Math Phys

Mon. Sept 10, 2001 3 M71 11:30 → 12:20 pm

Survey

Name:

Physics taken — Physics taking.

Math taken — Math taking

Major: Physics 4-yr
Honours

double major ?

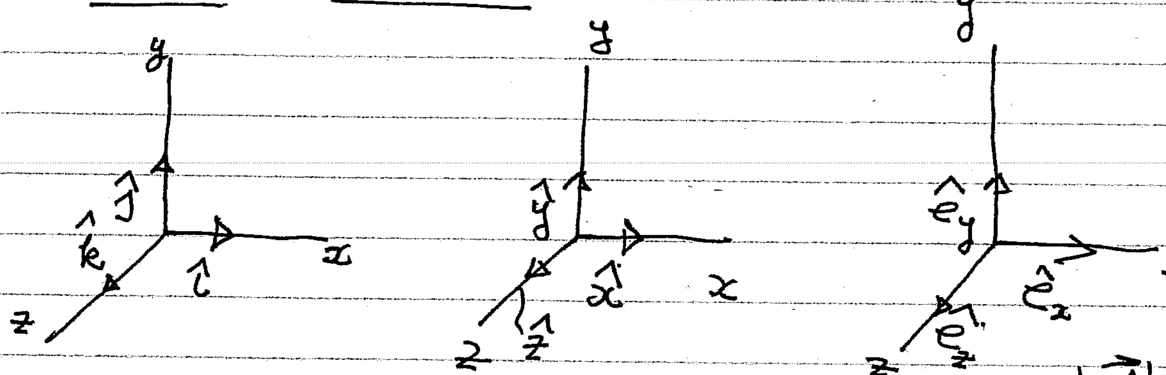
Textbook: — Comments.

Handbook — highly recommended

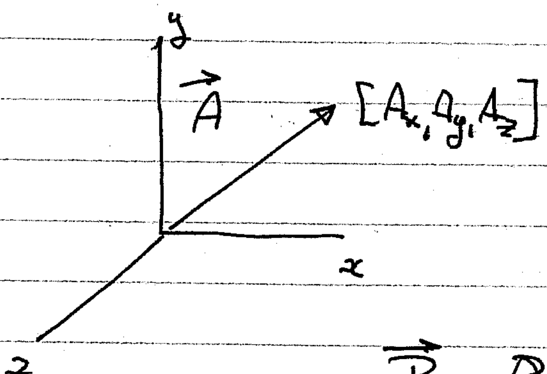
Coordinate Systems.

Calculus of Vectors.

Cartesian Coordinates



$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



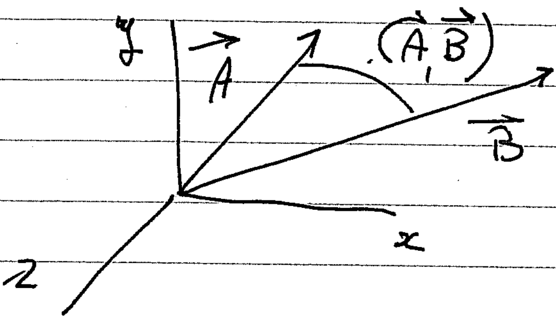
$$\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}$$

$$= A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z$$

$$\vec{B} = B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}$$

$$\vec{A} \pm \vec{B} = (A_x \pm B_x)\hat{\imath} + (A_y \pm B_y)\hat{\jmath} + (A_z \pm B_z)\hat{k}$$

$$\vec{A}\cdot\vec{B} \equiv |\vec{A}||\vec{B}| \cos(\vec{A},\vec{B}) = A_x B_x + A_y B_y + A_z B_z$$



$$\vec{A}\times\vec{B} = ? \qquad |\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}| \sin(\vec{A},\vec{B}).$$

r.h.s rule -

$\vec{A}\times\vec{B}$ $\perp$ to plane containing $(\vec{A},\vec{B})$

+ r h s rule

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \cancel{\begin{vmatrix} A_x & B_x & C_x \\ B_x & B_y & \\ & & \end{vmatrix}} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Mathematical Physics

Lecture 2: Wednesday September 12, 2001

Given: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

We know how to add/subtract.

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Also, given $\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

& $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$$

$$= A_xB_yC_z - A_xB_zC_y - A_yB_xC_z + A_yB_zC_x + A_zB_xC_y - A_zB_yC_x$$

$$\vec{C}\cdot(\vec{A}\times\vec{B}) = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= C_x(A_yB_z - A_zB_y) - C_y(A_xB_z - A_zB_x) + C_z(A_xB_y - A_yB_x)$$

$$= C_xA_yB_z - C_xA_zB_y - C_yA_xB_z + C_yA_zB_x + C_zA_xB_y - C_zA_yB_x$$

$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$ in general

We can show $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

{ 1st assignment has this problem }

Differentiation of Vector functions

Given: $\vec{A}(t) = A_x(t)\,\hat{\imath} + A_y(t)\,\hat{\jmath} + A_z(t)\,\hat{k}$

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t\to 0}\left\{\left[\frac{A_x(t+\Delta t)-A_x(t)}{\Delta t}\right]\hat{\imath} + \left[\frac{A_y(t+\Delta t)-A_y(t)}{\Delta t}\right]\hat{\jmath} + \left[\frac{A_z(t+\Delta t)-A_z(t)}{\Delta t}\right]\hat{k}\right\}$$

t is a parameter here

in mechanics it is the time.

$$\boxed{\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{\imath} + \frac{dA_y}{dt}\hat{\jmath} + \frac{dA_z}{dt}\hat{k}}$$

Also we may write

$\alpha = \alpha(t)$ scalar fn.

① $\frac{d}{dt}[\alpha \vec{A}(t)]$

$$\frac{d}{dt}\left(\alpha(t)\vec{A}(t)\right) = \frac{d\alpha}{dt}(\vec{A}) + \alpha\frac{d\vec{A}}{dt}$$ as we expect

Now suppose we have $\vec{A}(t)$, $\vec{B}(t)$

$$\frac{d}{dt}\left(\vec{A}\circ\vec{B}\right) = \frac{d\vec{A}}{dt}\circ\vec{B} + \vec{A}\circ\frac{d\vec{B}}{dt}$$

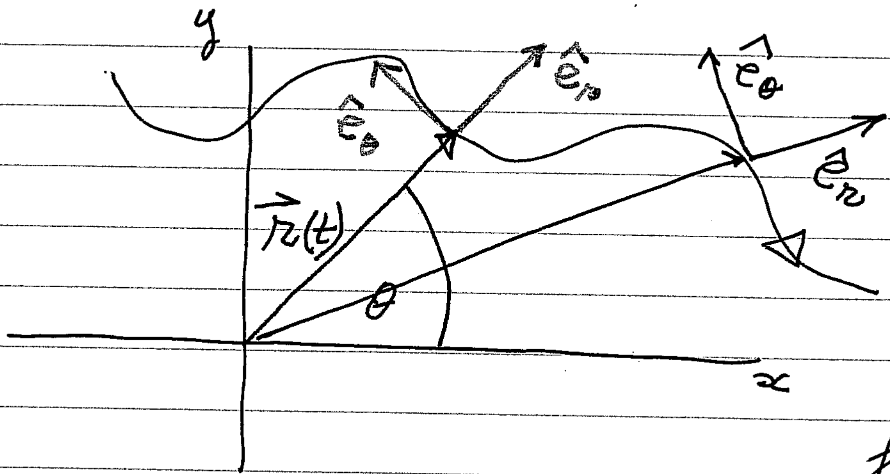
$$\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \cancel{\vec{A}\times\frac{d\vec{B}}{dt}} +$$

$$= \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}.$$

All this works well and is also straight-forward because we are talking about Cartesian Vectors

$\Rightarrow \hat{\imath}, \hat{\jmath}, \hat{k}$ are fixed

Plane Polar Coordinates / example from Mechanics



$\vec{r}$ is the position

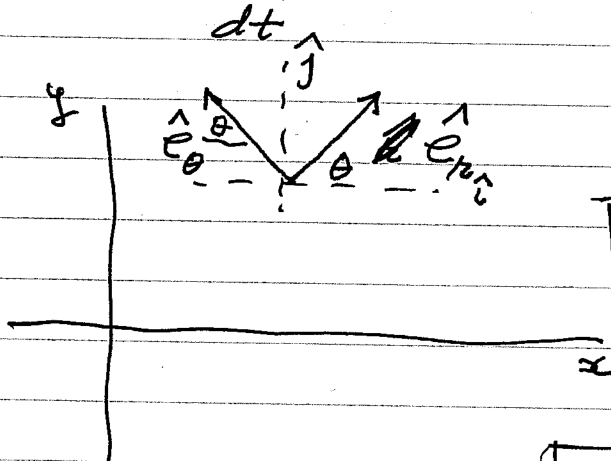
t is the time.

Note: $\hat{e}_r, \hat{e}_\theta$ are fn of time.

$$\vec{r}(t) = |\vec{r}(t)|\,\hat{e}_r$$
$$= r\hat{e}_r$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r)$$
$$= \frac{dr}{dt}\hat{e}_r + r\frac{d}{dt}\hat{e}_r$$

what is $\frac{d\hat{e}_r}{dt} = ?$



$$\hat{e}_r = |\hat{e}_r|\cos\theta\,\hat{i} + |\hat{e}_r|\sin\theta\,\hat{j}$$

$$\boxed{\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}}$$

$$\hat{e}_\theta = -|\hat{e}_\theta|\sin\theta\,\hat{i} + |\hat{e}_\theta|\cos\theta\,\hat{j}$$

$$\boxed{\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}}$$

Now what is $\frac{d}{dt}\hat{e}_1 = \frac{d}{dt}\left\{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right\}$

$$= -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}.$$

$$= \frac{d\theta}{dt}\left\{-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right\}.$$

$$= \frac{d\theta}{dt}\hat{e}_\theta$$

$\therefore$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r\hat{e}_r\right) = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta.$$

$$\vec{v}(t) = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta$$

radial velocity   tang. velocity

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left\{\frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta\right\}$$

$$= \frac{d^2r}{dt^2}e_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta + r\frac{d\theta}{dt}\frac{d\hat{e}_\theta}{dt}$$

we need $\frac{d}{dt}\hat{e}_\theta = \frac{d}{dt}\left\{-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right\}$

$$= -\cos\theta\frac{d\theta}{dt}\hat{i} + \sin\theta\frac{d\theta}{dt}\hat{j}$$

$$= \frac{d\theta}{dt}\left\{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right\}$$

but $\Rightarrow \frac{d}{dt}\hat{e}_\theta = -\frac{d\theta}{dt}\hat{e}_r$

Now put all together.

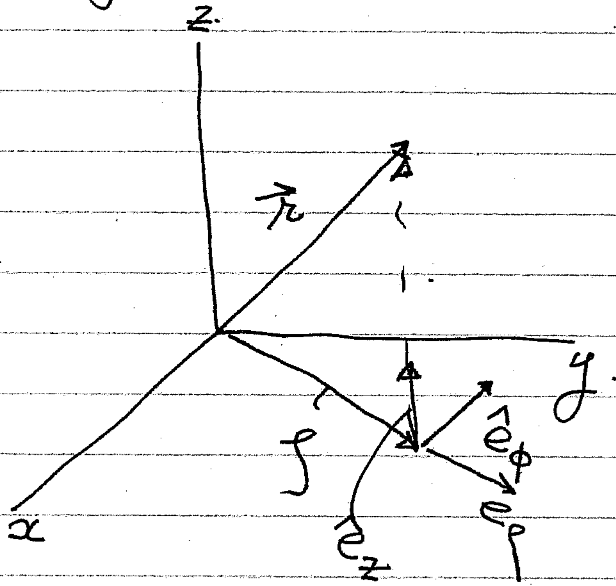
$$\vec{a}(t) = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{e}_\theta$$

$\frac{d}{dt} = \dot{}$

$\frac{d^2}{dt^2} = \ddot{}$

$$= \left[\underbrace{\ddot{r}}_{\text{radial}} - \underbrace{r(\dot{\theta})^2}_{\text{Centripetal}}\right]\hat{e}_r + \left[\underbrace{r\ddot{\theta}}_{\text{tangential}} + \underbrace{2\dot{r}\dot{\theta}}_{\text{Coriolis}}\right]\hat{e}_\theta$$

Cylindrical Coordinates:



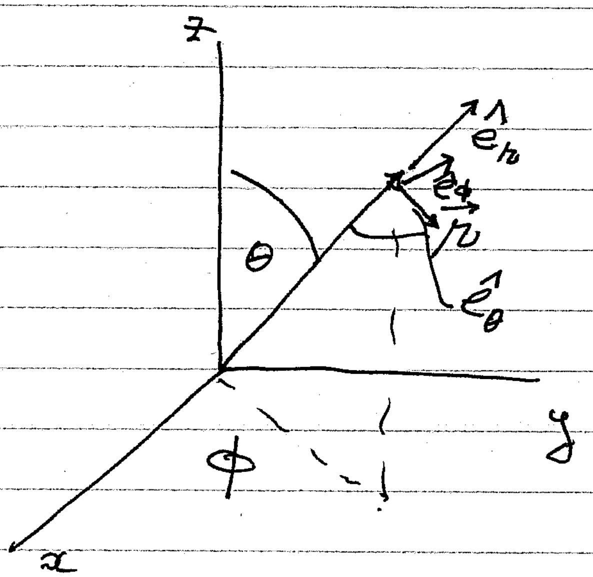
(ρ, ϕ, z)

$\hat{e}_z = \hat{k}$

$\vec{r} = \rho \hat{e}_\rho + z \hat{e}_z$

$\hat{e}_\rho = \cos\phi \hat{i} + \sin\phi \hat{j}$

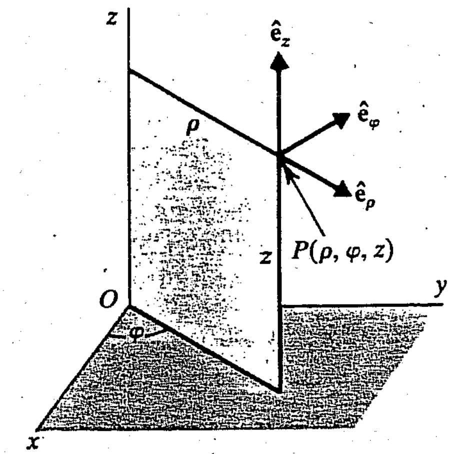
Spherical Coordinates



r, θ, ϕ

$\vec{r} = |\vec{r}| \hat{e}_r$

Cylindrical Coordinates



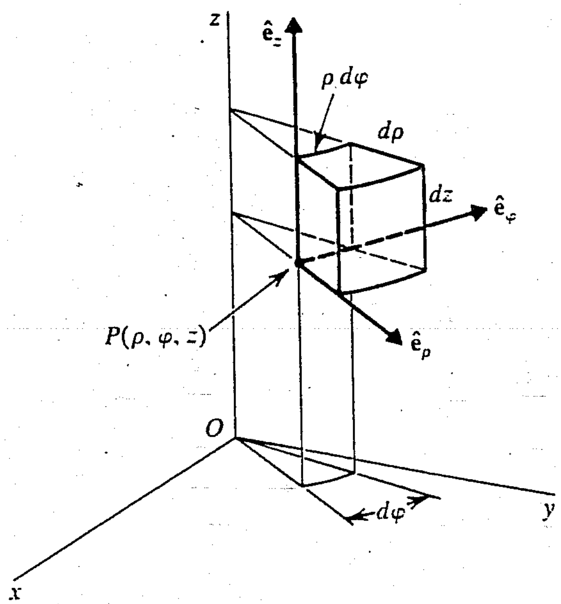
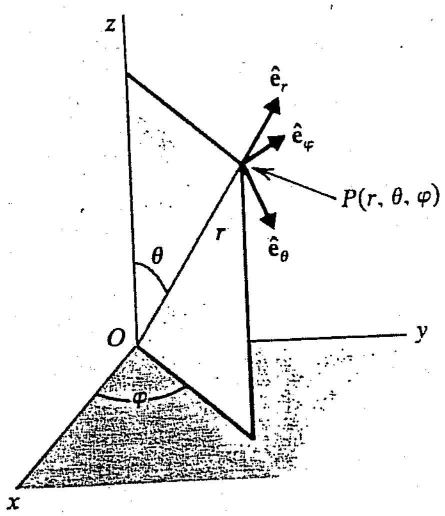


Figure The distances involved in changing the coordinates ρ, φ, and z by infinitesimal amounts. The volume element corresponding to these distances is also shown.

Spherical Coordinates



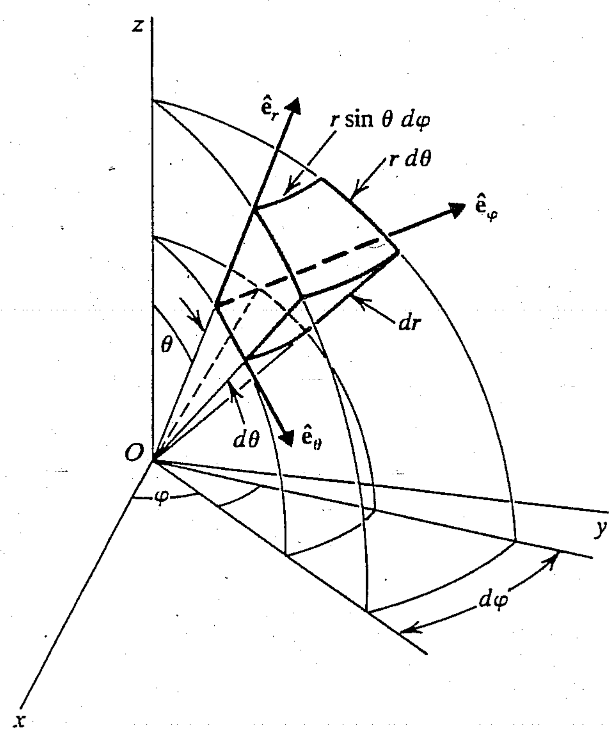


Figure . Displacements and volume element in the spherical coordinate system.

Mathematical Physics

Problems: Ch 1. Charlie Harper "yellow handout-Pg 38"

1. $\vec{A}+\vec{B}=?$
$\vec{A}-\vec{B}=?$

$$\vec{A}=2\hat{i}-\hat{j}+\hat{k} \qquad \vec{B}=\hat{i}-3\hat{j}-5\hat{k}$$

$$\vec{A}+\vec{B}=(2+1)\hat{i}+(-1-3)\hat{j}+(1-5)\hat{k}$$
$$=\underline{3\hat{i}-4\hat{j}-4\hat{k}}$$

$$\vec{A}-\vec{B}=(2-1)\hat{i}+(-1+3)\hat{j}+(1+5)\hat{k}$$
$$=\underline{\hat{i}+2\hat{j}+6\hat{k}}$$

as required

4. $\vec{A}=\hat{i}+4\hat{j}+3\hat{k}$ $\qquad \vec{A}\cdot\vec{B}=(1)(4)+(4)(2)-(3)(12)=0$
$\vec{B}=4\hat{i}+2\hat{j}-4\hat{k}$ $\qquad \vec{A}\perp\vec{B}$

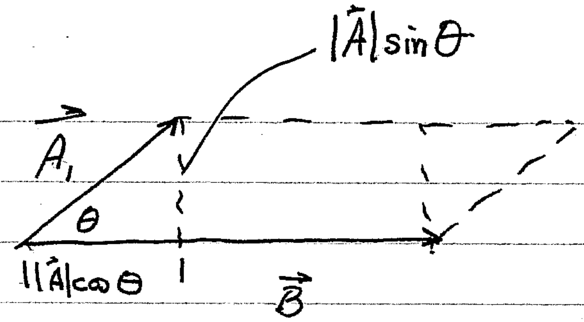
5. $\vec{A}=2\hat{i} \qquad \vec{B}=3\hat{i}+4\hat{j}$

$$\vec{A}\cdot\vec{B}=|\vec{A}||\vec{B}|\cos(\vec{A},\vec{B}) \Rightarrow \cos(\vec{A},\vec{B})=\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}=\frac{(2)(3)}{(2)(5)}=\frac{6}{10}$$

$$(\vec{A},\vec{B})=\cos^{-1}\left(\frac{6}{10}\right)=\cos^{-1}(.6)=53.1^{\circ}$$

as required

6.



Area = (base)(height)

$\therefore$ Area $= |\vec{B}||\vec{A}|\sin\theta \Rightarrow |\vec{A}\times\vec{B}|$

9. $\vec{A} = 3\hat{i} + 2\hat{j}$ $\quad\vec{B} = 2\hat{j} - 4\hat{k}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 0 & 2 & -4 \end{vmatrix} = \hat{i}(-8) - \hat{j}(-12) + \hat{k}6$$

$$= -8\hat{i} + 12\hat{j} + 6\hat{k}$$

$$|\vec{A}\times\vec{B}| = \sqrt{8^2 + 12^2 + 6^2} = \sqrt{64+144+36} = \sqrt{244} = 15.6\ m^2 = \text{area}$$

as required.

12.

$$\hat{k}\times\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0\hat{i} - \hat{j}(-1) + \hat{k}(0)$$

$$= \hat{j}$$

$\hat{k}\times\hat{i} = \hat{j}$ as required

13. $\vec{A} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\vec{B} = \hat{i} + 2\hat{j} - \hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \hat{i}(8) - \hat{j}(2) + \hat{k}(4).$$

$\vec{C} = \vec{A} \times \vec{B} = 8\hat{i} - 2\hat{j} + 4\hat{k}$ $\quad$ $\vec{C}$ is $\perp$ to plane containing $\vec{A}, \vec{B}$.

we want $\hat{C} = \dfrac{\vec{C}}{|\vec{C}|} = \dfrac{8\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{8^2 + 2^2 + 4^2}}$.

$$\hat{C} = \frac{8\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{84}}$$

14 $\vec{A} = a\hat{i} - 2\hat{j} + \hat{k}$ is $\perp$ to $\vec{B} = \hat{i} - 2\hat{j} - 3\hat{k}$ $\quad$ find a

$\vec{A} \cdot \vec{B} = a + 4 - 3 = 0 \Rightarrow a = 3 - 4$

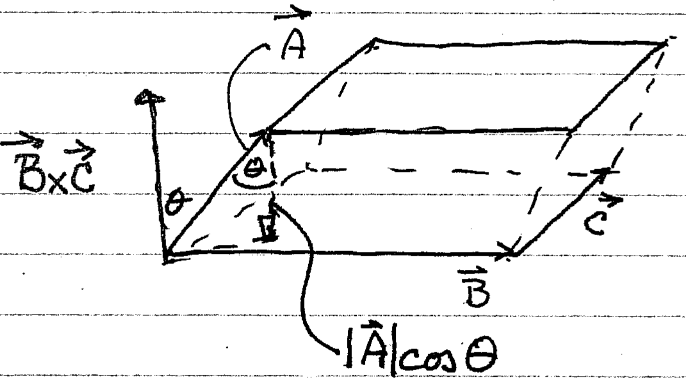
$\underline{\underline{a = 1}}$ as required.

16. find $\vec{A} \cdot (\vec{B} \times \vec{C})$ $\quad$ $\vec{A} = 2\hat{i}$ $\quad$ $\vec{B} = 3\hat{j}$ $\quad$ $\vec{C} = 4\hat{k}$.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 24 \quad \text{as required}$$

17.

Volume of parallelepiped = (area of base)(altitude.



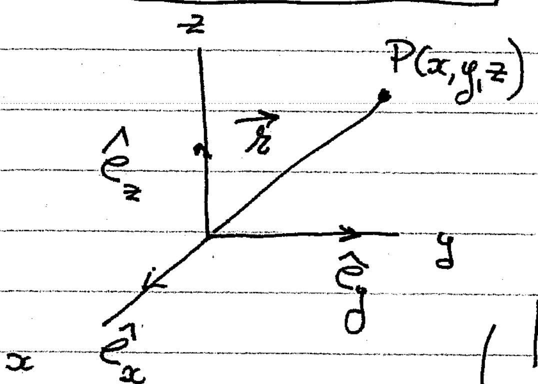
Volume = $|\vec{B}\times\vec{C}|\left(|\vec{A}|\cos\theta\right) = \left|(\vec{B}\times\vec{C})\cdot\vec{A}\right|$ as required.

Mathematical Physics

Lecture, Friday, Sept 14, 2001

Let us look at some useful details of the coordinate systems of interest to us. — these must be compared to the details of the well-known Cartesian System

Cartesian Coordinates



$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$d\vec{r} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$$

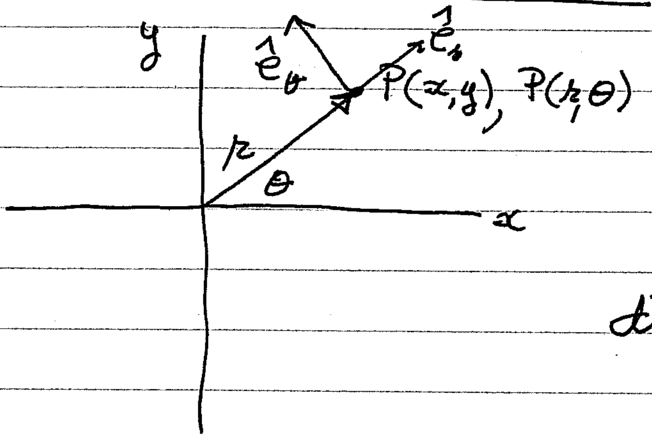
no problem

$$\left.\begin{aligned} |d\vec{r}| &= \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \\ d\,vol &= dx\,dy\,dz \\ d\,area &= dx\,dy,\ dy\,dz,\ dz\,dx \end{aligned}\right\}$$

dx, dy, dz each have units of length.

In Cartesian coordinates, the point $P(x,y,z)$ is described unambiguously. Also $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are fixed in space.

Plane Polar Coordinates



$\vec{r} = r\hat{e}_r$ — does not contain θ, not unique

$d\vec{r} = d(r\hat{e}_r) =$

$$d\vec{r}\Big|_{\hat{e}_r \text{ fixed}} = \hat{e}_r\, dr$$

$$d\vec{r}\Big|_{r_{\text{fixed}}} = r\, d\hat{e}_r = r\, d\theta\, \hat{e}_\theta$$

but $\hat{e}_r = \cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y$ $\qquad$ $\hat{e}_\theta = -\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y$

$$d\hat{e}_r = -\sin\theta\, d\theta\, \hat{e}_x + \cos\theta\, d\theta\, \hat{e}_y$$

$$= (-\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y)\, d\theta$$

$$= d\theta\, \hat{e}_\theta$$

$d\vec{r} = \hat{e}_r\, dr + r\, d\theta\, \hat{e}_\theta$

$$\therefore\ d\vec{r} = d(r\hat{e}_r) = dr\, \hat{e}_r + r\, d\theta\, \hat{e}_\theta$$

$|d\vec{r}| = \sqrt{(dr)^2 + (r\, d\theta)^2}$

$$|d\vec{r}| = \sqrt{(dr)^2 + (r\, d\theta)^2}$$

Notice r, θ do not play the equivalent roles of (x, y)

Cartesian

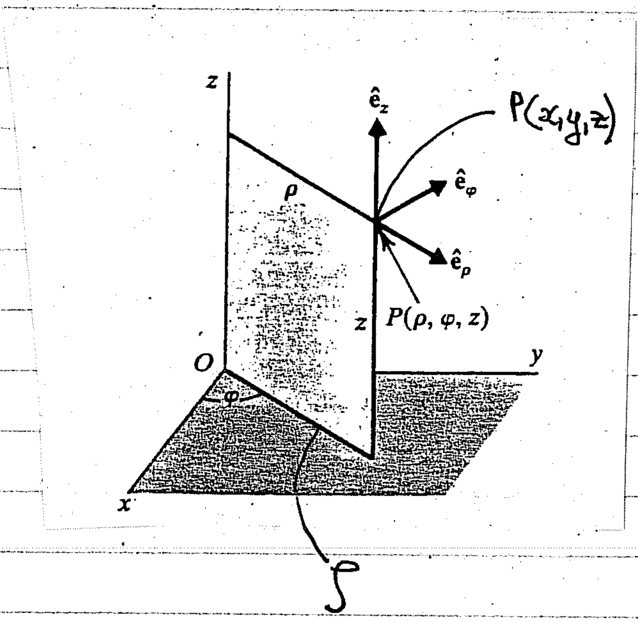
$$|d\vec{r}| = \sqrt{(dx)^2 + (dy)^2}$$

dr = length.
$r\, d\theta$ = length.

$d(\text{area}) = r\, d\theta\, dr$

in Cartesian
$d(\text{area}) = dx\, dy$

Cylindrical Coordinates



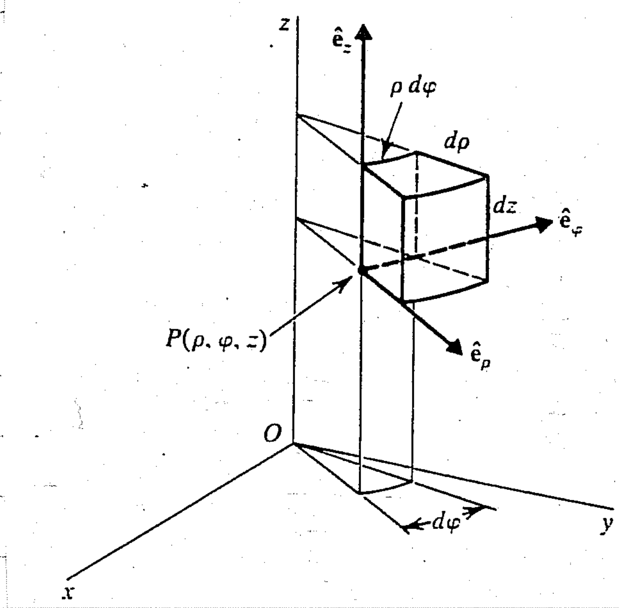
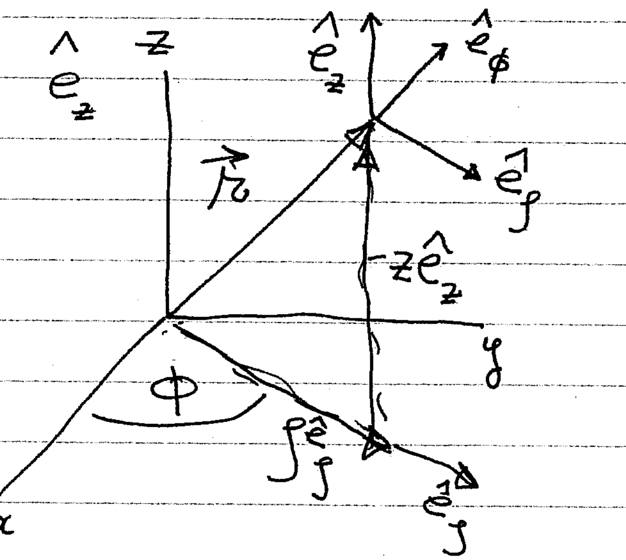


Figure The distances involved in changing the coordinates ρ, φ, and z by infinitesimal amounts. The volume element corresponding to these distances is also shown.

(ρ, ϕ, z)

Warning: if you look elsewhere, find different notation



Observe

$$\vec{r}\,(\text{position vector}) = \hat{e}_\rho\, \rho + \hat{e}_z\, z$$

Observe, the above does not involve ϕ, hence the determination of the point is not unique — just as with polar coordinates

element of length

$d\vec{r} = ?$

$$d\vec{r}\Big|_{\phi, z \text{ fixed}} = \hat{e}_\rho \, d\rho \quad , \quad d\vec{r}\Big|_{\rho, z \text{ fixed}} = \hat{e}_\phi \, \rho \, d\phi$$

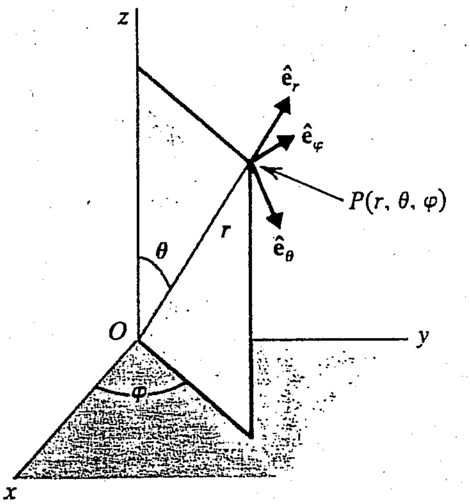
$$d\vec{r}\Big|_{\rho, \phi \text{ fixed}} = \hat{e}_z \, dz$$

hence $\cancel{d\vec{r}}$ $d\vec{r} = \hat{e}_\rho \, d\rho + \hat{e}_\phi \, \rho \, d\phi + \hat{e}_z \, dz$

$$|d\vec{r}| = \sqrt{(d\rho)^2 + (\rho \, d\phi)^2 + (dz)^2}$$

$$d\,vol = (\rho \, d\phi)(d\rho)(dz)$$

Spherical Coordinates



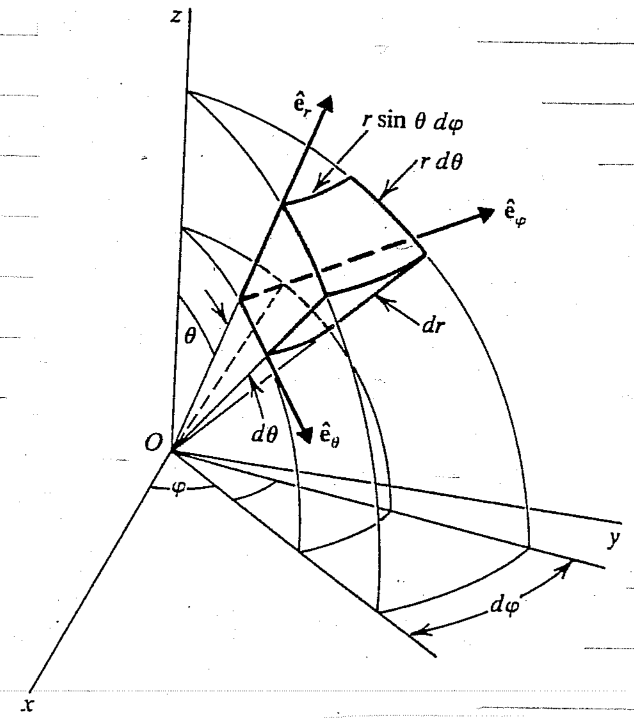
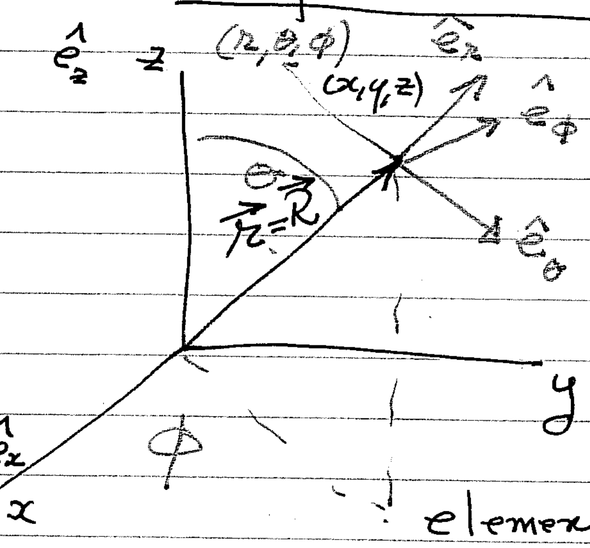


Figure Displacements and volume element in the spherical coordinate system.

The position Vector:



$$\vec{R} = \vec{r} = r\hat{e}_r = \vec{R}$$

Notice, θ, ϕ do not appear in position vector — hence $P(x,y,z)$ in spherical (r, θ, ϕ) is not uniquely determined.

element of length.

$$\boxed{d\vec{R} = d\vec{r}} \qquad \cancel{\text{use } d\vec{R}}$$

$$d\vec{r}\Big|_{\theta, \phi \text{ fixed}} = \hat{e}_r\, dr \quad , \quad d\vec{r}\Big|_{r, \phi \text{ fixed}} = r\, d\theta\, \hat{e}_\theta$$

$$d\vec{r}\Big|_{r, \theta} = r \sin\theta\, d\phi\, e_\phi$$

or $d\vec{r} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r \sin\theta \, d\phi$

$$|d\vec{r}| = \sqrt{(dr)^2 + (r d\theta)^2 + (r\sin\theta \, d\phi)^2}$$

$$dvol = dr\,(r\,d\theta)(r\sin\theta\,d\phi)$$
$$= r^2 \sin\theta \, dr\, d\theta\, d\phi$$

The relationship between Cartesian Coordinates & cylindrical, spherical.

Cylindrical (See hand-out)

$$\left.\begin{array}{l} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{array}\right\} \Rightarrow \begin{array}{l} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{array}$$

$$\boxed{\vec{r} = \hat{e}_x x + \hat{e}_y y + \hat{e}_z z = \hat{e}_x \rho\cos\phi + \hat{e}_y \rho\sin\phi + \hat{e}_z z}$$

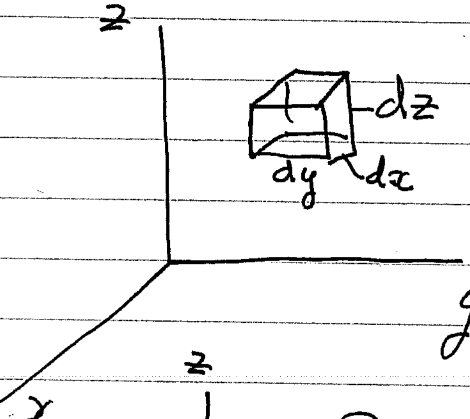
Spherical (See hand-out)

$$\begin{array}{lll} x = r\sin\theta\cos\phi & r = \sqrt{x^2+y^2+z^2} & \phi = \tan^{-1}(y/x) \\ y = r\sin\theta\sin\phi & \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) & \\ z = r\cos\theta & \text{or } \theta = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) & \end{array}$$

$$\vec{r} = \hat{e}_x x + \hat{e}_y y + \hat{e}_z z = \hat{e}_x r\sin\theta\cos\phi + \hat{e}_y r\sin\theta\sin\phi + \hat{e}_z r\cos\theta$$

It would be nice if we could write the unit vectors $(\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z)$ and $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ in terms of $\hat{e}_x, \hat{e}_y, \hat{e}_z$ —

How to do this?

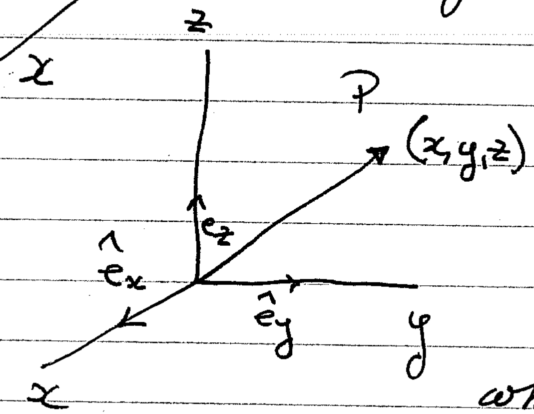


$d\,vol = dx\,dy\,dz.$

$x, y, z \to h_1\ h_2\ h_3$

$\rho\ \phi\ z \quad h_1\ h_2\ h_3$

$r, \theta, \phi \to h_1\ h_2\ h_3$



$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z.$

$d\vec{r}\Big|_{y,z\ \text{fixed}} = dx\,\hat{e}_x \Rightarrow \frac{d\vec{r}}{dx}\Big|_{y,z\ \text{fixed}} = \hat{e}_x$

what is $d\vec{r}\Big|_{y,z\ \text{fixed}} = \partial\vec{r}$

$$\boxed{\frac{\partial \vec{r}}{\partial x} = h_1\hat{e}_x}$$

$d\vec{r}\Big|_{x,z\ \text{fixed}} = dy\,\hat{e}_y$

$\frac{d\vec{r}}{dy}\Big|_{x,z\ \text{fixed}} = \frac{\partial \vec{r}}{\partial y} = h_2\hat{e}_y$

& finally $d\vec{r}\Big|_{x,y\ \text{fixed}} = dz\,\hat{e}_z$

or $\frac{d\vec{r}}{dz}\Big|_{x,y\ \text{fixed}} \Rightarrow \boxed{\frac{\partial \vec{r}}{\partial z} = h_3\hat{e}_z}$

Hence $\boxed{d\vec{r} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z}$

In the Cartesian case — clearly —

$$\boxed{\begin{array}{ll} \hat{e}_x = \dfrac{1}{h_1}\dfrac{\partial \vec{r}}{\partial x} & h_1 = 1 \\ \hat{e}_y = \dfrac{1}{h_2}\dfrac{\partial \vec{r}}{\partial y} & h_2 = 1 \\ \hat{e}_z = \dfrac{1}{h_3}\dfrac{\partial \vec{r}}{\partial z} & h_3 = 1 \end{array}}$$

Cylindrical Coordinates. ρ, ϕ, z.

$\vec{r} = \rho\,\hat{e}_\rho + z\,\hat{e}_z$ (see figure)

$$d\vec{r}\Big|_{\phi, z \text{ fixed}} = d\rho\,\hat{e}_\rho \Rightarrow \boxed{\frac{\partial \vec{r}}{\partial \rho} = h_1 \hat{e}_\rho} \Rightarrow h_1 = 1$$

$$d\vec{r}\Big|_{\rho, z \text{ fixed}} = \rho\,d\phi\,\hat{e}_\phi \Rightarrow \frac{\partial \vec{r}}{\partial \phi} = \rho\,\hat{e}_\phi = h_2\,\hat{e}_\phi \Rightarrow h_2 = \rho.$$

$$d\vec{r}\Big|_{\rho, \phi \text{ fixed}} = dz\,\hat{e}_z \Rightarrow \frac{\partial \vec{r}}{\partial z} = h_3\,\hat{e}_z \Rightarrow h_3 = 1$$

$\therefore$ Cylinder: $h_1 = 1$, $h_2 = \rho$ $h_3 = 1$

Spherical Coordinates:

$d\vec{r} = r\hat{e}_r$

but now see figure.

$$d\vec{r}\Big|_{\theta,\phi \text{ fixed}} = dr\,\hat{e}_r \Rightarrow \frac{\partial \vec{r}}{\partial r} = h_1 \hat{e}_r \Rightarrow h_1 = 1$$

$$d\vec{r}\Big|_{r,\phi \text{ fixed}} = r\,d\theta\,\hat{e}_\theta \Rightarrow \frac{\partial \vec{r}}{\partial \theta} = h_2 \hat{e}_\theta \Rightarrow h_2 = r$$

$$d\vec{r}\Big|_{r,\theta \text{ fixed}} = r\sin\theta\,d\phi\,\hat{e}_\phi \Rightarrow \frac{\partial \vec{r}}{\partial \phi} = h_3 \hat{e}_\phi \Rightarrow h_3 = r\sin\theta.$$

Cylindrical

also $\vec{r} = \hat{e}_x x + \hat{e}_y y + \hat{e}_z z = \hat{e}_x \rho\cos\phi + \hat{e}_y \rho\sin\phi + \hat{e}_z z$

$$\boxed{\hat{e}_\rho = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = \frac{1}{1}(\hat{e}_x\cos\phi + \hat{e}_y\sin\phi)}$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}\left(-\hat{e}_x\rho\sin\phi + \hat{e}_y\rho\cos\phi\right)$$

$$\boxed{\hat{e}_\phi = -\hat{e}_x\sin\phi + e_y\cos\phi.}$$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \frac{1}{1}\hat{e}_z$$

Spherical

$$\vec{r} = \hat{e}_x x + \hat{e}_y y + \hat{e}_z z = \hat{e}_x r\sin\theta\cos\phi + \hat{e}_y r\sin\theta\sin\phi + \hat{e}_z r\cos\theta$$

$$\hat{e}_r = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r} = \left(\hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z \cos\theta\right)\frac{1}{1}$$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta} = \frac{1}{r}\left[\hat{e}_x r\cos\theta\cos\phi + \hat{e}_y r\cos\theta\sin\phi - \hat{e}_z r\sin\theta\right]$$

$$\boxed{\hat{e}_\theta = \hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi - \hat{e}_z\sin\theta}$$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial\phi} = \frac{1}{r\sin\theta}\left[-\hat{e}_x r\sin\theta\sin\phi + \hat{e}_y r\sin\theta\cos\phi\right]$$

$$\boxed{\hat{e}_\phi = -\hat{e}_x\sin\phi + \hat{e}_y\cos\phi}$$

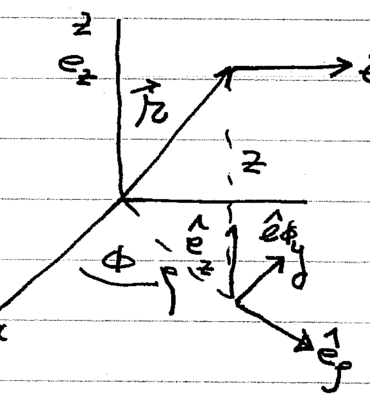
Mathematical Physics

Lecture: Monday, September 17, 2001

Finish up material on relationship amongst the different sets of unit vectors.

Now obtain the kinematic quantities in cylindrical and spherical coordinates:

Cylindrical



$$\vec{r} = \hat{e}_\rho \rho + \hat{e}_z$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\hat{e}_\rho \rho + \hat{e}_z z\right)$$

$$= \dot{\rho}\hat{e}_\rho + \rho \frac{d\hat{e}_\rho}{dt} + \dot{z}\hat{e}_z$$

$$\hat{e}_\rho = \cos\phi\, \hat{e}_x + \sin\phi\, \hat{e}_y$$

$$\hat{e}_\phi = -\sin\phi\, \hat{e}_x + \cos\phi\, \hat{e}_y$$

which suggests

$$\frac{d\hat{e}_\rho}{dt} = \left(-\sin\phi\, \hat{e}_x + \cos\phi\, \hat{e}_y\right)\dot{\phi} = \dot{\phi}\,\hat{e}_\phi$$

$$\frac{d\hat{e}_\phi}{dt} = \left(-\cos\phi\, \hat{e}_x - \sin\phi\, \hat{e}_y\right)\dot{\phi} = -\dot{\phi}\,\hat{e}_\rho$$

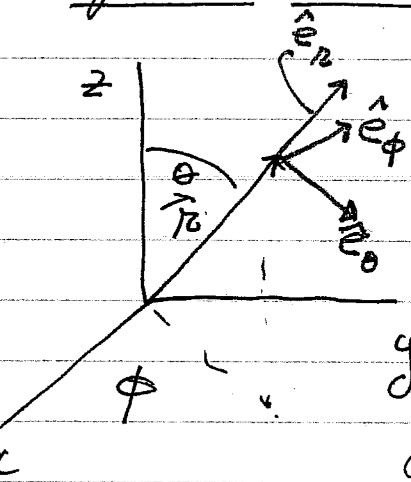
hence $\boxed{\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \dot{\rho}\,\hat{e}_\rho + \rho\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z}$ cylindrical

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{\rho}\,\hat{e}_\rho + \dot{\rho}\frac{d\hat{e}_\rho}{dt} + \dot{\rho}\dot{\phi}\,\hat{e}_\phi + \rho\ddot{\phi}\,\hat{e}_\phi + \rho\,\dot{\phi}\frac{d\hat{e}_\phi}{dt} + \ddot{z}\,\hat{e}_z$$

$$\boxed{\vec{a}(t) = \left(\ddot{\rho} - \rho(\dot{\phi})^2\right)\hat{e}_\rho + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{e}_\phi + \ddot{z}\,\hat{e}_z}$$ cylindrical

---

Spherical Coordinates:



$$\vec{r}(t) = r\,\hat{e}_r$$

$$\vec{v}(t) = \frac{d}{dt}\left(r\,\hat{e}_r\right) = \dot{r}\,\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

from page

$$\hat{e}_r = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z \cos\theta$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_x\left[\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi}\right] + \hat{e}_y\left[\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi}\right] - \hat{e}_z\sin\theta\,\dot{\theta}$$

$$= \left[\hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi\right]\dot{\theta} + \left[-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi\right]\sin\theta\,\dot{\phi} - \hat{e}_z\sin\theta\,\dot{\theta}$$

$$= \hat{e}_\theta\left[\dot{\theta}\right] + \hat{e}_\phi\sin\theta\,\dot{\phi}$$

$$\therefore \boxed{\frac{d\hat{e}_r}{dt} = \hat{e}_\theta\,\dot{\theta} + \hat{e}_\phi\sin\theta\,\dot{\phi}}$$

$$\therefore \boxed{\vec{v}(t) = \dot{r}\,\hat{e}_r + \hat{e}_\theta\, r\dot{\theta} + \hat{e}_\phi\, r\sin\theta\,\dot{\phi}}$$ Spherical coordinates.

finally $\boxed{\vec{v}(t) = \hat{e}_r \dot{r} + \hat{e}_\theta r \dot{\theta} + \hat{e}_\phi r \dot{\phi} \sin\theta}$

Now before we begin the very tedious calculation for $\vec{a}(t)$,

we will need:

$$\frac{d\hat{e}_\theta}{dt}, \quad \text{and} \quad \frac{d\hat{e}_\phi}{dt}$$

recall $\hat{e}_\theta = \hat{e}_x \cos\theta\cos\phi + \hat{e}_y \cos\theta\sin\phi - \hat{e}_z \sin\theta$

$$\frac{d\hat{e}_\theta}{dt} = \hat{e}_x\left[-\sin\theta\cos\phi\,\dot{\theta} - \cos\theta\sin\phi\,\dot{\phi}\right] + \hat{e}_y\left[-\sin\theta\sin\phi\,\dot{\theta} + \cos\theta\cos\phi\,\dot{\phi}\right] - \hat{e}_z\cos\theta\,\dot{\theta}$$

$$= -\left\{\left[\hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z\cos\theta\right]\dot{\theta}\right\}$$

$$+ \left\{-\cos\theta\sin\phi\,\hat{e}_x + \cos\theta\cos\phi\, e_y\right\}\dot{\phi}$$

or $\boxed{\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r\dot{\theta} + \hat{e}_\phi\cos\theta\,\dot{\phi}}$

$$\hat{e}_\phi = -\hat{e}_x\sin\phi + \hat{e}_y\cos\phi$$

finally $\frac{d\hat{e}_\phi}{dt} = \left(-\hat{e}_x\cos\phi - \hat{e}_y\sin\phi\right)\dot{\phi}$

$$\sin\theta \frac{d\hat{e}_\phi}{dt} = \left(-\hat{e}_x \sin\theta\cos\phi - \hat{e}_y \sin\theta \sin\phi\right)\dot{\phi}$$

$\frac{d\hat{e}_\phi}{dt}$ must be $\perp$ to $\hat{e}_\phi$; hence we should

be able to get $\frac{d\hat{e}_\phi}{dt}$ in terms of $\hat{e}_r$ & $\hat{e}_\theta$

Recall:

$$\hat{e}_r = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z \cos\theta \quad \Big| \sin\theta$$

$$\hat{e}_\theta = \hat{e}_x \cos\theta\cos\phi + \hat{e}_y \cos\theta\sin\phi - \hat{e}_z \sin\theta \quad \Big| \cos\theta$$

$$\hat{e}_r \sin\theta = \hat{e}_x \sin^2\theta\cos\phi + \hat{e}_y \sin^2\theta\sin\phi + \hat{e}_z \cos\theta\sin\theta$$

$$\hat{e}_\theta \cos\theta = \hat{e}_x \cos^2\theta\cos\phi + \hat{e}_y \cos^2\theta\sin\phi - \hat{e}_z \cos\theta\sin\theta$$

$+$

$$\hat{e}_r \sin\theta + \hat{e}_\theta\cos\theta = \hat{e}_x\cos\phi\left(\sin^2\theta+\cos^2\theta\right) + \hat{e}_y\sin\phi\left(\sin^2\theta+\cos^2\theta\right)$$

or $\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta = \hat{e}_x\cos\phi + \hat{e}_y\sin\phi$

or $-\left(\hat{e}_x\cos\phi + \hat{e}_y\sin\phi\right)\dot{\phi} = \boxed{-\dot{\phi}\left(\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta\right) = \frac{d\hat{e}_\phi}{dt}}$

We are ready!

$$\boxed{\vec{v}(t) = \hat{e}_r \dot{r} + \hat{e}_\theta r \dot{\theta} + \hat{e}_\phi r \dot{\phi} \sin\theta}$$

We have shown separately

$$\boxed{\begin{aligned} \frac{d\hat{e}_r}{dt} &= \hat{e}_\theta \dot{\theta} + \hat{e}_\phi \sin\theta \dot{\phi} \\ \frac{d\hat{e}_\theta}{dt} &= -\hat{e}_r \dot{\theta} + \hat{e}_\phi \cos\theta \dot{\phi} \\ \frac{d\hat{e}_\phi}{dt} &= -\left(\hat{e}_r \sin\theta + \hat{e}_\theta \cos\theta\right)\dot{\phi} \end{aligned}}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \cancel{\dot{r}\frac{d\hat{e}_r}{dt}} + \cancel{\ddot{r}\hat{e}_r} + \cancel{r\dot{\theta}\frac{d\hat{e}_\theta}{dt}} + \cancel{\dot{r}\dot{\theta}\hat{e}_\theta} + \cancel{r\ddot{\theta}\hat{e}_\theta}$$

$$+ \cancel{\frac{d\hat{e}_\phi}{dt} r\dot{\phi}\sin\theta} + \cancel{\hat{e}_\phi \dot{r}\dot{\phi}\sin\theta} + \cancel{\hat{e}_\phi r \ddot{\phi}\sin\theta} + \hat{e}_\phi r \dot{\phi}\cos\theta\,\dot{\theta}$$

Now, separately:

$$\dot{r}\frac{d\hat{e}_r}{dt} = \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\sin\theta\dot{\phi}\,\hat{e}_\phi$$

$$r\dot{\theta}\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}^2 r + \hat{e}_\phi r\dot{\theta}\dot{\phi}\cos\theta$$

$$r\dot{\phi}\sin\theta\frac{d\hat{e}_\phi}{dt} = -\hat{e}_r r\dot{\phi}\sin^2\theta\dot{\phi} - \hat{e}_\theta \cos\theta\sin\theta\, r\dot{\phi}^2$$

$$= -\hat{e}_r r\dot{\phi}^2\sin^2\theta - \hat{e}_\theta r\cos\theta\sin\theta\,\dot{\phi}^2$$

Now put it all together.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \hat{e}_r\Big[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta$$

$$+\hat{e}_\theta\Big[\dot{r}\dot{\theta} + \dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\cos\theta\sin\theta$$

$$+\hat{e}_\phi\Big[\dot{r}\sin\theta\dot{\phi} + r\dot{\theta}\dot{\phi}\cos\theta + \dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta\Big]$$

$$\boxed{\begin{aligned}\vec{a}(t) = \frac{d\vec{v}}{dt} = {} & \hat{e}_r\left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right] \\ & + \hat{e}_\theta\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right] \\ & + \hat{e}_\phi\left[r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta\right]\end{aligned}}$$

Right on the button

as required.

Mathematical Physics.

Lecture, Wednesday, September 19, 2001,

The Gradient

In one variable calculus, we deal with.

$f(x)$

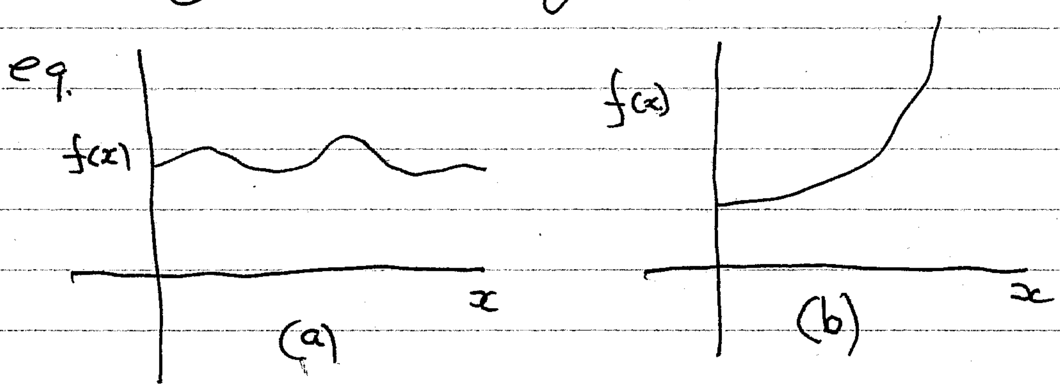
Question: what does $\frac{df}{dx}$ tell us?

It tells us how <u>rapidly</u> the fn' $f(x)$ varies when we change the argument x by a tiny amount dx.

i.e. $df = \frac{df}{dx}\, dx$.

In words: If we change x by an amount dx, then f changes by an amount df – i.e. the derivative is the <u>proportionality</u> factor.

eg.

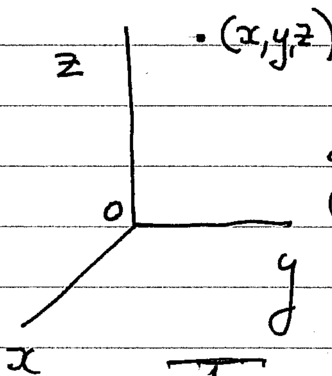


for example, in (a) above, the fn varies slowly with x & hence the derivative is correspondingly small.

In (b) however, $f(x)$ increases rapidly with x & hence the derivative is large

The Gradient

Now we are interested in fn's of three variables x, y, z. Say the temperature $T(x,y,z)$ in a room.



Start at one corner & set up a system of axes;

then for each point (x,y,z) in the room.

T gives the temperature at that spot

We want to generalize the notion of "derivative" to fn's like $T(x,y,z)$ which depend not on one, but on 3 - variables:

Now the derivative is supposed to tell us how fast the fn varies, if we move a tiny distance. However, this time the situation is more complicated, because it depends on what direction we move

$\boxed{\text{The Case of temperature}}$

If we go straight up, then the temperature T will probably increase fairly rapidly; but if we move horizontally, it may not change much at all. In fact the question

$\boxed{\text{"How fast does T vary"}}$

has an infinite number of answers, one for each direction we may wish to explore.

$\boxed{\text{The Concept of "directional derivative"}}$

Fortunately, the problem is not as bad as it looks

$\boxed{\text{" How does } T(x,y,z) \text{ Change ??}}$

Given $T(x,y,z)$

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

This tells us how $T(x,y,z)$ changes when we alter all three variables by the infinitesimal amounts dx, dy, dz

N.B. — we do not require an infinite # of derivatives — three will suffice.

Observation: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

Re-write dT as follows:

$$dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

or $dT = \overrightarrow{\nabla T}\cdot d\vec{r}$

where $\overrightarrow{\nabla T} \equiv \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$ — definition of the gradient of $T(x,y,z)$.

The gradient is a vector quantity, with three components; it is the generalized derivative which we have been looking for.

Compare

$$df = \frac{df}{dx}dx \qquad \text{for } f = f(x)$$

$$\& \quad dT = \vec{\nabla T} \cdot \vec{dr} \qquad \text{for } T = T(x, y, z)$$

Note the Structure here!!

Geometrical Interpretation of the Gradient

Like any vector, the gradient has magnitude and direction !!

To determine the geometric meaning let's write the dot product according to its definition:

$$dT = \vec{\nabla T} \cdot \vec{dr} = |\vec{\nabla T}||\vec{dr}| \cos(\vec{\nabla T}, \vec{dr})$$

where if you like set $(\vec{\nabla T}, \vec{dr}) = \theta$.

Now, fix the magnitude $|\vec{dr}|$ & search around various directions (i.e. vary θ);

the maximum change in T (i.e. dT)

evidently occurs when $\theta = 0$ (i.e. $\cos\theta = 1$).

That is, for a fixed distance $|d\vec{r}|$, dT is the

greatest when we move in the same direction as $\vec{\nabla T}$

Hence, the gradient $\vec{\nabla T}$ points in the direction

of maximum increase of the fn' $T(x,y,z)$

Moreover, the magnitude $|\vec{\nabla T}|$, gives the slope

(rate of change) along this maximal direction.

$\boxed{\text{The Directional Derivative}}$

Suppose now we begin with:

$$dT = \vec{\nabla T} \cdot d\vec{r}$$

Now ask, what is the change of T in the direction $\vec{r}$

Answer: $\dfrac{dT}{dr} = \vec{\nabla T} \cdot \dfrac{d\vec{r}}{dr}$ but $\dfrac{d\vec{r}}{dr}$ is a unit

vector — call it $\hat{n}$

hence $\dfrac{dT}{dn} = \vec{\nabla} T \cdot \hat{n}$

$$\frac{dT}{dr} = \vec{\nabla} T \cdot \hat{r}$$

definition of directional derivative

Example: Potter pg 335 [directional derivative]

Find the derivative of the fn' $\phi = x^2 - 2xy + z^2$ at the point $P(2, -1, 1)$ in the direction of the vector $\vec{A} = 2\hat{i} - 4\hat{j} + 4\hat{k}$

Soln: $\dfrac{dT}{dn} = \vec{\nabla} T \cdot \hat{n}$ or $\dfrac{d\phi}{dn} = \vec{\nabla}\phi \cdot \hat{n}$

The gradient

$$\vec{\nabla}\phi = (2x - 2y)\hat{i} - 2x\hat{j} + 2z\hat{k}$$

Now at the point $P(2, -1, 1)$.

$$\boxed{\vec{\nabla}\phi\Big|_P = (4+2)\hat{i} - 4\hat{j} + 2\hat{k}}$$

the unitvector $\hat{n}$ in the desired direction is given by

$$\hat{n} = \frac{\vec{A}}{|\vec{A}|} = \frac{2\hat{i} - 4\hat{j} + 4\hat{k}}{\sqrt{2^2 + 4^2 + 4^2}} = \frac{2\hat{i} - 4\hat{j} + 4\hat{k}}{6} = \tfrac{1}{3}\hat{i} - \tfrac{2}{3}\hat{j} + \tfrac{2}{3}\hat{k}$$

or $\boxed{\hat{n} = \tfrac{1}{3}\hat{i} - \tfrac{2}{3}\hat{j} + \tfrac{2}{3}\hat{k}}$

hence & finally, the derivative in the direction of $\vec{A}$ is:

$$\frac{d\phi}{dn} = \vec{\nabla}\phi \cdot \hat{n}$$

$$= (6\hat{\imath} - 4\hat{\jmath} + 2\hat{k}) \cdot (1/3\,\hat{\imath} - 2/3\,\hat{\jmath} + 2/3\,\hat{k})$$

$$= 2 + 8/3 + 4/3 = \frac{6+8+4}{3} = 18/3 = \underline{\underline{6}}$$

"Discussion of the Gradient"

Imagine that you are standing on a hillside.

Look around you & find the direction of steepest descent.

Answer: That is the direction of the gradient.

Now measure the slope in that direction

- that is the magnitude of the gradient

What would it mean for the gradient to vanish.

Answer:

If $\vec{\nabla}T = 0$ at (x,y,z) then $dT = 0$ for small displacements about the point (x,y,z)

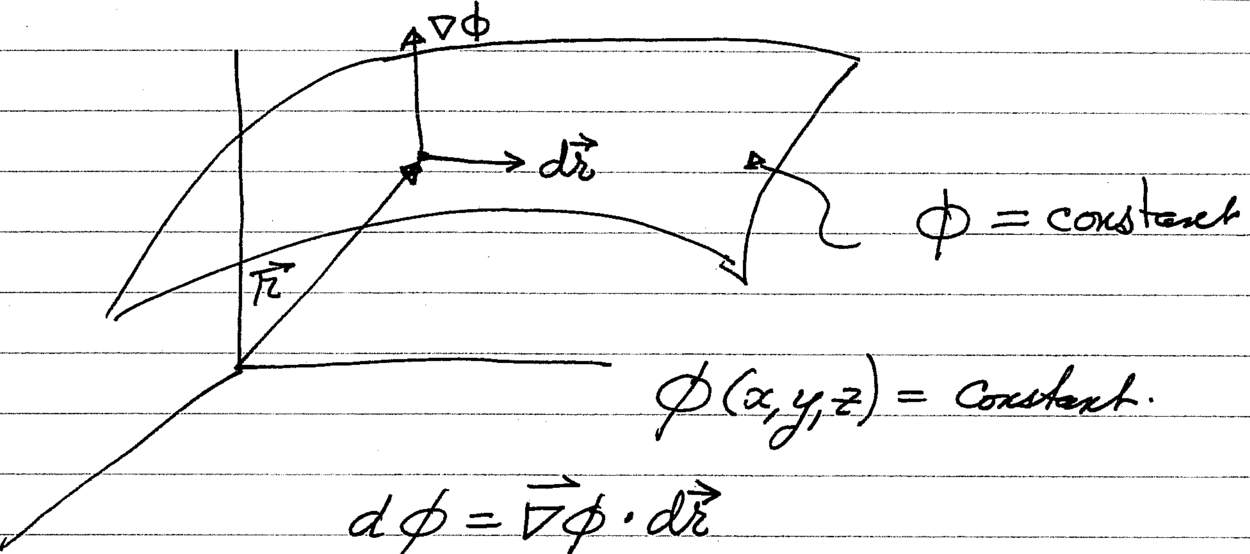
This is a <u>stationary point</u> of the fn' $T(x,y,z)$.

It could be a <u>maximum</u> (a summit), a <u>minimum</u> (a valley) or a saddle-point (a pass) or a shoulder.

This is analogous to the situation for fn's of one variable (i.e. $f(x)$), where the vanishing derivative signals a maximum, or minimum or an inflexion point.

<u>Specifically</u>: ⇒ If you want to locate the extrema of a fn' of three variables, set its gradient to zero!!

Normal to a curve in plane or surface in 3-dimensions



$$\phi(x,y,z) = \text{Constant}.$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$

but in this case $d\phi = d(\text{constant}) = 0$

$\Rightarrow \vec{\nabla\phi} \cdot d\vec{r} = 0 \Rightarrow \vec{\nabla\phi} \perp$ to surface which is constant

i.e. $\vec{\nabla}\phi \perp$ to a constant ϕ surface (or constant curve in plane)

To show this, consider a constant ϕ surface

and the differential vector $d\vec{r}$ (figure

If $\nabla\phi$ is normal to a constant ϕ surface, then

$\vec{\nabla\phi} \cdot d\vec{r} = 0$ since $d\vec{r}$ is a vector that lies

in the surface.

Hence a <u>unit normal</u> to a constant ϕ surface

is $\hat{n} = \dfrac{\vec{\nabla\phi}}{|\vec{\nabla}\phi|}$ as required

Potter 6.4.3 pg 337

Example: Find the equation of the plane which is tangent to surface $x^2+y^2-z^2=4$ at $P(1,2,-1)$

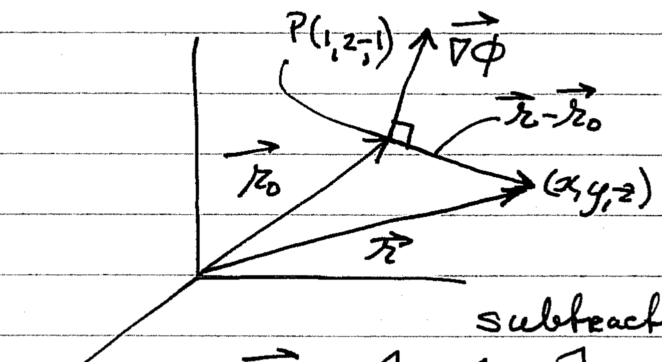
Soln:

the gradient of ϕ is normal to the constant surface

$$\phi = x^2+y^2-z^2=4$$

hence $\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

is normal to the given surface. At the point $P(1,2,-1)$ the normal vector is $\vec{\nabla\phi} = 2\hat{i}+4\hat{j}+2\hat{k}$



The vector to the given pt $\vec{r_0} = \hat{i}+2\hat{j}-\hat{k}$ subtracted from the vector to a general pt $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ is a vector in the desired plane.

$$\vec{r}-\vec{r_0} = x\hat{i}+y\hat{j}+z\hat{k} - (\hat{i}+2\hat{j}-\hat{k})$$

$$\boxed{\vec{r}-\vec{r_0} = (x-1)\hat{i} + (y-2)\hat{j} + (z+1)\hat{k}}$$

Some Problems - Vector Calculus September 2001

Set #1

Find the gradient of each scalar function. Use $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ if required.

1. $\phi = x^2 + y^2$ **2.** $\phi = 2xy$ **3.** $\phi = r^2$

4. $\phi = e^x \sin 2y$ **5.** $\phi = x^2 + 2xy - z^2$ **6.** $\phi = \ln r$

7. $\phi = 1/r$ **8.** $\phi = \tan^{-1} y/x$ **9.** $\phi = r^n$

Find a unit vector $\hat{n}$ normal to each surface at the point indicated.

10. $x^2 + y^2 = 5$, $(2, 1, 0)$ **11.** $r = 5$, $(4, 0, 3)$

12. $2x^2 - y^2 = 7$, $(2, 1, -1)$ **13.** $x^2 + yz = 3$, $(2, -1, 1)$

14. $x + y^2 - 2z^2 = 6$, $(4, 2, 1)$ **15.** $x^2y + yz = 6$, $(2, 3, -2)$

Determine the equation of the plane tangent to the given surface at the point indicated.

16. $x^2 + y^2 + z^2 = 25$, $(3, 4, 0)$ **17.** $r = 6$, $(2, 4, 4)$

18. $x^2 - 2xy = 0$, $(2, 2, 1)$ **19.** $xy^2 - zx + y^2 = 0$, $(1, -1, 2)$

The temperature in a region of interest is determined to be given by the function $T = x^2 + xy + yz$. At the point $(2, 1, 4)$, answer the following questions. What is the unit vector that points in the direction of maximum change of temperature? What is the value of the derivative of the temperature

20. In the x direction?

21. In the direction of the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$?

22. In the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$?

Find the divergence of each vector field at the point $(2, 1, -1)$.

23. $\mathbf{u} = x^2\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$ **24.** $\mathbf{u} = y\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

25. $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ **26.** $\mathbf{u} = xy\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

27. $\mathbf{u} = \mathbf{r}/r$ **28.** $\mathbf{u} = \mathbf{r}/r^3$

29. Show that $\nabla \cdot (\phi\mathbf{u}) = \phi\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\phi$ by expanding in rectangular coordinates.

Find the curl of each vector field at the point $(-2, 4, 1)$.

32. $\mathbf{u} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ **33.** $\mathbf{u} = y^2\mathbf{i} + 2xy\mathbf{j} + z^2\mathbf{k}$

34. $\mathbf{u} = xy\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$ **35.** $\mathbf{u} = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j}$

36. $\mathbf{u} = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + e^x\mathbf{k}$ **37.** $\mathbf{u} = \mathbf{r}/r^3$

Using the vector functions $\mathbf{u} = xy\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ and $\mathbf{v} = x^2\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$, evaluate each function at the point $(-1, 2, 2)$.

38. $\nabla \cdot \mathbf{u}$ **39.** $\nabla \cdot \mathbf{v}$ **40.** $\nabla \times \mathbf{u}$

41. $\nabla \times \mathbf{v}$ **42.** $\nabla \cdot \mathbf{u} \times \mathbf{v}$ **43.** $(\nabla \times \mathbf{u}) \times \mathbf{v}$

44. $\nabla \times (\mathbf{u} \times \mathbf{v})$ **45.** $\mathbf{u} \times (\nabla \times \mathbf{v})$ **46.** $(\mathbf{u} \times \nabla) \times \mathbf{v}$

47. $(\mathbf{u} \cdot \nabla)\mathbf{v}$ **48.** $\nabla(\mathbf{u} \cdot \mathbf{v})$ **49.** $(\mathbf{v} \cdot \nabla)\mathbf{v}$

Determine if each vector field is solenoidal and/or irrotational.

50. $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ **51.** $x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$

52. $y\mathbf{i} + x\mathbf{j}$ **53.** $x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

54. $y^2\mathbf{i} + 2xy\mathbf{j} + z^2\mathbf{k}$ **55.** $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

56. $\sin y\,\mathbf{i} + \sin x\,\mathbf{j} + e^3\mathbf{k}$

57. $x^2y\mathbf{i} + y^2x\mathbf{j} + z^2\mathbf{k}$ **58.** $\mathbf{r}/r^3$

Determine the scalar potential function ϕ, provided that one exists, associated with each vector field.

67. $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

68. $\mathbf{u} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

69. $\mathbf{u} = y^2\mathbf{i} + 2xy\mathbf{j} + z\mathbf{k}$

70. $\mathbf{u} = e^x \sin y\, \mathbf{i} + e^x \cos y\, \mathbf{j}$

71. $\mathbf{u} = 2x \sin y\, \mathbf{i} + x^2 \cos y\, \mathbf{j} + z^2\mathbf{k}$

72. $\mathbf{u} = 2xz\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$

September 2001

Set #1

Answers to even problems

Vector Calculus

2. $2y\mathbf{i} + 2x\mathbf{j}$ **4.** $e^x(\sin 2y\mathbf{i} + 2\cos 2y\mathbf{j})$ **6.** $\mathbf{r}/r^2$
8. $(-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)$ **10.** $(2/\sqrt{5})\mathbf{i} + (1/\sqrt{5})\mathbf{j}$
12. $0.907\mathbf{i} - 0.243\mathbf{j}$ **14.** $0.174\mathbf{i} + 0.696\mathbf{j} - 0.696\mathbf{k}$
16. $3x + 4y = 25$
18. $y = 2$ **20.** 5 **22.** $4\sqrt{3}$ **24.** 0
26. 1 **28.** 0
30. $\nabla \cdot \mathbf{v} = 0$. No. 1
32. 0 **34.** $-\mathbf{j} + 2\mathbf{k}$ **36.** $-0.1353\mathbf{j}$ **38.** 7
40. $\mathbf{k}$ **42.** 4
44. $-5\mathbf{i} + 10\mathbf{j} - 38\mathbf{k}$ **46.** $10\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ **48.** $14\mathbf{i} - 9\mathbf{j} + 8\mathbf{k}$
50. irrotational **52.** irrotational, solenoidal
54. irrotational **56.** neither

58. irrotational, solenoidal **68.** $(x^3 + y^3 + z^3)/3 + C$
70. $e^x \sin y + C$ **72.** $x^2 z + y^3/3 + C$

Mathematical Physics

Lecture Friday, September 21, 2001

The Gradient

$\Phi(x, y, z)$

$d\Phi = \vec{\nabla}\Phi \cdot \vec{dr}$ $\qquad$ $\vec{\nabla}\phi = \dfrac{\partial\phi}{\partial x}\hat{e}_x + \dfrac{\partial\phi}{\partial y}\hat{e}_y + \dfrac{\partial\phi}{\partial z}\hat{e}_z$

$d\phi$ is maximum along the direction of $\vec{\nabla}\phi$

$\boxed{\text{directional derivative}}$

$\dfrac{d\Phi}{dr} = \vec{\nabla}\Phi \cdot \dfrac{\vec{dr}}{dr}$ $\qquad$ $\dfrac{\vec{dr}}{dr} = \hat{n}$

$\boxed{\dfrac{d\Phi}{dn} = \vec{\nabla}\Phi \cdot \hat{n}}$ directional derivative.

Also if $\phi(x, y, z) = $ Constant

then $\vec{\nabla}\phi$ is <u>normal</u> to such a surface (curve)

& $\hat{n}$ (~~so~~ unit vector normal to surface).

$$\hat{n} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}$$

Unit normal to surface/curve $\phi(x,y,z) = \text{const.}$

$\boxed{10.}$ handout

$$\phi(x,y,z) = x^2 + y^2 = 5, \quad \text{at } P(2,1,0)$$

$$\hat{n} = \frac{\nabla\phi}{|\vec{\nabla}\phi|}. \qquad \vec{\nabla}\phi = 2x\hat{e}_x + 2y\hat{e}_y.$$

$$\vec{\nabla}\phi\Big|_{P(2,1,0)} = 4\hat{e}_x + 2\hat{e}_y.$$

$$|\vec{\nabla}\phi| = \sqrt{16+4} = \sqrt{20}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\nabla\phi|} = \frac{4\hat{e}_x + 2\hat{e}_y}{\sqrt{20}} = \frac{4\hat{e}_x + 2\hat{e}_y}{2\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}\hat{e}_x + \frac{1}{\sqrt{5}}\hat{e}_y$$

as required.

The equation of a tangent plane – via example

Find the equation of the plane which is tangent to the surface $\phi(x,y,z) = x^2 + y^2 - z^2 = 4$ at $P(1,2,-1)$

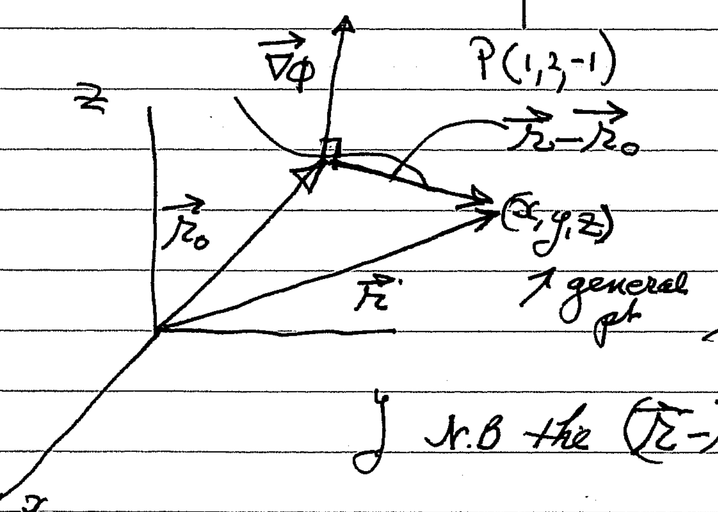
Soln: what do we know?

$\phi(x,y,z) = 4 = \text{constant}$

$\Rightarrow \vec{\nabla}\phi$ is $\perp$ to surface.

in our case $\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

& at $P(1,2,-1)$ $\left.\vec{\nabla}\phi\right| = 2\hat{i} + 4\hat{j} + 2\hat{k}$



the vector to the given point $\vec{r}_0 = +\hat{i} + 2\hat{j} - \hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is a vector to a general point.

N.B the $(\vec{r} - \vec{r}_0)$ is a vector in the desired plane $\vec{r} - \vec{r}_0 = (x-1)\hat{i} + (y-2)\hat{j} + (z+1)\hat{k}$

This vector, when dotted with the vector normal to it i.e. $\vec{\nabla}\Phi$, must yield zero, i.e.

$$\vec{\nabla}\Phi\cdot(\vec{r}-\vec{r}_0)=(2\hat{i}+4\hat{j}+2\hat{k})\cdot\left[(x-1)\hat{i}+(y-2)\hat{j}+(z+1)\hat{k}\right]$$

$$=2(x-1)+4(y-2)+2(z+1)=0$$

Thus the tangent plane

$$2x-2+4y-8+2z+2=0$$

$$2x+4y+2z=8+2-2=8$$

$$\boxed{x+2y+z=4}$$ as required.

$\boxed{16}$ hand out

$$\Phi(x,y,z)=x^2+y^2+z^2=25\ ,\quad P(3,4,0)$$

$$\vec{\nabla}\Phi=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\nabla\Phi\Big|_{P(3,4,0)}=6\hat{i}+8\hat{j}$$

$$\vec{r}_0=3\hat{i}+4\hat{j}\qquad \vec{r}-\vec{r}_0=(x-3)\hat{i}+(y-4)\hat{j}+z\hat{k}$$

$$\vec{\nabla}\Phi\cdot(\vec{r}-\vec{r}_0)=(6\hat{i}+8\hat{j})\cdot([x-3]\hat{i}+(y-4)\hat{j}+z\hat{k})=0$$

$$=6[x-3]+8[y-4]=0\qquad \boxed{3x+4y=25}$$

$$6x-18+8y-32=0$$

$$6x+8y=32+18\Rightarrow 6x+8y=50$$ as request

What else can we do with

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Apply to a vector fn two-ways

$$\vec{A} = A_x(x,y,z)\,\hat{\imath} + A_y(x,y,z)\,\hat{\jmath} + A_z(x,y,z)\,\hat{k}$$

ie. $\vec{A}(x,y,z)$

"dot" operation "Divergence"

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z.$$ scalar fn

$$\vec{A} = 2xy^3\,\hat{\imath} - 3y^5z^3\,\hat{\jmath} + 2xyz\,\hat{k}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}\left(2xy^3\right) + \frac{\partial}{\partial y}\left(-3y^5z^3\right) + \frac{\partial}{\partial z}\left(2xyz\right)$$

$$= 2y^3 - 15y^4z^3 + 2xy.$$

Solenoidal Vector field:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z k \qquad A(x, y, z)$$

if $\vec{\nabla} \cdot \vec{A} = 0$ $\quad \vec{A}$ is solenoidal.

$$\vec{\nabla} \cdot \vec{A} = \text{div } \vec{A}$$

— — —

$$\vec{\nabla} \times \vec{A} = \text{curl } \vec{A}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{A} = 2xy^3 \hat{i} - 3y^5 z^3 \hat{j} + 2xyz \hat{k}$$

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \left(2xz + 9y^5 z^2\right)$$

$$\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = \left(2yz - 0\right)$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0 - 6xy^2$$

$$\therefore \vec{\nabla}\times\vec{A} = \hat{i}\left(2xz+9y^5z^2\right) - \hat{j}\,2yz + \hat{k}\left(-6xy^2\right)$$

if $\vec{\nabla}\times\vec{A} = 0$ $\quad \vec{A}$ called irrotational or conservative.

Integrals - integration.

if $\vec{A} = \vec{A}(t)$ $\quad t =$ a parameter

$$\int \vec{A}(t)\,dt = \hat{e}_x\int A_x(t)\,dt + \hat{e}_y\int A_y(t)\,dt + \hat{e}_z\int A_z(t)\,dt + \vec{C}$$

say $\vec{A}(t) = 4t^3\hat{e}_x - 6t^{-2/3}\hat{e}_y + (2t - t^3)\hat{e}_z$

$$\int \vec{A}(t)\,dt = \hat{e}_x\int 4t^3\,dt + \hat{e}_y\int -6t^{-2/3}\,dt + \hat{e}_z\int (2t - t^3)\,dt$$

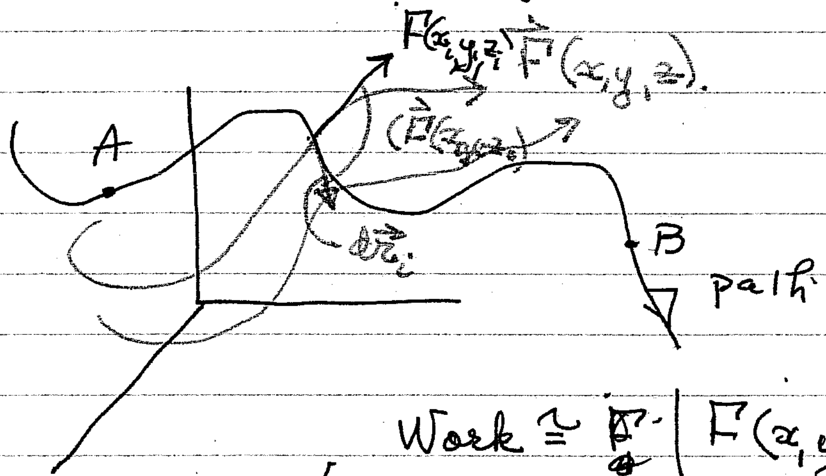
acc. $\vec{a}(t) = t^3\hat{e}_x + 2t^2\hat{e}_y + t\hat{e}_z$

obtain $\vec{v}(t) = \int \vec{a}(t)\,dt$

$$= \hat{e}_x\int t^3\,dt + \hat{e}_y\int 2t^2\,dt + \hat{e}_z\int t\,dt + \vec{C}$$

$$\vec{v}(t) = \frac{t^4}{4}\hat{e}_x + \frac{2}{3}t^3\hat{e}_y + \frac{t^2}{2}\hat{e}_z + \vec{C}$$

The line integral



$$\text{Work} \simeq \left|\vec{F}(x_1,y_1,z_1)\right|\left|\vec{\Delta r}_1\right|\cos\left(\vec{F}(x_1,y_1,z_1),\vec{\Delta r}_1\right)$$

$$\text{Work} = \sum_{i}^{N}\left|\vec{F}(x_i,y_i,z_i)\right|\left|\vec{\Delta r}_i\right|\cos\left(\vec{F}(x_i,y_i,z_i),\vec{\Delta r}_i\right)$$

$$= \lim_{N\to\infty}\sum_{i=1}^{N}\left(\qquad\right) = \int_A^B \vec{F}(x,y,z)\cdot d\vec{r} \equiv \text{Work}$$

Mathematical Physics.

Lecture, Monday, September 24, 2001

Integration of Vector fn's

Given $\vec{A} = \vec{A}(t)$ $\quad t =$ a parameter, in mechanics $t =$ time

$$\int \vec{A}(t)dt = \hat{e}_x \int A_x(t)\cancel{dt}\ \hat{e}_y \int A_y(t)dt + \hat{e}_z \int A_z(t)dt + \vec{C}$$

e.g. say $\vec{A}(t) = 4t^3 \hat{e}_x - 6t^{-2/3}\hat{e}_y + (2t - t^3)\hat{e}_z$

$$\int \vec{A}(t)dt = \hat{e}_x \int 4t^3 dt + \hat{e}_y \int -6t^{-2/3}dt + \hat{e}_z \int (2t - t^3)dt + \vec{C}$$

$$= \hat{e}_x t^4 - 18t^{1/3}\hat{e}_y + \hat{e}_z\left(\frac{2t^3}{3} - \frac{t^4}{4}\right) + \vec{C}$$

Example: from mechanics given $\vec{a}(t) =$ acceleration vector

$$\vec{a}(t) = t^3\hat{e}_x + 2t^2\hat{e}_y + t\hat{e}_z$$

obtain $\vec{v}(t) = \int \vec{a}(t)dt$

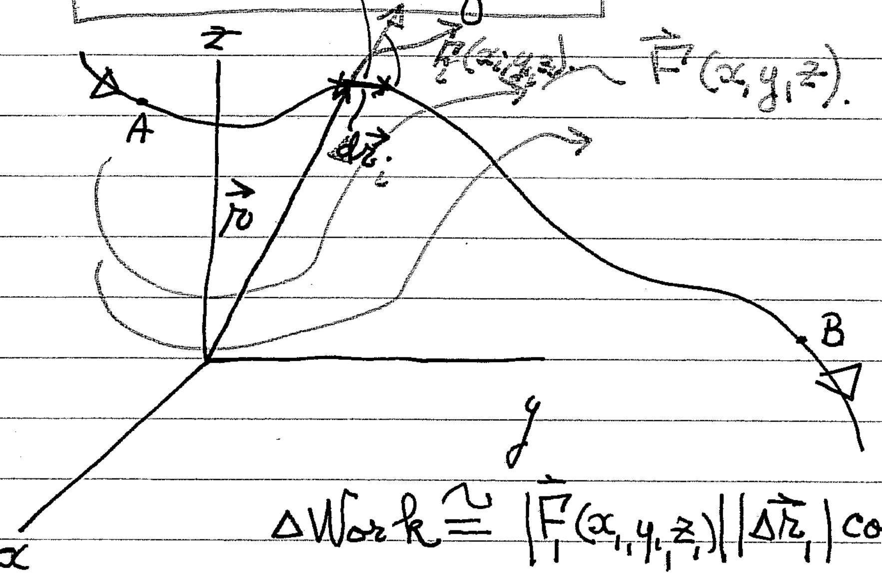
$$= \hat{e}_x \int t^3 dt + \hat{e}_y \int 2t^2 dt + \hat{e}_z \int t\,dt + \vec{C}$$

$$\vec{v}(t) = \frac{t^4}{4}\hat{e}_x + \frac{2}{3}t^3\hat{e}_y + \frac{t^2}{2}\hat{e}_z + \vec{C}$$

etc

$(\vec{F}_i, \vec{\Delta r}_i)$

The Line Integral — Work in Mechanics.



$$\Delta \text{Work} \simeq |\vec{F}_1(x_1,y_1,z_1)||\vec{\Delta r}_1|\cos(\vec{F}_1,\vec{\Delta r}_1) + |\vec{F}_2(x_2,y_2,z_2)||\vec{\Delta r}_2|\cos(\vec{F}_2,\vec{\Delta r}_2)$$

$$+ \cdots |\vec{F}_i(x_i,y_i,z_i)||\vec{\Delta r}_i|\cos(\vec{F}_i,\vec{\Delta r}_i)$$

$$+ \cdots$$

$$\text{Work} \simeq \sum_{i=1}^{N} |\vec{F}_i(x_i,y_i,z_i)||\vec{\Delta r}_i|\cos(\vec{F}_i,\vec{\Delta r}_i)$$

~~ΔWork~~

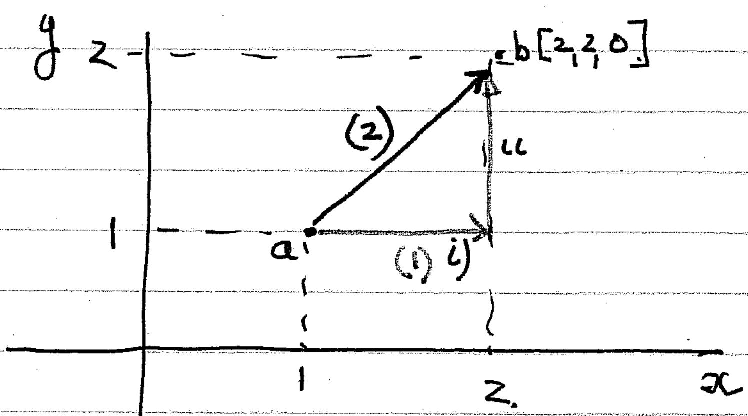
$$\text{Work} = \lim_{N\to\infty} \sum_{i=1}^{N} |\vec{F}_i(x_i,y_i,z_i)||\vec{\Delta r}_i|\cos(\vec{F}_i,\vec{\Delta r}_i)$$

$$\boxed{\text{Work} = \int \vec{F}(x,y,z)\cdot d\vec{r}}$$

line integral

Examples of line integral evaluations.

e.g evaluate $\vec{v} = y^2\hat{i} + 2x(y+1)\hat{j}$ $\quad a(1,1,0) \to b(2,2,0)$



I: evaluate Line integral along (1), (2)

II: evaluate a $\to$ b along (1) & return along (2)

Soln:

I) path (1): consists of two parts: i) along x, & then ii) along y.

along i) $d\vec{r} = dx\,\hat{i}$, $y = 0$ $\quad dy = dz = 0$

So $\vec{v}\cdot d\vec{r} = y^2 dx \Rightarrow \int \vec{v}\cdot d\vec{r} = \int_1^2 (1)\,dx = 1$

Now along ii) $dx = dz = 0$

ii) $d\vec{r} = dy\,\hat{j}$, $x = 2$ $\quad \vec{v}\cdot d\vec{r} = 2x(y+1)\Big|_{x=2} = 4(y+1)\,dy$

So $\int \vec{v}\cdot d\vec{r} = 4\int_1^2 (y+1)\,dy = 4\left(\frac{y^2}{2}+y\right)\Big|_1^2 = 4(2+2) - 4\left(\frac{1}{2}+1\right)$

$= 16 - 6 = 10$

Hence via path(1) we obtain $\int^{b} \vec{v}\cdot d\vec{r} = 1 + 10 = 11$

Path (2) here $x = y$. $dx = dy$ & $dz = 0$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{v}\cdot d\vec{r} = x^2 dx + 2x(x+1)\,dx = (3x^2 + 2x)\,dx \quad \text{so}$$

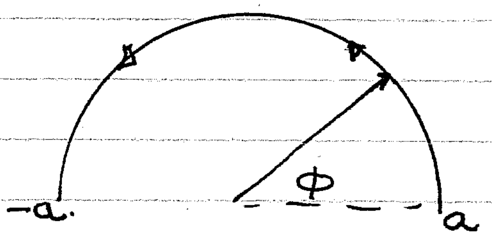
$$\int_a^b \vec{v}\cdot d\vec{r} = \int_1^2 (3x^2+2x)\,dx = \left(x^3 + x^2\right)\Big|_1^2 = 8 + 4 - (1+1) = 10$$

N.B: The strategy here is to get everything in terms of one variable, here x; we could just as well have eliminated x in favor of y.

b) $\oint \vec{v}\cdot d\vec{r} = 11 - 10 = 1$ unit

Example: Given $\vec{F_1} = \hat{e}_x x^2 + \hat{e}_y y^2$ along semi-circle S

do this in polar | cartesian coordinates.



Polar Coordinates.

$x = a\cos\phi$ hence $\vec{F_1} = \hat{e}_x a^2\cos^2\phi + \hat{e}_y a^2\sin^2\phi$

$y = a\sin\phi$

$d\vec{r} = dx\hat{e}_x + dy\hat{e}_y$ $\quad dx = -a\sin\phi\, d\phi$

$dy = a\cos\phi\, d\phi$

$$\int \vec{F_1}\cdot d\vec{r} = \int_0^{\pi} \left(\hat{e}_x a^2\cos^2\phi + \hat{e}_y a^2\sin^2\phi\right)\cdot\left(-\hat{e}_x a\sin\phi\, d\phi + \hat{e}_y a\cos\phi\, d\phi\right)$$

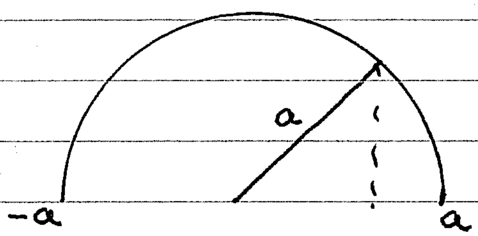
$$= a^3\int_0^{\pi}\left(-\cos^2\phi\sin\phi + \sin^2\phi\cos\phi\right)d\phi$$

$$= a^3\left\{\left[\tfrac{1}{3}\cos^3\phi\right]\Big|_0^{\pi} + \left[1/3\sin^3\phi\right]\Big|_0^{\pi}\right\}$$

$$= \frac{a^3}{3}\left[-1-1\right] = -2/3\, a^3$$

as required

b) Cartesian Coordinates.



on semi-circle.

$$x^2+y^2=a^2 \Rightarrow y=+\sqrt{a^2-x^2}$$

$$dy=-\tfrac{1}{2}\,(a^2-x^2)^{-1/2}\,2x\,dx.$$

hence

$$\int \vec{F}_1\cdot d\vec{r} = \int_{+a}^{-a}\left(\hat{e}_x x^2+\hat{e}_y y^2\right)\cdot\left(dx\,\hat{e}_x+dy\,\hat{e}_y\right)$$

$$= \int_{+a}^{-a} x^2\,dx+(a^2-x^2)(-\tfrac{1}{2})(a^2-x^2)^{-1/2}\,2x\,dx$$

$$= \int_{+a}^{-a}\left\{\left[x^2-x\left[a^2-x^2\right]^{1/2}\right\}dx\right.$$

$$= \int_{+a}^{-a} x^2\,dx - \underbrace{\cancel{\int_{+a}^{-a} x\left[a^2-x^2\right]^{1/2}dx}}_{SI}$$

0 odd integrand symmetric limits

$$= \left.\frac{x^3}{3}\right|_{+a}^{-a} = -\frac{a^3}{3}-\frac{a^3}{3} = \underline{\underline{-\frac{2a^3}{3}}}$$

Same as before

or S.I. $\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}(a^2-x^2)^{3/2}$

SI yields $\left.\tfrac{1}{3}(a^2-x^2)^{3/2}\right|_{a}^{-a} = \tfrac{1}{3}(a^2-a^2)^{3/2}-\tfrac{1}{3}(a^2-a^2)^{3/2}=0$

~~une~~ unnecessary work

Mathematical Physics

Lecture, Wednesday, September 26, 2001 & Friday 28, 2001

Vector Fields, line integrals (continued).

Question: when & under what conditions is.

$$\int_A^B \vec{F}\cdot d\vec{r} \quad \text{independent of path?}$$

$\vec{F}$ here is <u>any</u> vector fn

Recall: $d\Phi = \vec{\nabla}\Phi \cdot d\vec{r}$

hence $$\int_A^B d\Phi = \int_A^B \vec{\nabla}\Phi \cdot d\vec{r}$$

$$\Phi(B) - \Phi(A) = \int_A^B \vec{\nabla}\Phi \cdot d\vec{r}$$

depends only on end point

So if $\vec{F} = \vec{\nabla}\Phi$ then $\int_A^B \vec{F}\cdot d\vec{r}$ is path independent

then $$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{\nabla}\Phi \cdot d\vec{r} = \int_A^B d\Phi = \Phi(B) - \Phi(A)$$

Yes! but we would like to know $\phi(x,y,z)$

Note the following.

$\vec{\nabla}\times\vec{F}$ if $\vec{F}=\vec{\nabla}\phi$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \hat{e}_x\left(\cancelto{0}{\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}}\right)$$

$$-\hat{e}_y\left(\cancel{\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}}\right)$$

$$\hat{e}_z\left(\cancelto{0}{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}}\right)$$

if $\vec{\nabla}\times\vec{F}=0$ then $\phi(x,y,z)$ exists

Motivation for this discussion

Mechanics:

Suppose $F=F(x)$ one dimensional force depending on x

i.e. Hooke Spring

$F(x) = m\frac{dv}{dt}$ Newt II

$m\frac{dv}{dx}\frac{dx}{dt} = F(x)$ Chain rule

$mv\frac{dv}{dx} = F(x)$

or $m v\,dv = F(x)\,dx$

$$\int_{v_1}^{v_2} m v\,dv = \int_{x_1}^{x_2} F(x)\,dx$$

$$\downarrow$$

$$\int_{v_1}^{v_2} d\left(\tfrac{1}{2}mv^2\right) = \int_{x_1}^{x_2} F(x)\,dx$$

$$\tfrac{1}{2}mv^2\Big|_{v_1}^{v_2} = \int_{x_1}^{x_2} F(x)\,dx$$

$$\downarrow$$

$K = \tfrac{1}{2}mv^2$

$$\Delta K = \int_{x_1}^{x_2} F(x)\,dx$$ Work-energy theorem in one-dimension

$$\text{Work} = W_{1\to 2} \equiv \int_{x_1}^{x_2} F(x)\,dx$$ definition of Work one dim

Define Potential energy

$$V(x) \equiv \int_{x}^{x_s} F(x)\,dx$$ x_s (ref. pt)

or $$V(x) = -\int_{x_s}^{x} F(x)\,dx$$

Then we may write

$$\int_{x_1}^{x_2} F(x)\,dx = \int_{x_1}^{x_s} F(x)\,dx + \int_{x_s}^{x_2} F(x)\,dx.$$

$$= V(x_1) - V(x_2).$$

$$= -\left[V(x_2) - V(x_1)\right] = -\Delta V$$

Hence

$$\Delta K = \text{Work} = -\Delta V$$

$$\Delta(K+V) = 0 \qquad K+V = \underline{\underline{\text{constant}}}. \quad \text{Conservation of energy.}$$

Carry this over to 3-dim.

$$\vec{F} = \vec{F}(x,y,z).$$

$$\vec{v} \cdot \left\{ \vec{F} = m\frac{d\vec{v}}{dt} \right\}$$

$$\vec{F}\cdot\vec{v} = m\vec{v}\cdot\frac{d\vec{v}}{dt}.$$

$$\text{or } \vec{F}\cdot\frac{d\vec{r}}{dt} = m\vec{v}\cdot\frac{d\vec{v}}{dt}.$$

$$\text{or } \vec{F}\cdot d\vec{r} = m\vec{v}\cdot d\vec{v}$$

or integrate

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = m\int_{\vec{v}_1}^{\vec{v}_2} \vec{v}\cdot d\vec{v}$$

$$= m\int_{\vec{v}_1}^{\vec{v}_2} d\left(\tfrac{1}{2}\vec{v}\cdot\vec{v}\right)$$

$d\left(\tfrac{1}{2}\vec{v}\cdot\vec{v}\right)$
$= \tfrac{1}{2}\left[\vec{v}\cdot d\vec{v} + d\vec{v}\cdot\vec{v}\right]$
$\tfrac{1}{2}\, 2\vec{v}\cdot d\vec{v} = \vec{v}\cdot d\vec{v}$

$$= m\int_{\vec{v}_1}^{\vec{v}_2} d\left(\tfrac{1}{2}\cancel{m}v^2\right)$$

$$= \tfrac{1}{2}mv^2\Big|_{\vec{v}_1}^{\vec{v}_2} = \Delta K$$

Now look at

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \underset{\vec{r}_1 \to \vec{r}_2}{\text{Work}}$$

$$\text{define } V(\vec{r}) \equiv \int_{\vec{r}}^{\vec{r}_s} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_s}^{\vec{r}} \vec{F}\cdot d\vec{r}$$

$$\text{Work} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_s} \vec{F}\cdot d\vec{r} + \int_{\vec{r}_s}^{\vec{r}_2} \vec{F}\cdot d\vec{r}$$

$$= V(\vec{r}_1) - V(\vec{r}_2)$$

$$= -\left[V(\vec{r}_2) - V(\vec{r}_1)\right] = -\Delta V$$

Hence $\Delta K = -\Delta V \Rightarrow \Delta(K+V) = 0$

$K+V$ = Cons. of mech en.

$$\therefore \boxed{V(\vec{r}) - V(\vec{r}_s) = -\int_{\vec{r}_s}^{\vec{r}} \vec{F}\cdot d\vec{r}}$$

$$\Delta V(\vec{r}_s \rightarrow \vec{r}) = -\int_{\vec{r}_s}^{\vec{r}} \vec{F}\cdot d\vec{r}$$

if $\vec{F} = -\nabla V$

$$V(x,y,z) - V(x_s y_s z_s) = \int_{x_s y_s z_s}^{x,y,z} \vec{\nabla} V \cdot d\vec{r}$$

We try to arrange things such that

$\vec{r}_s$, $V(\vec{r}_s) = 0$

Given $\vec{F}$, if $\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F} = -\Delta \vec{V}$

e.g. $\vec{F} = ay(y^2 - 3z^2)\hat{e}_x + 3ax(y^2 - z^2)\hat{e}_y - 6axyz\hat{e}_z$

a) if $\vec{F}$ cons.

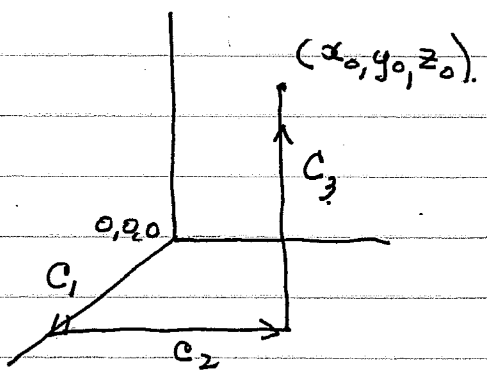
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay(y^2 - 3z^2) & 3ax(y^2 - z^2) & -6axyz \end{vmatrix}$$

$$= \hat{e}_x\left(-6xz + 6axz\right) - \hat{e}_y\left(-6ayz + 6ayz\right) + \hat{e}_z\left(3a(y^2 - z^2) - 3a(y^2 - z^2)\right)$$

$= 0$ $\vec{F}$ is con

$$V(\vec{r}) - V(\vec{r}_s) = -\int_{\vec{r}_s}^{\vec{r}} \vec{F}\cdot d\vec{r}$$

This is path independent – choose what we like



$$\vec{r}_s = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$V(\vec{r}_s) = 0$$

$$V(\vec{r}) - V(\vec{r}_s) = -\int_{0,0,0}^{x_0 y_0 z_0} \vec{F}\cdot d\vec{r} = -\left\{ \int_0^{x_0} F_x\,dx + \int_0^{y_0} F_y\,dy + \int_0^{z_0} F_z\,dz \right.$$

$$= -\left[\int_0^{x_0} a\cancel{y}(y^2 - 3z^2)\,dx \right. \quad \begin{matrix} y = 0 \\ z = 0 \end{matrix} \text{ on } C_1$$

$$+ \int_0^{y_0} 3ax_0(y^2)\,dy = ax_0 y_0^3 \qquad C_2 \begin{cases} x = x_0 \\ z = 0 \end{cases}$$

$$+ \int_0^{z_0} -6ax_0 y_0 z\,dz = -3ax_0 y_0 z_0$$

$$C_3 \quad \begin{matrix} x = x_0 \\ y = y_0 \end{matrix}$$

$$V(x_0, y_0, z_0) - \cancel{V(0,0,0)}^{0} = -\left\{ a x_0 y_0^3 - 3 a x_0 y_0 z_0^2 \right\}$$

$$\boxed{V(x, y, z) = 3axyz^2 - axy^3}$$

$$\vec{F} = -\vec{\nabla} V$$

$$= -\left\{ (3ayz^2 - ay^3)\hat{e}_x + (3axz^2 - 3axy^2)\hat{e}_y + 6axyz\,\hat{e}_z \right.$$

or finally,

$$\boxed{\vec{F} = (ay^3 - 3ayz^2)\hat{e}_x + (3axy^2 - 3axz^2)\hat{e}_y - 6axyz\,\hat{e}_z}$$

Compare with what we started?!

Started with: $\vec{F}(xyz) = ay(y^2 - 3z^2)\hat{e}_x + 3ax(y^2 - z^2)\hat{e}_y - 6axyz\,\hat{e}_z$

Now clean this up & write

$$\boxed{\vec{F} = ay(y^2 - 3z^2)\hat{e}_x + 3ax(y^2 - z^2)\hat{e}_y - 6axyz\,\hat{e}_z}$$

right on the button — as required.

Another way!

$$\vec{F} = -\vec{\nabla} V$$

$$\boxed{\begin{aligned} \text{We are given } F_x &= ay(y^2-3z^2) \\ F_y &= 3ax(y^2-z^2) \\ F_z &= -6axyz \end{aligned}}$$

$\downarrow$

Now

$$F_x \hat{e}_x + F_y \hat{e}_y + F_z \hat{e}_z = -\left\{ \frac{\partial V}{\partial x}\hat{e}_x + \frac{\partial V}{\partial y}\hat{e}_y + \frac{\partial V}{\partial z}\hat{e}_z \right\}$$

pull out - sign, so we don't lose it!

hence, we must conclude.

$$\frac{\partial V}{\partial x} = ay(y^2-3z^2)$$

integrating partially w.r.t. x we can write:

Step I

$$V = \int ay(y^2-3z^2)\,dx + f(y,z)$$

$$\boxed{V = ayx(y^2-3z^2) + f(y,z).}$$

Now $\dfrac{\partial V}{\partial y} = F_y = 3ax(y^2-z^2)$

$$\frac{\partial}{\partial y}\left(ayx\left[y^2-3z^2\right] + f(y,z) \right) = 3ax(y^2-z^2)$$

$$\text{or } \frac{\partial}{\partial y}\left(axy^3 - 3ayxz^2 + f(y,z)\right) = 3ax\,(y^2 - z^2).$$

differentiating: we obtain

$$3axy^2 - 3axz^2 + \frac{\partial f(y,z)}{\partial y} = 3ax\,(y^2 - z^2)$$

$$3ax(y^2 - z^2) + \frac{\partial f}{\partial y} = 3ax(y^2 - z^2)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 0 \quad \text{ok!}$$

finally,

$$\frac{\partial}{\partial z}\left(ayx(y^2 - 3z^2) + g(z)\right) = -6axyz$$

$$\text{or } -6axyz + \frac{\partial g(z)}{\partial z} = -6axyz$$

$$\Rightarrow \partial g/\partial z = 0$$

$$\therefore V(x,y,z) = ayx\,(y^2 - 3z^2)$$

Now put in the $-$ sign

$$V(x,y,z) = -ayx\,(y^2 - 3z^2)$$

$$\boxed{V(x,y,z) = 3axyz^2 - axy^3 \quad \text{as before}}$$

Mathematical Physics

Line integrals $\boxed{\text{Friday, Sept 26, 2001}}$

Hand books are in elevator displayed in case assignments - Marg office

Example $\boxed{71}$ handout

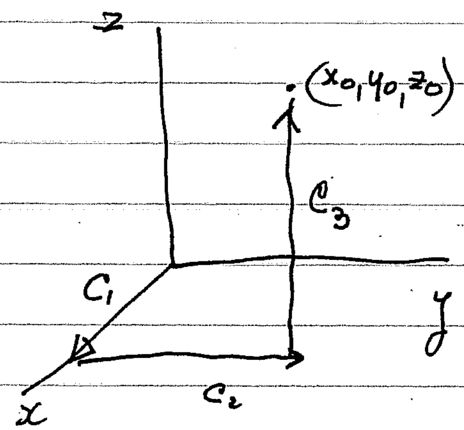
Given:

$$\vec{u}(x,y,z) = 2x\sin y\,\hat{i} + x^2\cos y\,\hat{j} + z^2\hat{k}$$

$$\vec{\nabla}\times\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\sin y & x^2\cos y & z^2 \end{vmatrix}$$

$$= \hat{i}[0] - \hat{j}[0] + \hat{k}[0] \quad \text{conservative}$$

$$V(x_0 y_0 z_0) - V(x_s, y_s, z_s) = -\int_{x_s, y_s, z_s}^{x_0 y_0 z_0} \vec{u}\cdot d\vec{r}$$



$$= -\left[\int_{0\;C_1}^{x_0} 2x\cancel{\sin y}^{\,0}\,dx + \int_{0\;C_2}^{y_0} x_0^2\cos y\,dy + \int_0^{z_0} z^2 dz\right]$$

C_1: dy, dz; $y, z = 0$ $\quad$ C_2: $x = x_0$, $dx = 0$, $dz = 0$ $\quad$ $dx = dy = 0$

$$= -\left[x_0^2 \sin y\Big|_0^{y_0} + \frac{z_0^3}{3}\right]$$

$$V(x,y,z) = -x^2\sin y - \frac{z^3}{3}$$

$$= -x_0^2\sin y_0 - \frac{z_0^3}{3} \Rightarrow$$

②

$$\vec{u} = -\nabla \vec{V} = +\left\{ 2x\sin y\,\hat{i} + x^2\cos y\,\hat{j} + z^2\hat{k} \right\}$$

$$\vec{u} = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} = 2x\sin y\,\hat{i} + x^2\cos y\,\hat{j} + z^2\hat{k}$$

$$\frac{\partial V}{\partial x} = 2x\sin y$$

$$V = \int 2x\sin y\,dx + f(y,z)$$

$$\boxed{V = x^2\sin y + f(y,z)}$$

$$\cancel{\frac{\partial V}{\partial y}} =$$

$$\frac{\partial V}{\partial y} = x^2\cos y + \frac{\partial f}{\partial y} = x^2\cos y \Rightarrow \frac{\partial f}{\partial y} = 0 \quad \therefore f = f(z)$$

$$\frac{\partial V}{\partial z} = \frac{\partial f}{\partial z} = z^2 \Rightarrow f = \frac{z^3}{3}$$

– sign.

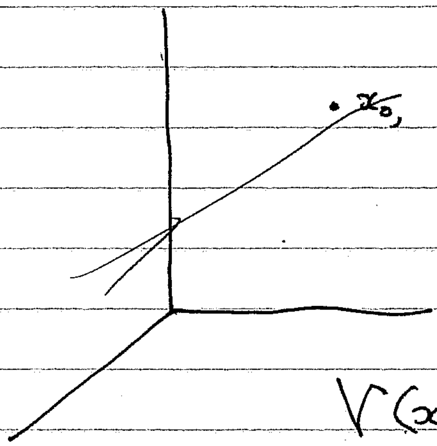
$$\therefore V(x,y,z) = x^2\sin y + z^3/3$$

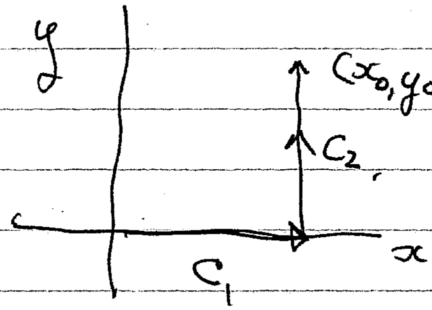
$$V(x,y,z) = -\left[x^2\sin y + z^3/3 \right]$$

$\boxed{70}$ $\vec{u} = e^{x}\sin y\,\hat{i} + e^{x}\cos y\,\hat{j}$

$$\overrightarrow{\nabla \times u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x}\sin y & e^{x}\cos y & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}\, e^{x}\cos y - e^{x}\cos y = 0.$$





$$V(x_0,y_0,z_0) - V(x_s,y_s,z_s) = -\left[\int_0^{x_0} e^{x}\sin y\,dx \Big|_{\substack{C_1\\ y=0}} + \int_0^{y_0} e^{x_0}\cos y\,dy\right.$$

$$= -\left[e^{x_0}\sin y\right|_0^{y_0}$$

$$= -e^{x_0}\sin y_0 \Rightarrow V(x,y) - e^{x}\sin y.$$

$$\vec{u} = -\vec{\nabla} V = +e^{x}\sin y\,\hat{i} + e^{x}\cos y\,\hat{j}$$

$$\vec{u} = -\left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right] = e^{x}\sin y\,\hat{i} + e^{x}\cos y\,\hat{j}$$

$$\frac{\partial V}{\partial x} = e^{x}\sin y$$

$$V = \int e^{x}\sin y\,dx + f(y)$$

$$V = e^{x}\sin y + f(y)$$

$$\frac{\partial V}{\partial y} = e^{x}\cos y + \frac{\partial f}{\partial y} = e^{x}\cos y \qquad \frac{\partial f}{\partial y} = 0 \qquad f = \text{const.}$$

$$\therefore \quad V = -e^{x}\sin y + \text{Const.}$$

$$\vec{u} = +\overrightarrow{\nabla V}$$