{\Delta y}{2}\right\}\right.\right\}\Delta Z$$

(II) $$= \frac{\partial F_z}{\partial y}.$$

add (I) + (II) & take limit.

$$\lim_{\Delta S\to 0}\left(\frac{1}{\Delta S}\oint \vec{F}\cdot d\vec{r}\right) = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}.$$

hence $$(\vec{\nabla}\times\vec{F})\cdot\hat{\imath} = (\vec{\nabla}\times\vec{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

Now put everything together:

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \lim_{\Delta S\to 0}\left(\frac{1}{\Delta S}\oint \vec{F}\cdot d\vec{r}\right)$$

$$= \left(\frac{\text{circulation}}{\text{area}}\right)_\perp \text{ to area}$$

or $$\vec{\nabla}\times\vec{F} = \hat{\imath}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

~~who~~

→ over

which of course may be written as:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

as required

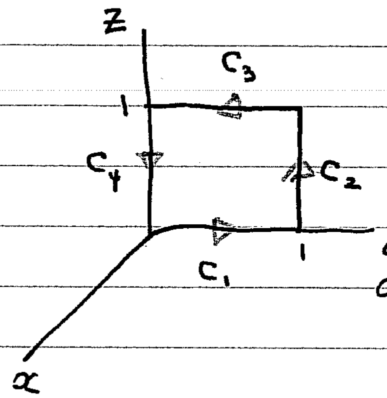
Lecture, Friday, October 5, 2001

Examples of divergence/Stoke's theorems

Ex. 8 Griffiths 2nd ed — pg 37.

Given $\vec{v} = (2xz + 3y^2)\hat{j} + (4yz^2)\hat{k}$.

$$\oint_{\text{Closed}} \vec{v} \cdot d\vec{r} = \iint_{\text{Cap surface}} \vec{\nabla} \times \vec{v} \cdot \hat{n}\, ds.$$



r.h.s. $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (2xz + 3y^2) & 4yz^2 \end{vmatrix}$

$$= \hat{i}\left(4z^2 - 2x\right) - \hat{j}(0) + \hat{k}\left(2z\right)$$

$\hat{n}\, ds = \hat{i}\, dy\, dz$. but $x = 0$ for this surface.

$$\therefore \text{rhs} = \iint (4z^2\hat{i} + \hat{k}\,2z) \cdot \hat{i}\, dy\, dz.$$

$$= \int_0^1\int_0^1 4z^2\, dy\, dz = \int_0^1 dy \int_0^1 4z^2 dz$$

$$= \int_0^1 dy \left[\frac{4z^3}{3}\right]\Big|_0^1$$

$$= 4/3$$

→ over

Now, what about the line integral — we must integrate on four sides C_1, C_2, C_3, C_4

C_1

$$\int_{C_1} \vec{v}\cdot d\vec{r} = \int_{C_1} (2xz + 3y^2)\,dy + 4yz^2\,dz$$

on C_1 $x=0$, $z=0$, $dz=0$

$$= \int_0^1 3y^2\,dy = y^3\Big|_0^1 = 1$$

C_2

$$\int_{C_2} \vec{v}\cdot d\vec{r} = \int_{C_2} (2xz+3y^2)\,dy + 4yz^2\,dz.$$

on C_2 : $x=0$, $y=1$, $dy=0$.

$$= \int_0^1 4(1)z^2\,dz = \int_0^1 4z^2\,dz = \frac{4z^3}{3}\Big|_0^1 = 4/3$$

C_3

$$\int_{C_3} \vec{v}\cdot d\vec{r} = \int_{C_3} (2xz+3y^2)\,dy + 4yz^2\,dz.$$

here $x=0$ $z=1$, $dz=0$

$$= \int_1^0 3y^2\,dy = \frac{3y^3}{3}\Big|_1^0 = -1.$$

C_4

$$\int_{C_4} \vec{v}\cdot d\vec{r} = \int_{C_4} (2xz+3y^2)\,dy + 4yz^2\,dz$$

on C_4 $x=0$ $z=1$ $y=0$, $dy=0$.

$$= \int_1^0 0\,dz = 0$$

$$\oint \vec{v}\cdot d\vec{r} = 1 + 4/3 - 1 + 0 = 4/3 \quad \text{as required}$$

Mathematical Physics

Wednesday, October 10, 2001

Curvilinear Coordinates:

Cylindrical: $\overset{q_1}{\rho}, \overset{q_2}{\phi}, \overset{q_3}{z}$ $\quad h_1 = 1,\ h_2 = \rho \quad h_3 = 1$

Gradient: $\vec{\nabla} f = \hat{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial q_3}$

$$= \hat{e}_\rho \frac{1}{1}\frac{\partial f}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{1}\frac{\partial f}{\partial z}$$

$$\therefore \vec{\nabla} f = \hat{e}_\rho \frac{\partial f}{\partial \rho} + \frac{\hat{e}_\phi}{\rho}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{\partial f}{\partial z}$$

divergence: $\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_1 h_3 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right].$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho \frac{A_\rho}{\cancel{\rho}}\right) + \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z}(\rho A_z)\right)$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

curl:

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\,\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{e}_\rho\left(\frac{\partial A_z}{\partial \phi} - \frac{\partial(\rho A_\phi)}{\partial z}\right) - \rho\,\hat{e}_\phi\left[\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z}\right] + \hat{e}_z\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)\right.$$

put it together, clean it up —

$$\boxed{\begin{aligned} \vec{\nabla}\times\vec{A} = {} & \left[\frac{1}{\rho}\left(\frac{\partial A_z}{\partial \phi}\right) - \frac{\partial A_\phi}{\partial z}\right]\hat{e}_\rho \\ & + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{e}_\phi \\ & + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right] \end{aligned}}$$

Finally, the Laplacian:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

in our case:

$$\nabla^2 f = \frac{1}{\rho}\left\{\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right]\right\}$$

$$\boxed{\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}}$$

The spherical Coordinates

$q_1 = r \quad q_2 = \theta \quad q_3 = \phi$

$h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$

Gradient:

$$\nabla f = \hat{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial q_3}$$

in our case:

$$\vec{\nabla} f = \hat{e}_r \frac{1}{1}\frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

or

$$\boxed{\vec{\nabla} f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}}$$

divergence:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_1 h_3 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right]$$

in our case

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \phi}(r A_\phi)\right]$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi}$$

curl:

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

in our case,

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\,\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\Big\{ \hat{e}_r\left[\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)\right]$$

$$- \hat{e}_\theta r\left[\frac{\partial}{\partial r}(r\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_r\right]$$

$$+ r\sin\theta\,\hat{e}_\phi\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial}{\partial\theta}A_r\right]$$

Now clear this expression:

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)-\frac{\partial}{\partial\phi}A_\theta\right]\hat{e}_r \\ &+\frac{1}{r}\left[\frac{1}{\sin\theta}\cdot\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right]\hat{e}_\theta \\ &+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{e}_\phi\end{aligned}}$$

finally

the Laplacian:

$$\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

in our case, this can be written —

$$\nabla^2 f=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$$

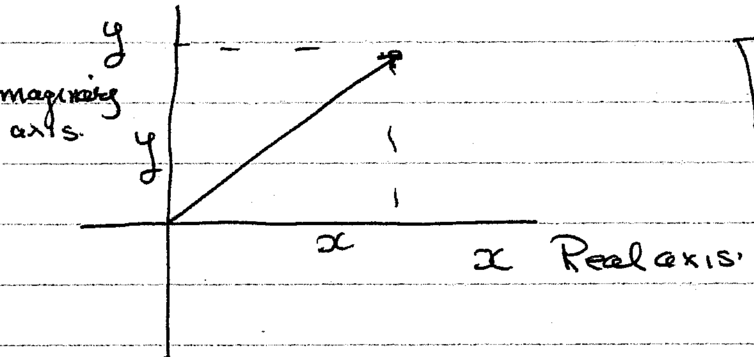
$$\boxed{\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}}$$

Mathematical Physics

Lecture, Monday, October 15, 2001

Complex numbers and their algebra

$x^2+1=0 \qquad x=\pm\sqrt{-1} = \pm i \qquad i \equiv \sqrt{-1}$ {use it as an operator}.



$$\boxed{z = x + iy}$$

Note: It turns out (amazingly!) that we don't need to introduce any other numbers to solve any algebraic equation.

In fact, the fundamental theorem of algebra states that all roots of any algebraic equation of the form

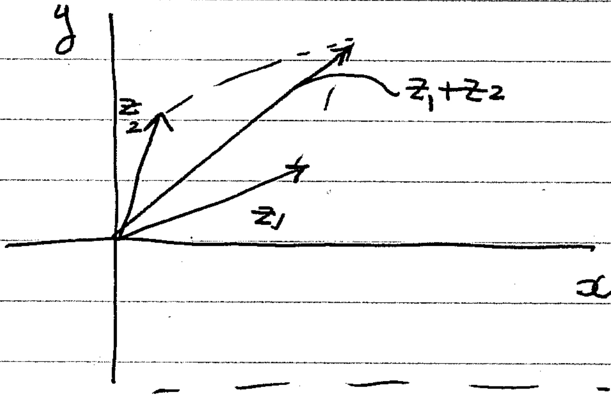
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with arbitrary (real or complex) coefficients $a_n, a_{n-1}, \cdots a_1, a_0$ are in the complex number system. In this sense, then, the complex number system is the most complete system.

$z = x + iy$ we write $\mathrm{Re}(z) = x \quad \mathrm{Im}(z) = y$

Arithmetic: $z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

add/subtract $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.



also $z - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\therefore\ z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$.

Multiplication

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_1 x_2 + i^2 y_2 y_1$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

& $Re(z_1 z_2) = x_1 x_2 - y_1 y_2$

$Im(z_1 z_2) = x_1 y_2 + y_1 x_2$

N.B $z_1 = z_2$ iff $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ & $y_1 = y_2$.

There are new operations for complex numbers which do not exist for real numbers.

Complex Conjugation

The complex conjugate z^* (or $\bar{z}$) of z

is $z^* \equiv (x + iy)^* \equiv x - iy$

which is obtained by replacing i with $-i$.

We note immediately

$$zz^* = (x+iy)(x-iy) = x^2+y^2$$

which is a positive real number.

The positive square root of zz^* (i.e. $\sqrt{zz^*}$) is called the absolute value or norm of z or (modulus)

$$\therefore \quad |z| = \sqrt{zz^*} = \sqrt{z^*z} = \sqrt{x^2+y^2}$$

<u>division</u> $\quad \frac{z_1}{z_2} = ?$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \; ? \qquad \text{multiply by } 1 = \frac{z_2^*}{z_2^*}$$

$$\frac{z_1}{z_2} = \left(\frac{x_1+iy_1}{x_2+iy_2}\right)\left(\frac{x_2-iy_2}{x_2-iy_2}\right) = \frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2}$$

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x^2+y^2} + i\frac{(y_1x_2 - x_1y_2)}{x^2+y^2}$$

$$\therefore \mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{x_1x_2+y_1y_2}{x^2+y^2}$$

$$\mathrm{Im}\left(\frac{z_1}{z_2}\right) = \frac{y_1x_2 - x_1y_2}{x^2+y^2}$$

Also, note $\frac{1}{z} = \frac{z^*}{|z|^2} = \frac{x-iy}{x^2+y^2}$

Note: the following.

$$(z_1+z_2)^* = z_1^* + z_2^*$$

$$(z_1z_2)^* = z_1^* z_2^*$$

$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$$

$$(z^*)^* = z$$

$$\mathrm{Re}(z) = \tfrac{1}{2}(z+z^*)$$

$$\mathrm{Im}(z) = \frac{1}{2i}(z-z^*)$$

$$(z^n)^* = (z^*)^n$$

Some properties of modulus (norm)

$$|z_1 z_2| = |z_1|\,|z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Examples:

a) $\dfrac{1}{1-i} - \dfrac{1}{1+i} = i$

because $\dfrac{1}{1-i} - \dfrac{1}{1+i} = \dfrac{(1+i)-(1-i)}{(1-i)(1+i)} = \dfrac{2i}{|1-i|^2} = \dfrac{2i}{2} = i$

we have used $z = 1-i$

$|z|^2 = zz^* = x^2+y^2$

b) $(i+1)^{-4} = -\frac{1}{4}$

because $(i+1)^{-4} = \dfrac{1}{(i+1)^4} = \dfrac{1}{(i+1)^2(i+1)^2} = \dfrac{1}{(2i)(2i)} = \dfrac{1}{4i^2} = -\dfrac{1}{4}$

c) $\dfrac{2+i}{3-i} = \frac{1}{2}(1+i)$

because $\dfrac{2+i}{3-i} = \dfrac{2+i}{(3-i)}\dfrac{(3+i)}{(3+i)} = \dfrac{6+5i+i^2}{9-i^2} = \dfrac{5+5i}{10} = \frac{1}{2}(1+i)$

d) $\left|\dfrac{2i-1}{i-2}\right| = 1$

→ over

$$\left|\frac{2i-1}{i-2}\right| = \frac{|2i-1|}{|i-2|} = \frac{\sqrt{(2i-1)(-2i-1)}}{\sqrt{(i-2)(-i-2)}} = \frac{-4i^2+1}{} = \frac{\sqrt{5}}{\sqrt{-i^2+4}}$$

$$= \frac{\sqrt{5}}{\sqrt{5}} = 1$$

(e) Show $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

proof $\left|\frac{z_1}{z_2}\right| = \sqrt{\frac{z_1}{z_2}\left(\frac{z_1}{z_2}\right)^*} = \sqrt{\frac{z_1}{z_2}\frac{z_1^*}{z_2^*}} = \sqrt{\frac{|z_1|^2}{|z_2|^2}} = \frac{|z_1|}{|z_2|}$

(f) show $|z_1 z_2| = |z_1||z_2|$

$$|z_1 z_2| = \sqrt{(z_1 z_2)(z_1 z_2)^*} = \sqrt{z_1 z_2 z_1^* z_2^*} = \sqrt{(z_1 z_1^*)(z_2 z_2^*)}$$

$$= \sqrt{|z_1|^2 |z_2|^2} = |z_1||z_2|$$

g) The equation $|z-a| = b$ where a and b are fixed numbers (b real & positive) describes a circle of radius b with center at $a = a_x + i a_y$

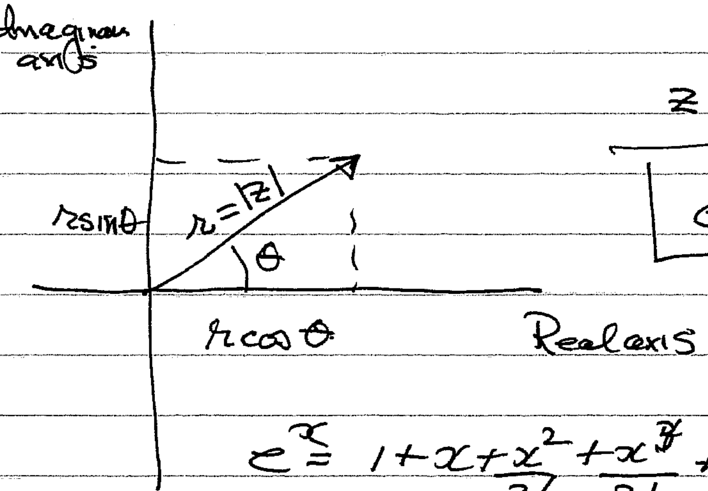
$$|z-a|^2 = |[(x+iy)-(a_x+ia_y)]|^2 = |(x-a_x)+i(y-a_y)|^2$$

$$= (x-a_x)^2 + (y-a_y)^2 = b^2$$

Note: $|z-a|$ is the distance between two complex numbers z, a.

Therefore $|z-a|=b$, where a = constant & z is variable, is collection of all points z that are a distance b from a

Polar Coordinates & Complex Numbers



$$z = x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler identity

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

thus $z = |z| e^{i\theta}$

$$r = \sqrt{x^2+y^2} = \sqrt{zz^*} = |z|.$$

$$\tan\theta = y/x.$$

Note: $\boxed{\theta \text{ not uniquely determined}}$

$$z^* = x - iy = r\cos\theta - ir\sin\theta = r\left[\cos(-\theta) + i\sin(-\theta)\right]$$
$$= r e^{-i\theta}$$

Example $r = 1$

a) $1 = e^{i\theta} e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta$

b) ~~$e^{i(\theta_1+\theta_2)} = (\cos\theta_1 + i\sin\theta_1)(\cos$~~

$$e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

on the other hand.

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right] + i\left(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)$$

$$\therefore \cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

c) $e^{3i\theta} = \cos 3\theta + i\sin 3\theta$

$$(a+b)^n = a^n + \frac{n a^{n-1} b}{1!} + \frac{n(n-1) a^{n-2} b^2}{2!} + \frac{n(n-1)(n-2) a^{n-3} b^3}{3!} + \dots$$

on the other hand

$$e^{3i\theta} = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3(\cos^2\theta)i\sin\theta + \frac{3\cdot 2\cos\theta(i\sin\theta)^2}{2!} + \frac{(i\sin\theta)^3\, 3\cdot 2\cdot 1}{3!}$$

$$= \cos^3\theta + 3(\cos^2\theta)i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3.$$

$$= \cos^3\theta - 3\cos\theta\sin\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$$\therefore \quad \cos 3\theta = \cos^3\theta - 3\cos\theta\sin\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

Example de Moivre theorem

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

$$\& \ e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

or $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

Also
$$\boxed{\begin{aligned} \cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{aligned}}$$
useful

The n^{th} roots of Unity

a) $$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

b) $$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

c) $$\sqrt{z_1} = \sqrt{r_1 e^{i\theta_1}} = (r_1 e^{i\theta_1})^{1/2} = r_1^{1/2}(e^{i\theta_1})^{\frac{1}{2}} = \sqrt{r_1}\, e^{i\theta/2}.$$

and so forth.

All these have interesting geometric interpretations.

b) says when you multiply a complex number, z_1, by another complex number, z_2, you dilate its magnitude by a factor r_2 and increase its angle by θ_2.

That is, multiplication involves both a dilation and a rotation. In particular, if we multiply a complex number by $e^{i\omega t}$, where t is a real variable (time), we get a vector of constant length in the xy plane that is rotating with velocity ω.

Roots of Unity:

We want to find all z's satisfying

$$z^n = 1$$

most general way to write unity.

$$1 = e^{2i\pi k} \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots \qquad \begin{array}{l} k=+1 \\ 1 = e^{2i\pi} = \cos 2\pi + i \sin 2\pi \\ \quad = 1 \end{array}$$

$$\text{thus } z^n = e^{2i\pi k} \qquad k = 0, \pm 1, \pm 2 \ldots \qquad \begin{array}{l} 1 = e^{-2i\pi} = \cos(-2\pi) + i\sin(-2\pi) \\ \quad = 1 \end{array}$$

taking the n^{th} root of both sides

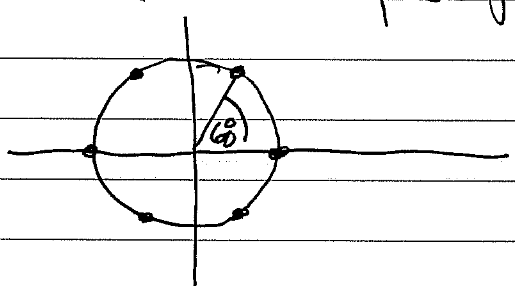
$$z = \left(e^{2i\pi k}\right)^{\frac{1}{n}} = e^{\frac{2i\pi k}{n}} \qquad k = 0, \pm 1, \pm 2 \ldots$$

& the <u>distinct roots</u> $\{z_k\}$ are.

$$z_k = e^{2i\pi k/n} \qquad \text{for } k = 0, 1, 2, \ldots n-1$$

We see that there are exactly $\underline{\underline{n}}$ $\underline{\underline{n^{th}}}$ roots of unity.

All of these roots are equally spaced on a unit circle in complex plane



six sixth roots

Example:

Various roots of unity are given in the following.

a) For $n=2$

$$z = e^{2\pi i k/2} = e^{\pi i k} \quad \text{where } k=0,1$$

$$\Rightarrow z_1 = e^0,\ z_2 = e^{\pi i} \Rightarrow z_1 = 1,\ z_2 = -1$$

as required.

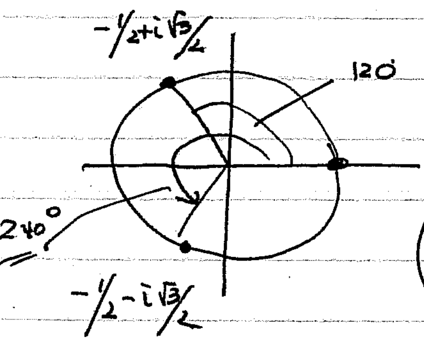
b) $n=3$.

$$z = e^{2\pi i k/3} \quad k=0,1,2 \Rightarrow z_1 = e^0,\ z_2 = e^{2\pi i/3},\ z_3 = e^{4\pi i/3}$$

$$e^{2\pi i/3} = \cos 2\pi/3 + i\sin 2\pi/3 = -1/2 + i\frac{\sqrt{3}}{2}$$

$$e^{4\pi i/3} = \cos 4\pi/3 + i\sin 4\pi/3 = -1/2 - i\sqrt{3}/2$$

as required.



Let us verify above as follows:

$$\left(-1/2 + i\frac{\sqrt{3}}{2}\right)^3 = \left(-1/2\right)^3 + 3\frac{\left(-1/2\right)^2\left(\frac{i\sqrt{3}}{2}\right)}{1!} + 3\cdot 2\,\frac{\left(-1/2\right)\left(\frac{i\sqrt{3}}{2}\right)^2}{2!}$$

$$+\frac{3\cdot 2\cdot 1\left(\frac{i\sqrt{3}}{2}\right)^3}{3!}$$

$$= -\frac{1}{8} + 3/4\,\frac{i\sqrt{3}}{2} - \frac{3}{2}\left(-\frac{3}{4}\right) + \left(-i\frac{3}{8}\right)\sqrt{3}$$

$$= -1/8 + \frac{i\,3\sqrt{3}}{8} - i\left(\frac{3\sqrt{3}}{8}\right) + \frac{9}{8} = -1/8 + \frac{9}{8} = 8/8 = 1$$

as required.

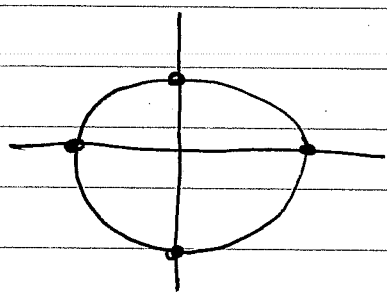
& of course one obtains a similar result for.

$$\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = 1 \quad \text{do it!!}$$

c) $n=4$ $\quad z = e^{2\pi i k/4} = e^{\pi i k/2}$ where $k = 0, 1, 2, 3$.

$z_1 = 1 \quad z_2 = e^{\pi i/2} \quad z_3 = e^{\pi i} \quad z_4 = e^{3\pi i/2}$

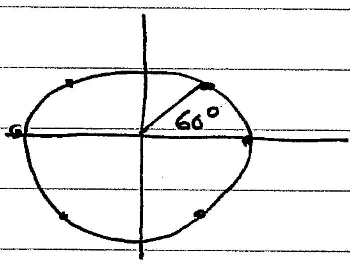
or $z_1 = 1 \quad z_2 = i \quad z_3 = -1 \quad z_4 = -i$



(d) $n=6$ $\quad z = e^{2\pi i k/6} = e^{\pi i k/3}$ where $k = 0, 1, 2, 3, 4, 5$

$z_1 = 1 \quad z_2 = e^{\pi i/3} \quad z_3 = e^{2\pi i/3} \quad z_4 = e^{\pi i} \quad z_5 = e^{4\pi i/3} \quad z_6 = e^{5\pi i/3}$

$z_1 = 1 \quad z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad z_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad z_4 = -1 \quad z_5 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \quad z_6 = \frac{1}{2} - \frac{\sqrt{3}}{2}$



etc

Example

Let us find the square root of $z = a + ib$ in Cartesian coordinates. The technique we will use also applies to finding the n^{th} root.

<u>Soln'</u>: we first write z in polar form.

Step ① $\quad Z = re^{i\theta} \quad r = \sqrt{a^2+b^2}, \quad \tan\theta = b/a$

raise both sides to the ½ power

$$Z^{1/2} = r^{1/2} e^{i\theta/2} = (a^2+b^2)^{1/4} e^{i\theta/2} = (a^2+b^2)^{1/4}\left(\cos\theta/2 + i\sin\theta/2\right)$$

see reference source

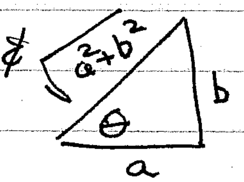
Now $\cos\theta/2 = \pm\left[\frac{1}{2}(1+\cos\theta)\right]^{1/2}$

trig identity

$\sin\theta/2 = \left[1/2(1-\cos\theta)\right]^{1/2}$

$\Rightarrow \cos\theta/2 = \pm\left[1/2\left(1+\frac{a}{\sqrt{a^2+b^2}}\right)\right]^{1/2}$

$\sin\theta/2 = \pm\left[1/2\left(1-\frac{a}{\sqrt{a^2+b^2}}\right)\right]^{1/2}$



$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$

$\therefore$

$$z^{1/2} = r^{1/2} e^{i\theta/2} = (a^2+b^2)^{1/4} e^{i\theta/2} = (a^2+b^2)^{1/4}\left(\cos\theta/2 + i\sin\theta/2\right)$$

$$= \frac{(a^2+b^2)^{1/4}}{\sqrt{2}}\left[\left(1+\frac{a}{\sqrt{a^2+b^2}}\right)^{1/2} + i\left(1-\frac{a}{\sqrt{a^2+b^2}}\right)^{1/2}\right]$$

Thus,

$$z^{1/2} = \left(\sqrt{a^2+b^2}\right)^{1/2} \frac{1}{\sqrt{2}} \left[\left(1 + \frac{a}{\sqrt{a^2+b^2}}\right)^{1/2} + i \left(1 - \frac{a}{\sqrt{a^2+b^2}}\right)^{1/2} \right]$$

$$\boxed{z^{1/2} = \frac{1}{\sqrt{2}} \left[\left(\sqrt{a^2+b^2} + a\right)^{1/2} + i \left(\sqrt{a^2+b^2} - a\right)^{1/2} \right]} \quad (1)$$

We see how complicated a simple square root calculation can become. Of course, when dealing with complex numbers (rather than symbols), in practice, we can directly evaluate r & θ & find the square root in polar coordinates, which of course is much simpler.

However, sometimes we need the analytic ~~not just~~ the numerical — expression for a square root.

Such a need arises, for instance, when considering the propagation of electromagnetic waves in conductors, where the square of the complex index of refraction ~~is~~ is given by Maxwell's equations ~~& the~~ and an expression for the index of refraction is desired.

equation (1) gives only one of the roots.

Clearly, there is another square root not included in (1).

To find this second root we must replace θ with $\theta + 2k\pi$ $k = 0, \pm 1, \pm 2 \cdots$ & $\theta/2$ with $\theta/2 + k\pi$ for $k = 0, \pm 1, \pm 2 \cdots$.

The distinct roots are obtained when ~~k = 0, 1, 2~~ $k = 0, 1$.

Thus, we have

$$\sqrt{z} = r^{1/2} e^{i(\theta/2 + k\pi)} = e^{ik\pi} \sqrt{r} e^{i\theta/2} \qquad k = 0, 1$$

$$= \pm \sqrt{r} e^{i\theta/2}.$$

The other root is simply the negative of that given by (1).

We can generalize this to the n^{th} root.

A general expression like (1) is ~~prohibitably~~ prohibitively difficult, however. On the other hand, the polar version is given by

$$z^{1/n} = r^{1/n} e^{i(\theta + 2k\pi)/n}$$

$$= r^{1/n} e^{i2k\pi/n} e^{i\theta/n} \qquad k = 0, 1, \cdots n-1$$

Examples

a) $\sqrt[5]{1+i} = (1+i)^{1/5} = (\,(1+i)\,)^{1/5} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{\frac{1}{5}}$

$$z = (1+i) = \sqrt{2}\, e^{i\pi/4}$$

$$\boxed{z = \sqrt{2}\, e^{i\pi/2 + 2\pi k}}$$

$$z^{\frac{1}{5}} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{\frac{1}{5}}$$

$$\boxed{z^{\frac{1}{5}} = (\sqrt{2})^{\frac{1}{5}}\, e^{i\pi/20}\, e^{i2k\pi/5}} \qquad k = 0, 1, 2, 3, 4$$

Now

$$z^{\frac{1}{5}} = (\sqrt{2})^{\frac{1}{5}} e$$

$z_1 \Rightarrow k = 0$

$$z_1 = (\sqrt{2})^{\frac{1}{5}} e^{i\pi/20} (1) = (\sqrt{2})^{\frac{1}{5}} (\cos \pi/20 + i \sin \pi/20)$$

$z_2 \Rightarrow k = 1$

$$z_2 = (\sqrt{2})^{1/5} e^{i\pi/20} e^{i2\pi/5} = (\sqrt{2})^{1/5} e^{i(\pi/20 + 2\pi/5)}$$

$$= (\sqrt{2})^{1/5} e^{i\, 9\pi/20}$$

$$z_2 = (\sqrt{2})^{1/5} (\cos 9\pi/20 + i \sin 9\pi/20)$$

$z_3 \Rightarrow k = 2$

$$z_3 = (\sqrt{2})^{1/5} e^{i\pi/20} e^{i\,4\pi/5} = (\sqrt{2})^{1/5} e^{i(\pi/20 + 4\pi/5)}$$

$$= (\sqrt{2})^{1/5} e^{i(17\pi/20)}$$

$$= \sqrt{2}^{1/5} (\cos 17\pi/20 + i \sin 17\pi/20)$$

$z_4 \Rightarrow k=3.$

$$z_4 = (\sqrt{2})^{1/5}\left(e^{i\pi/20}e^{i6\pi/5}\right) = (\sqrt{2})^{1/5} e^{i(\pi/20+6\pi/5)}.$$

$$= (\sqrt{2})^{1/5} e^{i\,25\pi/5}$$

$$= (\sqrt{2})^{1/5}\left(\cos 25\pi/\cancel{20} + i\sin 25\pi/20\right)$$

$$z_5 = (\sqrt{2})^{1/5}\left(e^{i\pi/20}e^{i8\pi/5}\right) = (\sqrt{2})^{1/5} e^{i(\pi/20+8\pi/5)}$$

$$(\sqrt{2})^{1/2} e^{i\,33\pi/\cancel{20}}$$

$$= (\sqrt{2})^{1/2}\left(\cos 33\pi/\cancel{20} + i\sin 33\pi/\cancel{20}\right).$$

$$z_5 = (\sqrt{2})^{\frac{1}{5}} e^{i\pi/20} e^{i\,8\pi/5} = (\sqrt{2})^{\frac{1}{5}} e^{i\left(\pi/20 + \frac{8\pi}{5}\right)}$$

$$z_5 = (\sqrt{2})^{1/5} e^{i\frac{33\pi}{20}} = (\sqrt{2})^{1/5}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$$

$$z_5^5 = \left(\sqrt{2}^{1/5} e^{i\frac{33\pi}{20}}\right)^5$$

$$\boxed{z_5^5 = \sqrt{2}\ e^{i\frac{33\pi}{4}}}$$ checks out

$$\cancel{\frac{33\pi}{4} + \frac{\pi}{4}} \qquad \frac{33\pi}{4} = \frac{32\pi}{4} + \pi/4$$

$$= 8\pi \cdot$$

$i^i = ?$ $\qquad e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$

$$= i$$

$$\therefore i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$$

Mathematical Physics. (1)

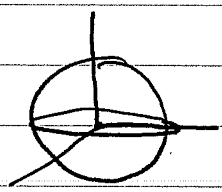
Lecture October 2001 { both lecture material & problems for students }

Problem 1.37

a) Check the divergence Theorem for the fn

$$\vec{v}_1 = r^2\hat{r} = r^2\hat{e}_r \quad \text{(spherical)}$$

using volume of sphere centered at the origin



$$\iint_{\text{Closed}} \vec{v}_1\cdot\hat{n}\,ds = \iiint_{\substack{\text{vol}\\ \text{encl}}} \vec{\nabla}\cdot\vec{v}_1\cdot dvol$$

$\vec{\nabla}\cdot\vec{v}_1$

in spherical: $\vec{\nabla}\cdot\vec{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

but here $v_\theta = v_\phi = 0$

$$\vec{\nabla}\cdot\vec{v}_1 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\right) = \frac{1}{r^2}4r^3 = 4r \quad \text{fn of } r \text{ only}$$

$$\int \vec{\nabla}\cdot\vec{v}\,dvol = \int (4r)\underbrace{(r^2\sin\theta\,dr\,d\theta\,d\phi)}_{dvol}$$

$$= 4\int_0^R r^3 dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= 4\frac{R^4}{4}(2)(2\pi) = 4\pi R^4$$

N.B. the volume integral could also be stated as

$$\int_0^R 4r\underbrace{(4\pi r^2 dr)}_{dvol} = 16\pi\int_0^R r^3 dr = 16\pi\frac{R^4}{4} = \underline{\underline{4\pi R^4}}$$

as before

Now calculate.

$$\iint_{\text{Closed}} \vec{v}_1 \cdot \hat{n}\, ds = \int r^2 \hat{r} \cdot \left(r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right) \qquad r = R \text{ at surface}$$

$$= \int r^4 \sin\theta\, d\theta\, d\phi$$

$$= r^4 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= R^4 (2)(2\pi) = 4\pi R^2.$$

as required

Problem 1.38

Compute divergence of the fn.

$$\vec{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$$
$$= (r\cos\theta)\hat{e}_r + (r\sin\theta)\hat{e}_\theta + (r\sin\theta\cos\phi)\hat{\phi}$$

in spherical coordinates

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2[r\cos\theta]\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\cos\theta\right) = \frac{1}{r^2}3r^2\cos\theta = \underline{3\cos\theta}$$

$$\therefore \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,[r\sin\theta]\right) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin^2\theta)\, r\right] = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin^2\theta)\right)$$
$$= \frac{1}{\sin\theta}\left[2\sin\theta\cos\theta\right]$$
$$= 2\cos\theta$$

$$\therefore \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r\sin\theta\cos\phi\right)$$
$$= \frac{1}{\cancel{r\sin\theta}}\left(\cancel{r\sin\theta}\right)\frac{\partial}{\partial\phi}\left(\cos\phi\right) = -\sin\phi$$

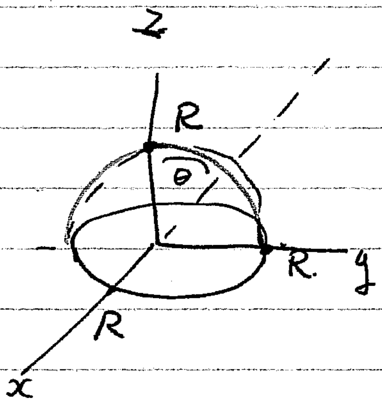
$$\therefore\ \boxed{\vec{\nabla}\cdot\vec{v} = 3\cos\theta + 2\cos\theta - \sin\phi}$$

$$\boxed{\vec{\nabla}\cdot\vec{v} = 5\cos\theta - \sin\phi}$$ as required.

Check the divergence theorem:

$$\iiint \vec{\nabla}\cdot\vec{v}\, d\text{vol} = \iiint (5\cos\theta - \sin\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^R r^2 dr \int_0^{\pi/2} \left[\int_0^{2\pi} (5\cos\theta - \sin\phi)\, d\phi \right] d\theta \sin\theta$$



$$= \frac{R^3}{3} \int_0^{\pi/2} \left[5\cos\theta(2\pi) + 0 \right] d\theta \sin\theta.$$

$$= \frac{R^3}{3} \int_0^{\pi/2} 2\pi (5\cos\theta\sin\theta)\, d\theta$$

$$= \frac{R^3}{3}(10\pi) \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta$$

$$= \frac{R^3}{3}(10\pi) \underbrace{\left\{ \frac{\sin^2\theta}{2} \Big|_0^{\pi/2} \right\}}_{=1/2}$$

$$= \frac{5\pi R^3}{3}$$

Now let us look at the surface integral.

We must integrate over two surfaces — one, the hemisphere

$$d\vec{A} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r} \qquad r = R$$

ϕ: $0 \to 2\pi$,

θ: $0 \to \pi/2$

$$\int \vec{v}\cdot d\vec{A} = \int (r\cos\theta)\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= R^3 \left(\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta \right) \int_0^{2\pi} d\phi$$

$$= R^3 \left(\frac{\sin^2\theta}{2} \right)\Big|_0^{\pi/2} 2\pi$$

$$= R^3 \left(\tfrac{1}{2}\right)(2\pi) = \pi R^3$$

the other surface is the flat bottom:

$\sin\pi/2 = 1$

$$\vec{dA} = 2\pi r\,dr\,\hat{\theta} = (dr)(r\sin\theta)\,d\phi\,\hat{\theta}$$
$$= 2\pi r\,dr\,\hat{\theta} = 2\pi r\,dr\,\hat{e}_\theta$$

$\sin\theta = 1 \Rightarrow \sin\pi/2 = 1$

$$\therefore \int \vec{v}\cdot\vec{dA} = \int (r\sin\theta)(2\pi r\,dr)$$
$$= \int_0^R r^2 dr\, 2\pi = 2\pi \frac{R^3}{3}$$

Total: $\int \vec{v}\cdot\vec{dA} = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$ it checks!!

as required.

Problem 1.39

Compute the gradient & Laplacian.

for $T = r(\cos\theta + \sin\theta\cos\phi)$.

the Gradient in spherical Coordinates:

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{e}_\phi$$

hence in our case we write.

$$\vec{\nabla}T = (\cos\theta + \sin\theta\cos\phi)\hat{e}_r + \frac{1}{r}r(-\sin\theta + \cos\theta\cos\phi)\hat{e}_\theta$$

$$+\frac{1}{r\sin\theta}r(-\sin\theta\sin\phi)\hat{e}_\phi$$

or collecting all our terms:

$$\boxed{\vec{\nabla}T = (\cos\theta + \sin\theta\cos\phi)\hat{e}_r + (-\sin\theta + \cos\theta\cos\phi)\hat{e}_\theta - \sin\phi\,\hat{e}_\phi}$$

Now we need the Laplacian in spherical coordinates