Mathematical Physics

Lecture, Monday, October 29, 2001

Matrices (continued) diagonalization

finish up the coupled oscillator problem.

Example (non-symmetric) Potter pg 274

Given $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ $\Rightarrow$ non-symmetric.

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0 \Rightarrow \lambda_1 = -1, \ \lambda_2 = 3$$

Eigenvectors:

$\lambda_1 = -1$

$$\begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = 0 \quad \begin{array}{l} \Rightarrow 2x_1^1 + x_2^1 = 0 \Rightarrow 2x_1^1 = -x_2^1 \\ \rightarrow 4x_1^1 + 2x_2^1 = 0 \end{array}$$

set $x_1^1 = 1 \Rightarrow x_2^1 = -2$.

$$\therefore X^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boxed{\hat{X}^{(1)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$\underline{\underline{\lambda_2 = 3}}$

$$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = 0 \qquad \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -2x_1^2 + x_2^2 = 0 \\ 4x_1^2 - 2x_2^2 = 0 \end{matrix}$$

$$\to 2x_1^2 = x_2^2$$

set $x_1^2 = 1 \Rightarrow x_2^2 = 2$

or $X^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\boxed{\hat{X}^{(2)} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Check orthogonality of eigenvectors

$$\hat{X}^{(1)} \cdot \hat{X}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{5} - \frac{4}{5} \neq 0$$

$\underline{\underline{\text{not}}}$ orthogonal.

Hence construct $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = M$

Now calculate $M^{-1} = \dfrac{\text{adj}\, M}{\det M}$ $\qquad \det M = \begin{vmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{vmatrix} = \frac{2}{5} + \frac{2}{5} = \frac{4}{5}.$

$$M^c = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \qquad (M^c)^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$M^{-1} = \frac{5}{4}\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{5}{2\sqrt{5}} & -\frac{5}{4\sqrt{5}} \\ \frac{5}{2\sqrt{5}} & \frac{5}{4\sqrt{5}} \end{bmatrix}$$

Now form $M^{-1}AM = \lambda I$

$$AM = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} \frac{-2}{\sqrt{5}} & \frac{4}{\sqrt{5}} + \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}} \end{bmatrix}$$

finally.

$$M^{-1}AM = \begin{bmatrix} \frac{5}{2\sqrt{5}} & \frac{-5}{4\sqrt{5}} \\ \frac{5}{2\sqrt{5}} & \frac{5}{4\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-5}{10} - \frac{\cancel{10}}{(4)(5)} & \frac{15}{10} - \frac{30}{20} \\ \frac{-5}{10} + \frac{10}{20} & \frac{15}{10} + \frac{30}{20} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$ right-on the button.

Complex Matrices:

example: Given $A = \begin{pmatrix} 1 & 0 & 5i \\ -2i & 2 & 0 \\ 1 & 1+i & 0 \end{pmatrix}$

Is A Hermitian

if A is Hermitian $\Rightarrow A = A^{+}$

we should compute A^{+}

$$A^{+} = (A^{*})^{T} = \begin{pmatrix} 1 & 0 & -5i \\ 2i & 2 & 0 \\ 1 & 1-i & 0 \end{pmatrix}^{T} = \begin{pmatrix} 1 & 2i & 1 \\ 0 & 2 & 1-i \\ -5i & 0 & 0 \end{pmatrix} \neq A$$

$\therefore$ A is not Hermitian

example

Given $A = \begin{pmatrix} 0 & 2i & -1 \\ -i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

is A Hermitian

$$A^{+} = (A^{*})^{T} = \begin{pmatrix} 0 & -2i & -1 \\ i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}^{T} = \begin{pmatrix} 0 & i & 3 \\ -2i & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \neq A$$

find A^{-1} in this case

$$A^{-1} = \frac{\operatorname{adj} A}{\det A} \qquad \det A = \begin{vmatrix} 0 & 2i & -1 \\ -i & 2 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 0 - 2i(0) - 1(-6) = 6$$

$$\operatorname{adj} A = (A^c)^T$$

$$A^c = \begin{pmatrix} 0 & 0 & -6 \\ 0 & 3 & 6i \\ 2 & i & -2 \end{pmatrix} \Rightarrow \operatorname{adj} A = (A^c)^T = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & i \\ -6 & 6i & -2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{i}{6} \\ -1 & i & -\frac{1}{3} \end{pmatrix}$$

check

$$AA^{-1} = \begin{pmatrix} 0 & 2i & -1 \\ -i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & i/6 \\ -1 & i & -1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as required

<u>Interesting results:</u>

A matrix may be diagonalized.

if

a) if <u>all</u> its eigenvalues are distinct.

b) if it is Hermitian or symmetric

c) if it is unitary or orthogonal.

Recall: Symmetric $A^T = A$

anti (skew) symmetric $A^T = -A$

orthogonal $A^T = A^{-1}$

Hermitian $A^\dagger = A = (A^*)^T$

skew Hermitian $A^\dagger = -A$

Unitary $A^\dagger = A^{-1}$

Problem 3.12 Griffiths

Show
a) $(AB)^T = B^T A^T$

b) $(AB)^{-1} = B^{-1} A^{-1}$

c) $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$

Soln:
a) $(AB)^T = B^T A^T$

proof $(AB)^T_{ki} = (AB)_{ik} = \sum_{j=1} A_{ij} B_{jk} = \sum_j B^T_{kj} A^T_{ji} = (B^T A^T)_{ki}$

So $(AB)^T = B^T A^T$

b) $(AB)^{-1} = B^{-1} A^{-1}$

associativity

proof form $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$

So $(AB)^{-1} = B^{-1} A^{-1}$

c). $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$

$(AB)^{\dagger} = \left[(AB)^{*}\right]^{T} = B^{*T} A^{*T} = B^{\dagger} A^{\dagger}$

as required

Problem:

a) Show that the product of two unitary matrices is unitary (it should also be true for orthogonal matrices)

b) Under what conditions is the product of two Hermitian matrices Hermitian

c) Is the sum of two unitary matrices unitary?

d) Is the sum of two Hermitian matrices Hermitian?

a) Suppose $U^{+} = U^{-1}$ & $W^{+} = W^{-1}$ (i.e. U, W are unitary)

then $(WU)^{+} = U^{+}W^{+} = U^{-1}W^{-1} = (WU)^{-1}$ so (WU) is unitary

b) Suppose $H = H^{+}$ $J = J^{+}$ (i.e. H, J are Hermitian)

then $(HJ)^{+} = J^{+}H^{+} = JH$ the product is Hermitian.

provided $JH = HJ$ i.e. they commute

~~c) $(U+W)^{+} \neq U^{+} + W^{+} \overset{?}{=} (U+W)^{+}$ No! the sum of two unitary matrices is not unitary.~~

~~d) $(H+J)^{+}$~~

over

c) $(U+W)^{+} = U^{+}+W^{+} = U^{-1}+W^{-1} \overset{?}{=} (U+W)^{-1}$ No! the sum of two unitary matrices is not unitary

d) $(H+J)^{+} = H^{+}+J^{+} = (H+J)$ yes: the sum of two Hermitian matrices is Hermitian

$\boxed{\text{Problem: Griffiths pg 88}}$

Given

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$$ Obtain eigenvalues/eigenvectors

$$\det|M-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & -2 \\ -2i & i-\lambda & 2i \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(2-\lambda)(i-\lambda)+2(i-\lambda)=0$$

$$(i-\lambda)\left[(-1-\lambda)(2-\lambda)+2\right]=0$$

$$(i-\lambda)(-2-\lambda+\lambda^2+2]=0$$

$$\lambda(i-\lambda)(-1+\lambda)=0 \quad \therefore \lambda_1=0 \quad \lambda_2=1 \quad \lambda_3=i$$

Eigen vectors:

$\boxed{\lambda_1 = 0}$ use $a_1\, a_2\, a_3$ as dummies

$$\begin{pmatrix} 2-0 & 0 & -2 \\ -2i & i-0 & 2i \\ 1 & 0 & -1-0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \quad \begin{array}{ll} 2a_1 + 0a_2 - 2a_3 = 0 \Rightarrow & a_1 - a_3 = 0 \\ -2ia_1 + ia_2 + 2ia_3 = 0 \Rightarrow & -2a_1 + 2a_3 + a_2 = 0 \Rightarrow a_2 = 0 \\ a_1 + 0a_2 - a_3 = 0 & a_1 - a_3 = 0 \end{array}$$

hence set $a_1 = 1 \Rightarrow a_3 = 1 \quad a_2 = 0$

$$\boxed{X^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \hat{X}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

N.B. $\boxed{\lambda_2 = 1}$ use a_1, a_2, a_3 as dummies

$$\begin{pmatrix} 2-1 & 0 & -2 \\ -2i & i-1 & 2i \\ 1 & 0 & -1-1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \quad \begin{array}{l} \Rightarrow a_1 + 0a_2 - 2a_3 = 0 \\ -2ia_1 + (i-1)a_2 + 2ia_3 = 0 \\ a_1 + 0a_2 - 2a_3 = 0 \text{ (redundant).} \end{array}$$

$$\boxed{a_1 = 2a_3}$$

$$-2ia_1 + ia_2 - a_2 + 2ia_3 = 0$$

$$-2ia_1 + ia_2 + 2ia_3 = a_2.$$

but $2a_3 = a_1$ $\quad -2ia_1 + ia_2 + ia_1 = a_2$

$$-ia_1 + ia_2 = a_2$$

$$-ia_1 = a_2 - ia_2 = a_2(1-i)$$

$$a_2 = \frac{-ia_1}{1-i} = \frac{-ia_1(1+i)}{(1-i)(1+i)} = \frac{-ia_1(1+i)}{2}$$

$$\boxed{a_2 = \frac{a_1}{2}(1-i)} \quad \text{N.B.}$$

This time pick $\boxed{a_1 = 2}$

hence $\underline{a_3 = 1}$ $\quad a_2 = (1-i)$

N.B

$$X^{(2)} = \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix} \Rightarrow \boxed{\hat{X}^{(2)} = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix}}$$

N.B. $\boxed{\text{finally}}$ $\boxed{\lambda_3 = i}$ $\underline{a_1, a_2, a_3,}$ dummies

$$\begin{pmatrix} 2-i & 0 & -2 \\ -2i & i-i & 2i \\ 1 & 0 & -1-i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$(2-i)a_1 + 0a_2 - 2a_3 = 0$ ①

$-2ia_1 + 0a_2 + 2ia_3 = 0$ ②

$a_1 + 0a_2 - (1+i)a_3 = 0$ ③

① becomes $2a_1 - ia_1 + 0a_2 - 2a_3 = 0$

② becomes $-2ia_1 + 0a_2 + 2ia_3 = 0$

③ becomes $a_1 + 0a_2 - (1+i)a_3 = 0$

What can we say about these $\underline{\underline{3}}$ statements?

① implies: $2a_1 - 2a_3 = ia_1$

② implies: $-2ia_1 + 0a_2 + 2ia_3 = 0$

③ implies $a_1 + 0a_2 - (1+i)a_3 = 0$ what to do?

$\longrightarrow$ over

Hence, we should look at

(1) $2a_1 - 2a_3 = ia_1 \Rightarrow a_1 = a_3 = 0$

(2) $-2i\,a_1 + ia_2 + 2ia_3 = ia_2 \qquad a_1 = a_3 = 0$

$\Rightarrow ia_2 = ia_2. \qquad a_2$ is anything; say it is 1

$a_1 + 0a_2 - (1+i)a_3 = 0 \Rightarrow a_1 + 0a_2 - a_3 - ia_3 = 0.$

$a_1 + 0a_2 - a_3 = ia_3$

$\Rightarrow a_1 = a_3 = 0.$

Hence we must conclude.

$$\boxed{X^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \hat{X}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.}$$

N.B.

$\boxed{\text{N.B.}}$ $\hat{X}^{(1)} \cdot \hat{X}_2 = \left(\frac{1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\right)\begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix}\frac{1}{\sqrt{7}}$

$= \frac{2}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} \neq 0 \Rightarrow \hat{X}^{(1)}$ not orthogonal to $\hat{X}^{(2)}$

also $\hat{X}^{(1)} \cdot \hat{X}^{(3)} = \left(\frac{1}{\sqrt{2}}\ 0\ \frac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

$$\hat{X}^{(2)} \cdot \hat{X}^{(3)} = \frac{1}{\sqrt{7}} \begin{pmatrix} 2 & 1+i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (1+i) \neq 0$$

$\Rightarrow \hat{X}^2$ <u>not</u> orthogonal to $\hat{X}^{(3)}$

hence, M is <u>not</u> Hermitian!

this could have been shown "a priori"

if M is to be Hermitian

then $M = M^\dagger = (M^*)^T$

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}; \quad \text{hence } M^* = \begin{pmatrix} 2 & 0 & -2 \\ 2i & -i & -2i \\ 1 & 0 & -1 \end{pmatrix}$$

$$\& \ (M^*)^T = \begin{pmatrix} 2 & 2i & 1 \\ 0 & -i & 0 \\ -2 & -2i & -1 \end{pmatrix} \neq \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$$

hence, M is <u>not</u> Hermitian

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Some Important Theorems – Hermitian / Unitary i.e. Symmetric / orthogonal Matrices

Theorem 1 (eigenvalues)

NB

a) The eigenvalues of a Hermitian matrix (symmetric matrix) are Real !!

PROOF:

Let A be Hermitian: $\Rightarrow A^{+} = A$.

hence $A x = \lambda x$ (1)

Now take complex conjugate & then transpose of (1)

$$\left[(A x)^{*}\right]^{T} = \lambda^{*} (x^{*})^{T}$$

or

$$\left[A^{*} x^{*}\right]^{T} = \lambda^{*} (x^{*})^{T}$$

hence

$$\Rightarrow (x^{*})^{T} (A^{*})^{T} = \lambda^{*} (x^{*})^{T} \Rightarrow x^{+} A^{+} = \lambda^{*} x^{+} \quad (2)$$

→ over →

Now multiply 1 by x^{+} from the left;

$$x^{+}Ax = \lambda x^{+}x \qquad \boxed{1'}$$

Now multiply 2 by x from the right

$$x^{+}A^{+}x = \lambda^{*}x^{+}x \qquad \boxed{2'}$$

Now form 1' − 2'

$$x^{+}Ax - x^{+}A^{+}x = \lambda x^{+}x - \lambda^{*}x^{+}x$$
$$= (\lambda - \lambda^{*})x^{+}x = 0$$

or $\underbrace{x^{+}Ax - x^{+}A^{+}x}_{0 \text{ because } A = A^{+}} = (\lambda - \lambda^{*})x^{+}x = 0$

Now $x^{+}x$ = dot product $\neq 0$ $\Rightarrow (\lambda - \lambda^{*}) = 0$

$\Rightarrow \lambda = \lambda^{*} \Rightarrow \lambda$'s are real.

as required.

b). The eigenvalues of a skew (anti)-Hermitian matrix.

(and thus skew-symmetric matrices) are ~~pure~~ imaginary or zero!

Proof.

$$\boxed{Ax = \lambda x \quad (1)} \qquad A^{\dagger} = -A$$

$$\therefore \left[(Ax)^*\right]^T = \lambda^*(x^*)^T$$

$$\Downarrow$$

$$(x^*)^T(A^*)^T = \lambda^*(x^*)^T \Rightarrow x^{\dagger}A^{\dagger} = \lambda^* x^{\dagger}$$

but $A^{\dagger} = -A$

or $\boxed{\boxed{-x^{\dagger}A = \lambda^* x^{\dagger}} \quad (2)}$

Multiply (1) from the left by $x^{\dagger}$

$$x^{\dagger}Ax = \lambda x^{\dagger}x \quad (1)'$$

Multiply (2) from the right by x.

$$\text{or } -x^{\dagger}Ax = \lambda^* x^{\dagger}x \quad (2)'$$

Now form (1)' + (2)' $\Rightarrow x^{\dagger}Ax - x^{\dagger}Ax = 0 = \lambda x^{\dagger}x + \lambda^* x^{\dagger}x$

$$= (\lambda + \lambda^*)x^{\dagger}x.$$

$$\Rightarrow \lambda = -\lambda^* \Rightarrow a + ib = -(a - ib) \quad a = 0$$

$$a + ib = -a + ib \quad \lambda = \text{pure imaginary}$$

c) The eigenvalues of a unitary matrix (thus of an orthogonal matrix) have absolute value 1.

Proof: Let A be unitary $\Rightarrow A^{+} = A^{-1}$

$$Ax = \lambda x \quad (1)$$

$$\left[(Ax)^{*}\right]^{T} = \lambda^{*}(x^{*})^{T}$$

$$\left[x^{*T} A^{*T}\right] = \lambda^{*} x^{*T}$$

or $x^{+}A^{+} = \lambda^{*}x^{+} \quad (2)$

multiply by (1) from the right & write

$$x^{+}\underbrace{A^{+}(A}x) = \lambda^{*} x^{+}(\lambda x)$$

associativity

$$x^{+}(A^{+}A)x = \lambda^{*}\lambda x^{+}x$$

$$x^{+}\underbrace{(A^{-1}A)}_{= I}x = |\lambda|^{2} x^{+}x$$

A is unitary

$$x^{+} I x = |\lambda^{2}| x^{+}x$$

$$x^{+}x = |\lambda^{2}| x^{+}x$$

$1 = |\lambda^{2}|$ as required.

Theorem: If two eigenvalues of a Hermitian (or real symmetric) matrix are different, the corresponding eigenvectors are orthogonal.

Proof: $Ax^{(1)} = \lambda_1 x^{(1)}$ ①

$Ax^{(2)} = \lambda_2 x^{(2)}$ ②

multiply ① by $x^{(2)\dagger}$ from the left

$x^{(2)\dagger} A x^{(1)} = \lambda_1 x^{(2)\dagger} x^{(1)}$ ①'

Now:

take Hermitian Conjugate of ② & then multiply by $x^{(1)}$ from the right

$[Ax^{(2)}]^\dagger = (\lambda_2 x^{(2)})^\dagger$

or $x^{(2)\dagger} A^\dagger = \lambda_2^* x^{(2)\dagger}$

but $A^\dagger = A$ & $\lambda_2^* = \lambda_2$.

hence $x^{(2)\dagger} A x^{(1)} = \lambda_2 x^{(2)\dagger} x^{(1)}$ ②'

①' − ②'

$\underbrace{x^{(2)\dagger} A x^{(1)} - x^{(2)\dagger} A x^{(1)}}_{0} = (\lambda_1 - \lambda_2) x^{(2)\dagger} x^{(1)}$

but $\lambda_1 \neq \lambda_2 \Rightarrow x^{(2)\dagger} x^{(1)} = 0 \Rightarrow x^{(2)} \perp x^{(1)}$ as required

Mathematical Physics

Wednesday, October 31, 2001

Some useful theorems regarding Hermitian (Symmetric) Unitary (Orthogonal)

First we need the following relationships

We must show

a) $(AB)^T = B^T A^T$

proof: $(AB)^T_{ki} = (AB)_{ik} = \sum_{j=1} A_{ij} B_{jk} = \sum B^T_{kj} A^T_{ji} = (B^T A^T)_{ki}$

hence $(AB)^T = B^T A^T$

b) $(AB)^{-1} = B^{-1} A^{-1}$

proof: form $(B^{-1}A^{-1})(AB) = B^{-1}\overbrace{(A^{-1}A)}^{\text{associatively}}B = B^{-1}B = I$, where $A^{-1}A = I$

so $(AB)^{-1} = B^{-1}A^{-1}$

c) $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$

proof $(AB)^{\dagger} = [(AB)^*]^T = (A^* B^*)^T = B^{*T} A^{*T} = B^{\dagger} A^{\dagger}$

as required

Some interesting Questions – with answers

a) Under what conditions is the *product* of *two* Hermitian matrices Hermitian?

Answer:

Suppose $H = H^\dagger$ $J = J^\dagger$ (i.e, H, J are Hermitian.

Then $(HJ)^\dagger = J^\dagger H^\dagger = JH$ $\Rightarrow$ the product is Hermitian provided $JH = HJ$ i.e. they commute.

b) Show that the product of *two* unitary (orthogonal) matrices is unitary (orthogonal).

Answer:

Suppose $U^\dagger = U^{-1}$ & $W^\dagger = W^{-1}$ (i.e. U, W are unitary)

then $(WU)^\dagger = U^\dagger W^\dagger = U^{-1} W^{-1} = (WU)^{-1}$ so WU is unitary

c) Is the *sum* of two unitary matrices unitary?

answer: $(U+W)^\dagger = U^\dagger + W^\dagger = U^{-1} + W^{-1} \stackrel{?}{=} (U+W)^{-1}$ *No*! – not unitary

d) Is the *sum* of two Hermitian matrices Hermitian?

answer: $(H+J)^\dagger = H^\dagger + J^\dagger = (H+J)$ *yes*; the sum of two Hermitian matrices is Hermitian

Vectors in matrix form

"Real Space"

$$X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad X^T = (a_1\ a_2\ a_3).$$

$$Y = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad Y^T = (b_1\, b_2\, b_3).$$

Say we want $|X|^2 = X \cdot X = X^T X = (a_1\ a_2\ a_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

$= a_1^2 + a_2^2 + a_3^2$

Also $X \cdot Y = X^T Y$

$$= (a_1\ a_2\ a_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Now suppose we are in Complex space

$$X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad Y = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

but here the a's, b's can be complex

what then is $|X|^2 = X^+ X = (a_1^*\ a_2^*\ a_3^*)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1^* a_1 + a_2^* a_2 + a_3^* a_3$

and it follows that

$$X \cdot Y = X^{+} Y = \left(a_1^{*} \; a_2^{*} \; a_3^{*} \right) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= a_1^{*} b_1 + a_2^{*} b_2 + a_3^{*} b_3$$

Some Useful theorems: Hermitian (symmetric), Unitary (orthogonal) matrices

a) The eigenvalues of a Hermitian (symmetric) matrix are real!

operate now in a complex vector space

Proof:

Let A be a Hermitian matrix $\Rightarrow A^{+} = A$.

then form $Ax = \lambda x$ (1) eigenvalue problem

Now take complex conjugate & then the transpose of (1)

$$\left[(Ax)^{*} \right]^{T} = \lambda^{*} (x^{*})^{T}$$

or $\left[A^{*} x^{*} \right]^{T} = \lambda^{*} (x^{*})^{T}$

hence $x^{*T} A^{*T} = \lambda^{*} (x^{*})^{T} \Rightarrow x^{+} A^{+} = \lambda^{*} x^{+}$ (2)

Now multiply ① by x^{+} from the left

$$x^{+}Ax = \lambda x^{+}x \quad ①'$$

Now multiply ② by x from the right

$$x^{+}A^{+}x = \lambda^{*}x^{+}x. \quad ②'$$

Now form ①' − ②'

$$\underbrace{x^{+}Ax - x^{+}A^{+}x}_{=0} = \lambda x^{+}x - \lambda^{*}x^{+}x.$$

because $A = A^{+}$

$\therefore \; x^{+}x(\lambda - \lambda^{*}) = 0$ Now $x^{+}x \neq 0$ = square of length.

$\Rightarrow \lambda - \lambda^{*} = 0 \quad \underline{\underline{\lambda = \lambda^{*}}}$ as required.

b) The eigenvalues of a skew (anti)-Hermitian matrix. (hence also skew symmetric) are pure imaginary or zero.

Proof: Let A be skew-Hermitian $\Rightarrow A^{+} = -A$.

$$Ax = \lambda x \quad ①$$

Now form $\left[(Ax)^{*}\right]^{T} = \lambda^{*}(x^{*})^{T}$

or $x^{*T} A^{*T} = \lambda^* (x^*)^T \Rightarrow x^+ A^+ = \lambda^* x^+$

but $A^+ = -A$.

hence $-x^+ A = \lambda^* x^+$ ②

Now multiply ① by x^+ from the left

$x^+ A x = \lambda x^+ x$ ①′

multiply ② by x from the right

$-x^+ A x = \lambda^* x^+ x$ ②′

Now form ①′ + ②′

$\Rightarrow x^+ A x - x^+ A x = \lambda x^+ x + \lambda^* x^+ x$

$\quad\quad\quad \parallel$

$0 = (\lambda + \lambda^*) x^+ x \qquad x^+ x \neq 0$

$\Rightarrow \lambda = -\lambda^* \Rightarrow a + ib = -(a - ib) \Rightarrow a = 0$

$= -a + ib$ $\quad$ λ is pure imaginary or zero.

as required.

c) The eigenvalues of a unitary matrix (orthogonal matrix) have absolute value 1.

Proof

Let A be a unitary matrix $\Rightarrow A^{+} = A^{-1}$

$$Ax = \lambda x \quad (1)$$

$$\left[(Ax)^{*}\right]^{T} = \lambda^{*}(x^{*})^{T}$$

$$x^{*T}A^{*T} = \lambda^{*}(x^{*})^{T}$$

$$x^{+}A^{+} = \lambda^{*}x^{+} \quad (2)$$

multiply (2) by (1) from the right.

$$x^{+}A^{+}(Ax) = \lambda^{*}x^{+}(\lambda x)$$

$$x^{+}(A^{+}A)x = \lambda^{*}\lambda x^{+}x$$

associativity

$$= |\lambda|^{2}x^{+}x$$

but $A^{+} = A^{-1}$

$$x^{+}(\underbrace{A^{-1}A}_{=I})x = |\lambda|^{2}x^{+}x,$$

$$\therefore x^{+}Ix = |\lambda|^{2}x^{+}x$$

or $x^{+}x = |\lambda|^{2}x^{+}x \Rightarrow \underline{\underline{1 = |\lambda|^{2}}}$ as required

Theorem: If two eigenvalues of a Hermitian (real symmetric) matrix are different, the corresponding eigenvectors are orthogonal.

Proof: $$A x^{(1)} = \lambda_1 x^{(1)} \quad ①$$

$$A x^{(2)} = \lambda_2 x^{(2)} \quad ②$$

multiply ① by $x^{(2)\dagger}$ from the left

$$x^{(2)\dagger} A x^{(1)} = \lambda_1 x^{(2)\dagger} x^{(1)} \quad ①'$$

Now take Hermitian Conjugate of ② & then multiply by $x^{(1)}$ from the right

$$[A x^{(2)}]^\dagger = [\lambda_2 x^{(2)}]^\dagger$$

$$= \lambda_2^* x^{(2)\dagger}$$

or

$$x^{(2)\dagger} A^\dagger = \lambda_2^* x^{(2)\dagger}$$

but $A^\dagger = A$ & $\lambda_2^* = \lambda_2$ Hermitian

$$\therefore x^{(2)\dagger} A x^{(1)} = \lambda_2 x^{(2)\dagger} x^{(1)} \quad ②'$$

Now form ①' − ②'

$$\underbrace{x^{(2)\dagger} A x^{(1)} - x^{(2)\dagger} A x^{(1)}}_{0} = (\lambda_1 - \lambda_2) x^{(2)\dagger} x^{(1)}$$ but $\lambda_1 \neq \lambda_2$

$\therefore x^{(2)\dagger} x^{(1)} = 0 \Rightarrow x^{(2)} \perp$ to $x^{(1)}$ as required

A final Summary:

Recall: Symmetric $A^T = A$. } Hermitian $A^\dagger = A$.
anti (skew) symmetric $A^T = -A$. } skew-Hermitian $A^\dagger = -A$.

orthogonal $A^T = A^{-1}$

Unitary $A^\dagger = A^{-1}$

Also, A matrix may be diagonalized.

if a) if **all** its eigenvalues are distinct.

b) if it is Hermitian or symmetric.

c) if it is unitary or othogonal.

Mathematical Physics

Lecture, Friday, November 2, 2001

Finish up - theorems etc — Hermitian matrices —

— deal next with:

Gram-Schmidt procedure —

Given: $[-1,1,1]$, $[1,-1,1]$, $[1,1,-1]$

find an orthonormal set of of vectors from above?

Let us write:

$$\vec{a}_1 = [-1,1,1] \quad \vec{a}_2 = [1,-1,1], \quad \vec{a}_3 = [1,1,-1]$$

or in the usual language of 3-dimensional vectors

$$\vec{a}_1 = [-\hat{i} + \hat{j} + \hat{k}], \quad \vec{a}_2 = [\hat{i} - \hat{j} + \hat{k}] \quad \vec{a}_3 = [\hat{i} + \hat{j} - \hat{k}]$$

Notice: $\vec{a}_1 \cdot \vec{a}_2 = (-1) - 1 + 1 \neq 0$

$\vec{a}_2 \cdot \vec{a}_3 = 1 - 1 - 1 \neq 0$

$\vec{a}_3 \cdot \vec{a}_1 = -1 + 1 - 1 \neq 0$

These vectors are independent (not-collinear) however —

they are <u>not</u> orthogonal

We wish to construct a set of orthonormal vectors

$\hat{a}_1', \hat{a}_2', \hat{a}_3'$

Step ①: normalize $\vec{a}_1 \Rightarrow \boxed{\hat{a}_1' = \dfrac{\vec{a}_1}{|\vec{a}_1|} = \dfrac{-\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}}$

Step ②: take $\vec{a}_2$ & subtract off the projection of $\vec{a}_2$ onto $\hat{a}_1'$

$\vec{a}_2 - (\hat{a}_1' \cdot \vec{a}_2)\hat{a}_1'$ where $\hat{a}_1' \cdot \vec{a}_2 = \left(\dfrac{-\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}\right) \cdot (\hat{\imath}-\hat{\jmath}+\hat{k})$

$$= \frac{-1-1+1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

& $(\hat{a}_1' \cdot \vec{a}_2)\hat{a}_1' = -\dfrac{1}{\sqrt{3}}\left(\dfrac{-\hat{\imath}+\hat{\jmath}+\hat{k}}{\sqrt{3}}\right)$

$$= \frac{\hat{\imath}-\hat{\jmath}-\hat{k}}{3}$$

& $\vec{a}_2 - (\hat{a}_1' \cdot \vec{a}_2)\hat{a}_1' = (\hat{\imath}-\hat{\jmath}+\hat{k}) - \left[\dfrac{\hat{\imath}-\hat{\jmath}-\hat{k}}{3}\right]$

$$\boxed{\vec{a}_2 - (\hat{a}_1' \cdot \vec{a}_2)\hat{a}_1 = 2/3\,\hat{\imath} - 2/3\,\hat{\jmath} + 4/3\,\hat{k}}$$

Now $\hat{a}_2' = \dfrac{\vec{a}_2 - (\hat{a}_1' \cdot \vec{a}_2)\hat{a}_1}{\| \vec{a}_2 - (\hat{a}_1' \cdot \vec{a}_2)\hat{a}_1 \|} = \dfrac{2/3\,\hat{\imath} - 2/3\,\hat{\jmath} + 4/3\,\hat{k}}{\left[(2/3)^2 + (2/3)^2 + (4/3)^2\right]^{1/2}}$

$$= \frac{2/3\,\hat{\imath} - 2/3\,\hat{\jmath} + 4/3\,\hat{k}}{(4/9 + 4/9 + 16/9)^{1/2}} = \frac{2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{3\sqrt{\frac{24}{9}}}$$

or $\hat{a}_2{}' = \dfrac{2\hat{i}-2\hat{j}+4\hat{k}}{2\sqrt{6}}$

$$\boxed{\text{or finally, } \hat{a}_2{}' = \frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}}$$

check: $\hat{a}_2{}' \cdot \hat{a}_2{}' = \left(\dfrac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}\right)\cdot\left(\dfrac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}\right) = \dfrac{1+1+4}{6} = 1$ right on the button.

also, we should check —

$$\hat{a}_1{}' \cdot \hat{a}_2{}' = \left(\frac{-\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)\cdot\left(\frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}\right) = \frac{-1-1+2}{\sqrt{18}} = 0$$

hence $\hat{a}_1{}' \perp \hat{a}_2{}'$ and <u>both</u> are normalized —

iii) Now subtract from $\vec{a}_3$ the projections of $\vec{a}_3$ onto both $\hat{a}_1{}', \hat{a}_2{}'$

i.e.

$$\vec{a}_3 - (\hat{a}_1{}' \cdot \vec{a}_3)\hat{a}_1{}' - (\hat{a}_2{}' \cdot \vec{a}_3)\hat{a}_2{}'$$

Now $\hat{a}_1{}' \cdot \vec{a}_3 = \left(\dfrac{-\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right)\cdot(\hat{i}+\hat{j}-\hat{k})$

$$= \frac{-1+1-1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

---

and $(\hat{a}_2{}' \cdot \vec{a}_3) = \left(\dfrac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}\right)\cdot(\hat{i}+\hat{j}-\hat{k}) = \dfrac{1-1-2}{\sqrt{6}} = \dfrac{-2}{\sqrt{6}}$

hence

$$\vec{a_3} - (\hat{a}_1' \cdot \vec{a_3})\hat{a}_1' - (\hat{a}_2' \cdot \vec{a_3})\hat{a}_2' =$$

$$= \hat{i} + \hat{j} - \hat{k} - \left[-\frac{1}{\sqrt{3}}\right]\left[\frac{-\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right] - \left[\frac{-2}{\sqrt{6}}\right]\left(\frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}\right)$$

$$= \hat{i} + \hat{j} - \hat{k} + \tfrac{1}{3}(-\hat{i}+\hat{j}+\hat{k}) + \frac{1}{3}(\hat{i}-\hat{j}+2\hat{k})$$

$$= \hat{i} + \hat{j} - \hat{k} \cancel{- \tfrac{1}{3}\hat{i}} \cancel{+ \tfrac{1}{3}\hat{j}} + \tfrac{1}{3}\hat{k} \cancel{+ \tfrac{1}{3}\hat{i}} \cancel{- \tfrac{1}{3}\hat{j}} + \tfrac{2}{3}\hat{k}.$$

$$= \hat{i} + \hat{j} - \hat{k} + \hat{k} = \hat{i} + \hat{j}.$$

$$\therefore \quad \hat{a}_3' = \frac{\hat{i}+\hat{j}}{\sqrt{2}}.$$

Check: $\hat{a}_1' \cdot \hat{a}_3' = \left(\frac{-\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right) \cdot \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) = 0$

$$\hat{a}_2' \cdot \hat{a}_3' = \left(\frac{\hat{i}-\hat{j}+2\hat{k}}{\sqrt{6}}\right) \cdot \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right) = 0$$

as required

i) $|e_1'\rangle = \dfrac{|e_1\rangle}{|\hat{e}_1|}$ Given e_1, e_2, e_3

ii) find projection of 2nd vector along the first & subtract it off

$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$$

then calculate

$$|e_2'\rangle = \frac{|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle}{\| \, |e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle \|}$$

iii subtract from $|e_3\rangle$ the projections of $|e_3\rangle$ onto $|e_1'\rangle, |e_2'\rangle$

$$|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle$$

& finally normalize $|e_3\rangle$

$$|e_3'\rangle = \frac{|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle}{\| \, |e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle \|}$$

do example over

$$|e_1\rangle = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad |e_2\rangle = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad |e_3\rangle = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$|e_1\rangle, |e_2\rangle, |e_3\rangle$ are <u>not</u> colinear.

$$\langle e_1|e_2\rangle = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = -1-1+1 \neq 0$$

$$\langle e_1|e_3\rangle = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -1+1-1 \neq 0$$

$$\langle e_2|e_3\rangle = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 1-1-1 \neq 0$$

they are <u>not</u> orthogonal

Gram - Schmidt.

$$|e_1'\rangle = \frac{|e_1\rangle}{|e_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{|e_1'\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

ii $|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

$$\langle e_1'|e_2\rangle = \left(-\frac{1}{\sqrt{3}}\ \ \frac{1}{\sqrt{3}}\ \ \frac{1}{\sqrt{3}}\right)\begin{pmatrix}1\\-1\\1\end{pmatrix} = -\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$\therefore$

$$\langle e_1'|e_2\rangle = -\frac{1}{\sqrt{3}}$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = -\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle = \begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{4}{3}\end{bmatrix}$$

$$|e_2'\rangle = \frac{|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle}{\left\| |e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle \right\|}$$

$$\left\| |e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle \right\| = \sqrt{4/9 + 4/9 + 16/9}$$

$$= \sqrt{\frac{24}{9}} = \frac{2}{3}\sqrt{6}$$

$$\therefore |e_2'\rangle = \frac{3}{2\sqrt{6}}\begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\4/3\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

finally

$$|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle$$

$$\langle e_1'|e_3\rangle = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{-1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{-2}{\sqrt{3}}$$

$$\langle e_2'|e_3\rangle = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$$

$$\langle e_1'|e_3\rangle|e_1'\rangle = \frac{-2}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{-2}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$\langle e_2'|e_3\rangle|e_2'\rangle = -\frac{2}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ +1/3 \\ -2/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} - \begin{bmatrix} -1/3 \\ +1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ \end{bmatrix}$$

Mathematical Physics

Lecture Monday, November 5, 2001

Tell class what is on November 16, 2001 examination

Example Given: $[-1, 1, 1]$, $[1, -1, 1]$ $[1, 1, -1]$

We have shown that they are independent but not orthogonal

Begin

Step ① $\boxed{|e_1'\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}}$

Step ② $|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

$$\langle e_1'|e_2\rangle = \begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix} = -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\begin{bmatrix}-1\\1\\1\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix}$$

$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle = \begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-1/3\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{4}{3}\end{bmatrix}$$

$$|e_2'\rangle = \frac{|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle}{\left\| |e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle \right\|}$$

$$= \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2}} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

$$= \frac{1}{\sqrt{4/9 + 4/9 + 16/9}} \begin{bmatrix} \\ \\ \end{bmatrix} = \frac{1}{\sqrt{\frac{24}{9}}} \begin{bmatrix} \\ \end{bmatrix} = \frac{3}{2\sqrt{6}} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\boxed{\therefore |e_2'\rangle = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}}$$

finally step 3 $\quad |e_3\rangle = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Obtain $|e_3'\rangle$

$$|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle$$

We need:

$$\langle e_1' | e_3 \rangle = \left[-\frac{1}{\sqrt{3}} \;\; \frac{1}{\sqrt{3}} \;\; \frac{1}{\sqrt{3}} \right] \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

$$\langle e_2' | e_3 \rangle = \left[\frac{1}{\sqrt{6}} \;\; -\frac{1}{\sqrt{6}} \;\; \frac{2}{\sqrt{6}} \right] \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$$

$$\langle e_1' | e_3 \rangle | e_1' \rangle = -\frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$\langle e_2' | e_3 \rangle | e_2' \rangle = \frac{-2}{\sqrt{6}} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{6} \\ \frac{2}{6} \\ -\frac{4}{6} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

finally, (next to)

$$|e_3\rangle - \langle e_1' | e_3 \rangle | e_1' \rangle - \langle e_2' | e_3 \rangle | e_2' \rangle =$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} - \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \; |e_3'\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

as required

Check orthogonality

$$\left.\begin{aligned} \langle e_1' | e_2' \rangle &= 0 \\ \langle e_1' | e_3' \rangle &= 0 \\ \langle e_2' | e_3' \rangle &= 0 \end{aligned}\right\} \text{ checks out.}$$

Mathematical Physics.

Lecture, Monday, November 5, 2001

Gram-Schmidt Procedure

Before proceeding, talk about exam —
November 16, 2001
Vector Calculus / algebra
coordinate systems.
Theorem of Gauss — divergence.
Theorem of Stokes —
line integrals —
Surface integrals —
complex number arithmetic/algebra
normals to surfaces etc

Example 1 do completely all steps

$[-1,1,1]$, $[1,-1,1]$, $[1,1,-1]$

stress the procedure (see attached calculations)

Example: $\boxed{\text{Complex set of vectors}}$

Given $|e_1\rangle = \begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix}$ $|e_2\rangle = \begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix}$ $|e_3\rangle = \begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix}$

Check $\left.\begin{matrix} \langle e_1|e_2\rangle \\ \langle e_1|e_3\rangle \\ \langle e_2|e_3\rangle \end{matrix}\right\} \neq 0$ they are not orthogonal

$$|e_1'\rangle = \frac{|e_1\rangle}{\||e_1\rangle\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} \Rightarrow |e_1'\rangle = \begin{bmatrix} \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

form

Step② $|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

$$\langle e_1'|e_2\rangle = \frac{1}{\sqrt{3}}(1-2i)$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = \begin{bmatrix} \frac{2+i}{3} \\ \frac{1-2i}{3} \\ \frac{-1+2i}{3} \end{bmatrix}$$

finally.

$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle = \begin{bmatrix} \frac{1-i}{3} \\ -\frac{(1+i)}{3} \\ -2\frac{(1+i)}{3} \end{bmatrix}$$

Normalization

$$\Rightarrow \underline{\underline{\frac{2\sqrt{3}}{3}}}$$

$$\boxed{|e_2'\rangle = \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ -\frac{(1+i)}{2\sqrt{3}} \\ -\frac{(1+i)}{\sqrt{3}} \end{bmatrix}}$$

Step 3 form.

$$|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle$$

$$\langle e_1'|e_3\rangle = -\frac{1}{\sqrt{3}}$$

$$\langle e_2'|e_3\rangle = \frac{2+i}{\sqrt{3}}$$

$$\langle e_1'|e_3\rangle|e_1'\rangle = \begin{bmatrix} -\frac{i}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \qquad \langle e_2'|e_3\rangle|e_2'\rangle = \begin{bmatrix} \frac{3-i}{6} \\ -\left(\frac{1+3i}{6}\right) \\ -\left(\frac{1+3i}{3}\right) \end{bmatrix}$$

Now form.

$$\begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix} - \begin{bmatrix} -\frac{i}{3} \\ -1/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} \frac{3-i}{6} \\ -\left(\frac{1+3i}{6}\right) \\ -\left(\frac{1+3i}{3}\right) \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \\ 0 \end{bmatrix} = |e_3'\rangle}$$

normalization $= 1$

$\boxed{\text{Example}}$ — $\boxed{\text{Complex set of vectors}}$

outline this only !

$$|e_1\rangle = \begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} \qquad |e_2\rangle = \begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix} \qquad |e_3\rangle = \begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix}$$

$$\langle e_1 | e_2 \rangle = \begin{bmatrix} -i & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix} = -\hat{i} - \hat{i} + 1 \neq 0$$

$$\langle e_1 | e_3 \rangle = \begin{bmatrix} -i & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix} = -i - 1 + i \neq 0$$

$$\langle e_2 | e_3 \rangle = \begin{bmatrix} 1 & i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix} = 1 - i + i \neq 0$$

step 1

$$|e_1'\rangle = \frac{|e_1\rangle}{\| |e_1\rangle \|} = \frac{1}{\sqrt{3}}\begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} \Rightarrow \boxed{|e_1'\rangle = \begin{bmatrix} \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}}$$

Step (2) $|e_2\rangle - \langle e_1'|e_2\rangle |e_1'\rangle$

$$\langle e_1'|e_2\rangle = \begin{bmatrix} \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix} = \frac{-i}{\sqrt{3}} - \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} - \frac{2i}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}(1-2i)$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = \frac{1}{\sqrt{3}}(1-2i)\frac{1}{\sqrt{3}}\begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix}$$

$$= \tfrac{1}{3}(1-2i)\begin{bmatrix} i \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} i+2 \\ 1-2i \\ -1+2i \end{bmatrix}$$

$$\therefore \langle e_1'|e_2\rangle|e_1'\rangle = \begin{bmatrix} \frac{2+i}{3} \\ \frac{1-2i}{3} \\ \vdots \end{bmatrix}$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = \begin{bmatrix} \frac{2+i}{3} \\ \frac{1-2i}{3} \\ \frac{-1+2i}{3} \end{bmatrix}$$

$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle =$$

$$= \begin{bmatrix} 1 \\ -i \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{2+i}{3} \\ \frac{1-2i}{3} \\ \frac{-1+2i}{3} \end{bmatrix} = \begin{bmatrix} 1-\left(\frac{2+i}{3}\right) \\ -i-\left(\frac{1-2i}{3}\right) \\ -1-\left(\frac{-1+2i}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3-2-i}{3} \\ \frac{-3i-1+2i}{3} \\ \frac{-3+1-2i}{3} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{3} \\ -\frac{(1+i)}{3} \\ -2\frac{(1+i)}{3} \end{bmatrix}$$

$$\therefore \; |e_2'\rangle = \frac{3}{2\sqrt{3}} \begin{bmatrix} \frac{1-i}{3} \\ \frac{-(1+i)}{3} \\ -2\frac{(1+i)}{3} \end{bmatrix}$$

$$\boxed{|e_2'\rangle = \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ \frac{-(1+i)}{2\sqrt{3}} \\ -\frac{(1+i)}{\sqrt{3}} \end{bmatrix}}$$

right on the button

check $\langle e_2' | e_2' \rangle = \begin{bmatrix} \frac{1+i}{2\sqrt{3}} & -\frac{(1-i)}{2\sqrt{3}} & -\frac{(1-i)}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ \frac{-(1+i)}{2\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{3}} \end{bmatrix}$

$$= \frac{2}{(2\sqrt{3})^2} + \frac{2}{(2\sqrt{3})^2} + \frac{2}{(\sqrt{3})^2} = \frac{2}{12} + \frac{2}{12} + \frac{2}{3}$$

$$= \frac{2}{12} + \frac{2}{12} + \frac{8}{12} = 12/12 = 1$$

check orthogonality.

$$\langle e_1' | e_2' \rangle = \begin{bmatrix} \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ \frac{-(1+i)}{2\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{3}} \end{bmatrix}$$

$$= \frac{-i-1}{(2)(3)} \; \frac{-1-i}{(2)(3)} + \frac{(1+i)}{3}$$

$$= \frac{-\cancel{i}-1 \; -1-\cancel{i}+2+\cancel{2i}}{6} = 0$$ right on the button

Now form

$$|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle$$

$$\langle e_1'|e_3\rangle = \begin{bmatrix} \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix}$$

$$= \frac{-i}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{i}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$$

& obtain

$$\langle e_2'|e_3\rangle = \begin{bmatrix} \frac{1+i}{2\sqrt{3}} & -\frac{(1-i)}{2\sqrt{3}} & -\frac{(1-i)}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix}$$

$$= \frac{1+i}{2\sqrt{3}} + \frac{(1-i)}{2\sqrt{3}} + \frac{i(1-i)}{\sqrt{3}}$$

$$\cancel{= 1+i+1 \quad i \quad +1}$$

$$= \frac{1+i+1-i+2i+2}{2\sqrt{3}} = \frac{4+2i}{2\sqrt{3}} = \frac{2+i}{\sqrt{3}}$$

→ over

Now form:

$$\langle e_1' | e_3 \rangle | e_1' \rangle = \frac{-1}{\sqrt{3}} \begin{bmatrix} \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{-i}{3} \\ -\frac{1}{3} \\ +\frac{1}{3} \end{bmatrix}$$

$$\langle e_2' | e_3 \rangle | e_2' \rangle = \left(\frac{2+i}{\sqrt{3}}\right) \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ \frac{-(1+i)}{2\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(2+i)(1-i)}{6} \\ \frac{-(2+i)(1+i)}{6} \\ \frac{-(2+i)(1+i)}{3} \end{bmatrix} = \begin{bmatrix} \frac{2-i+1}{6} \\ -\frac{[2+i+2i-1]}{6} \\ -\frac{[2+i+2i-1]}{3} \end{bmatrix} = \begin{bmatrix} \frac{3-i}{6} \\ -\frac{(1+3i)}{6} \\ -\frac{(1+3i)}{3} \end{bmatrix}$$

Now form.

$$\begin{bmatrix} 1 \\ -1 \\ -i \end{bmatrix} - \begin{bmatrix} \frac{-i}{3} \\ -\frac{1}{3} \\ +\frac{1}{3} \end{bmatrix} - \begin{bmatrix} \frac{3-i}{6} \\ -\frac{(1+3i)}{6} \\ -\frac{(1+3i)}{3} \end{bmatrix}$$

$$= 1 + \frac{i}{3} - \frac{(3-i)}{6} = \frac{6+2i-3+i}{6} = \frac{3+3i}{6} = \frac{1+i}{2}$$

$$-1 + \frac{1}{3} + \frac{(1+3i)}{6} = \frac{-6+2+1+3i}{6} = \frac{-3+3i}{6} = \frac{-1+i}{2}$$

$$-i - \frac{1}{3} + \frac{(1+3i)}{3} = \frac{-3i-1+1+3i}{3} = 0$$

Normalization:

$$\frac{1}{\sqrt{\left(\frac{1+1}{2}\right)^2 + \left(\frac{1+1}{2}\right)^2}} = \cancel{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{\frac{2}{4} + \frac{2}{4}}} = 1$$

$$|e_3'\rangle = \cancel{\sqrt{2}} \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \\ 0 \end{bmatrix}$$

$$\boxed{|e_3'\rangle = \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \\ 0 \end{bmatrix}}$$

Hence, putting all together :

$$|e_1'\rangle = \begin{bmatrix} \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} \qquad |e_2'\rangle = \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ \frac{-(1+i)}{2\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{3}} \end{bmatrix}$$

$$|e_3'\rangle = \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \\ 0 \end{bmatrix}$$

check normality:

$$\langle e_1' | e_1\rangle = \begin{bmatrix} \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} = \cancel{\frac{1}{3} + \frac{1}{9} +}$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 3/3 = 1$$

$$\langle e_2' | e_2\rangle = \begin{bmatrix} \frac{1+i}{2\sqrt{3}} & -\frac{(1-i)}{2\sqrt{3}} & -\frac{(1+i)}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ -\frac{(1+i)}{2\sqrt{3}} \\ -\frac{(1+i)}{\sqrt{3}} \end{bmatrix}$$

$$= \frac{(1+i)(1-i)}{4\cdot 3} + \frac{(1-i)(1+i)}{4\cdot 3} + \frac{(1-i)(1+i)}{3}$$

$$= \frac{2}{12} + \frac{2}{12} + \frac{2}{3} = \frac{2+2+8}{12} = 1$$

$$\langle e_3' | e_3' \rangle = \begin{bmatrix} \frac{1-i}{2} & \frac{-1-i}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ \\ \frac{-1+i}{2} \\ \\ 0 \end{bmatrix}$$

$$= \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) + \left(\frac{-1-i}{2}\right)\left(\frac{-1+i}{2}\right) + 0$$

$$= \frac{2}{4} + \frac{2}{4} = \frac{4}{4} = 1 \quad \checkmark$$

— — — —

Orthogonality check:

$$\langle e_1' | e_2' \rangle = \begin{bmatrix} \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2\sqrt{3}} \\ \\ \frac{-(1+i)}{2\sqrt{3}} \\ \\ \frac{-(1+i)}{\sqrt{3}} \end{bmatrix}$$

$$= \frac{-i(1-i)}{2 \cdot 3} + \frac{1}{\cancel{3}}\left(\frac{-1-i}{2 \cdot 3}\right) + \frac{(1+i)}{3}$$

$$= \frac{-i-1}{6} \; \frac{-1-i}{6} + \frac{2+2i}{6} = 0 \quad \checkmark$$

$$\langle e_1' | e_3' \rangle = \begin{bmatrix} \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{(1+i)}{2} \\ \frac{-1+i}{2} \\ 0 \end{bmatrix}$$

$$= \frac{-i(1+i)}{2\sqrt{3}} + \frac{(-1+i)}{2\sqrt{3}} + 0$$

$$= \frac{-i+1-1+i}{2\sqrt{3}} = 0 \checkmark$$

$$\langle e_2' | e_3' \rangle = \begin{bmatrix} \frac{1+i}{2\sqrt{3}} & -\frac{(1-i)}{2\sqrt{3}} & -\frac{(1-i)}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} \\ \frac{-1+i}{2} \\ 0 \end{bmatrix}$$

$$= \frac{(1+i)(1+i)}{4\sqrt{3}} - \frac{(1-i)(-1+i)}{4\sqrt{3}} + 0$$

$$= \frac{1+2i-1 - [-1+2i+1]}{4\sqrt{3}}$$

$$= \frac{\cancel{1}+\cancel{2i}-\cancel{1}+\cancel{1}-\cancel{2i}-\cancel{1}}{4\sqrt{3}} = 0$$

right on the button

Mathematical Physics.

Lecture Wednesday, November 7, 2001

Hermitian Matrices:

Let us look at $T = \begin{pmatrix} 1 & 1-i \\ 1+i & 0 \end{pmatrix}$

a) Is it Hermitian?

$$T^{+} = (T^{*})^{T} = \begin{bmatrix} 1 & 1+i \\ 1-i & 0 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 1-i \\ 1+i & 0 \end{bmatrix} = T$$

$\therefore T^{+} = T \Rightarrow T$ is Hermitian

b) eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1-i \\ 1+i & -\lambda \end{vmatrix} = 0 = -\lambda(1-\lambda) - (1-i)(1+i) = 0$$

$$0 = -\lambda + \lambda^2 - 2$$

$$0 = \lambda^2 - \lambda - 2$$

$$\therefore (\lambda+1)(\lambda-2) = 0 \Rightarrow \lambda_1 = 2 \quad \lambda_2 = -1$$

c) eigenvectors

$\lambda_1 = 2$ use dummies $a_1, a_2, a_3 \ldots$

$$\begin{pmatrix} -1 & (1-i) \\ 1+i & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow \begin{aligned} -a_1 + (1-i)a_2 &= 0 \quad (1) \\ (1+i)a_1 - 2a_2 &= 0 \quad (2) \end{aligned}$$

Question: are these the same?

look at (1) $a_1 = (1-i)a_2$

(2) $2a_2 = (1+i)a_1$

from 2 $\quad a_1 = \frac{2a_2}{1+i} = \frac{2a_2}{(1+i)(1-i)}(1-i) = \frac{2a_2}{2}(1-i)$

$$\left.\begin{array}{ll} \therefore\ a_1 = a_2(1-i) & (2) \\ a_1 = (1-i)a_2 & (4) \end{array}\right\} \text{ they are the same.}$$

Set $a_2 = 1 \Rightarrow a_1 = 1-i$

$\therefore\ X^{(1)} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ normalize $\boxed{\hat{X}^{(1)} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1-i \\ 1 \end{bmatrix}}$

Now look at $\lambda_2 = -1$

$$\begin{pmatrix} 2 & 1-i \\ 1+i & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow \left.\begin{array}{ll} (1) & 2a_1 + (1-i)a_2 = 0 \\ (2) & (1+i)a_1 + a_2 = 0 \end{array}\right\} \text{ are they the same?}$$

check: (1) $\quad a_1 = \frac{-(1-i)a_2}{2}$

(2) $\quad a_1 = \frac{-a_2}{1+i} = \frac{-a_2(1-i)}{(1+i)(1-i)}$

$a_1 = \frac{-a_2(1-i)}{2}$ $\quad$ the same

Let $a_2 = 2$

hence $a_1 = -\underline{(1-i)} = (i-1)$

$\therefore\ X^{(2)} = \begin{bmatrix} i-1 \\ 2 \end{bmatrix}$ normalize $\Rightarrow \hat{X}^{(2)} = \frac{1}{\sqrt{6}}\begin{bmatrix} i-1 \\ 2 \end{bmatrix}$

$$\boxed{\hat{X}^{(2)} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1+i \\ 2 \end{bmatrix}}$$

check orthogonality –

$$\hat{X}^{(1)\dagger} \cdot \hat{X}^{(2)} = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{-1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} = \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{-1+i}{\sqrt{6}}\right) + \frac{2}{\sqrt{3}\sqrt{6}}$$

$$\langle X^{(1)} | X^{(2)} \rangle = \quad = \frac{-1-i+i-1+2}{\sqrt{3}\sqrt{6}} = 0$$

as required

$\therefore$ they are orthogonal.

d) form the matrix S which is the similar matrix:

$$S = \begin{pmatrix} \frac{1-i}{\sqrt{3}} & \frac{-1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Question: is S unitary.

this implies $S^\dagger = S^{-1}$

check it out: $S^\dagger = (S^*)^T = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{-1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}^T = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$

check $S^\dagger S = S S^\dagger = I$ if S is unitary.

Check $S^{-1}S = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1-i}{\sqrt{3}} & \frac{-1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

as required

① $\Rightarrow \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{1-i}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} = \frac{2}{3} + \frac{1}{3} = 1$

② $\Rightarrow \left(\frac{1+i}{\sqrt{3}}\right)\left(\frac{-1+i}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right)$

$= \frac{-1-i+i-1}{\sqrt{3}\sqrt{6}} + \frac{2}{\sqrt{3}\sqrt{6}} = 0$

③ $\Rightarrow \left[-\frac{(1+i)}{\sqrt{6}}\right]\left[\frac{1-i}{\sqrt{3}}\right] + \frac{2}{\sqrt{6}\sqrt{3}} =$

$\Rightarrow \frac{-2}{\sqrt{6}\sqrt{3}} + \frac{2}{\sqrt{6}\sqrt{3}} = 0$

④ finally, $\left[-\frac{(1+i)}{\sqrt{6}}\right]\left[\frac{-1+i}{\sqrt{6}}\right] + \frac{4}{6}$

$\Rightarrow -\frac{[-1-i+i-1]}{\sqrt{6}\sqrt{6}} + 2/3 = \frac{-[-2]}{6} + 2/3 = 1/3 + 2/3 = 1$

$\therefore$ $S^{-1}S = 1 \Rightarrow S^{\dagger} = S^{-1}$ S is unitary.

Now form.

$$S^{-1}TS = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & (1-i) \\ (1+i) & 0 \end{bmatrix} \begin{bmatrix} \frac{1-i}{\sqrt{3}} & \frac{i-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= S^{-1} \begin{bmatrix} \frac{1-i}{\sqrt{3}} + \frac{1-i}{\sqrt{3}} & \frac{i-1}{\sqrt{6}} + \frac{2(1-i)}{\sqrt{6}} \\ \frac{(1+i)(1-i)}{\sqrt{3}} & \frac{(1+i)(i-1)}{\sqrt{6}} \end{bmatrix}$$

$$= S^{-1} \begin{bmatrix} \frac{2(1-i)}{\sqrt{3}} & \frac{(1-i)}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

finally

$$\begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-(1+i)}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{2(1-i)}{\sqrt{3}} & \frac{(1-i)}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

amazing!

Example:

Given $T = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$ is it Hermitian? – yes

$T = T^{+}$

a) eigenvalues

$$|T - \lambda I| = \begin{vmatrix} 2-\lambda & i & 1 \\ -i & 2-\lambda & i \\ 1 & -i & 2-\lambda \end{vmatrix} = 0$$

$$= (2-\lambda)^3 - (2-\lambda) - i\left(-2i + i\lambda - i\right) + 1\left(i^2 - (2-\lambda)\right)$$

$$= (2-\lambda)^3 - (2-\lambda) - 3 + \lambda - 1 + \lambda - 2$$

$$= (2-\lambda)^3 - 5 + 3\lambda - 3 = 0$$

$$= 2^3 + 3(2)^2(-\lambda) + \frac{3\cdot 2\,(2)}{2!}(-\lambda)^2 + \frac{3\cdot 2\cdot 1}{3!}(-\lambda)^3 - 5 + 3\lambda - 3 = 0$$

$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 5 + 3\lambda - 3 = 0$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$= -\lambda\left[\lambda^2 - 6\lambda + 9\right] = 0$$

$\lambda_1 = 0 \quad (\lambda-3)(\lambda-3) = 0 \Rightarrow \lambda_2 = \lambda_3 = 3$

b) eigenvectors.

a_1, a_2, a_3 dummies

$X^{(1)} \quad \lambda_1 = 0$

$$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \Rightarrow \begin{array}{ll} 2a_1 + a_2 i + a_3 = 0 & (1) \\ \left[-i a_1 + 2a_2 + i a_3 = 0\right] \frac{1}{i} = -i & (2) \\ a_1 - i a_2 + 2a_3 = 0 & (3) \end{array}$$

$$\frac{-ia_1 + 2a_2 + ia_3 = 0}{i}$$

$$= -a_1 + \frac{2a_2}{i} + a_3 = 0$$

$$= a_1 + 2ia_2 - a_3 = 0$$

$$2a_1 + ia_2 + a_3 = 0$$

$$\begin{array}{r} 2a_1 + ia_2 + a_3 = 0 \\ + \quad a_1 + 2ia_2 - a_3 = 0 \\ \hline 3a_1 + 3ia_2 = 0 \end{array}$$

or $ia_2 = -a_1$

$\boxed{a_2 = ia_1}$ $\longrightarrow$ put this into (3)

$$2a_1 + i(ia_1) + a_3 = 0$$

$$2a_1 - a_1 + a_3 = 0$$

$$a_1 + a_3 = 0$$

$$a_1 = -a_3 \Rightarrow \boxed{a_3 = -a_1}$$

set $a_1 = 1$

$a_2 = i$

$a_3 = -1$

$$X^{(1)} = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \Rightarrow \boxed{\hat{X}^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}}$$

Now get $X^{(2)}$ $X^{(3)}$ $\lambda_{2,3} = 3$

$$\begin{pmatrix} (2-3) & i & 1 \\ -i & (2-3) & i \\ 1 & -i & 2-3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & i & 1 \\ -i & -1 & i \\ 1 & -i & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \quad \left.\begin{aligned} -a_1 + a_2 i + a_3 &= 0 \\ -a_1 i - a_2 + a_3 i &= 0 \\ a_1 - a_2 i - a_3 &= 0 \end{aligned}\right\}$$

you get same relationship from all 3

hence there is but one condition:

$$a_1 - i a_2 - a_3 = 0$$

we could choose $\boxed{a_3 = 0 \quad a_2 = -i a_1}$ choice (1) $a_1 = 1$

or we could chose $\boxed{a_2 = 0 \quad \& \quad a_1 = a_3.}$ $a_1 = 1$

Lets call these: $X_0^{(2)} = \begin{pmatrix} a_1 \\ -i a_1 \\ 0 \end{pmatrix}$ set $a_1 = 1$ $\Rightarrow X_0^{(2)} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$

& $X_0^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{\hat{X}_0^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$ $\boxed{\hat{X}_0^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}$

Note: $\hat{X}_0^{(2)} \cdot \hat{X}_0^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} \neq 0$

these two vectors are not orthogonal

$\boxed{\text{Let us use Gram-Schmidt.}}$

$$\hat{X}_0^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \hat{X}_0^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$X_0^{(2)} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad X_0^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Step ①

$$|e_1\rangle = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \boxed{|e_1'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}$$

$$|e_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Step ② $|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

$$\langle e_1'|e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & +i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}.$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{-i}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{i}{2} \\ 1 \end{bmatrix}$$

Now normalize $\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1\right]^{1/2}$.

$$\left(\frac{1}{4} + \frac{1}{4} + 1\right)^{1/2} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$$

$$\boxed{\therefore |e_2'\rangle = \frac{2}{\sqrt{6}} \begin{bmatrix} \frac{1}{2} \\ i/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix}}$$

hence we now have

constructed

$$\boxed{\hat{X}^{(1)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix} \qquad \hat{X}^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = |e_1'\rangle}$$

$$\boxed{\hat{X}^{(3)} = |e_3'\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix}}$$

Now check orthogonality

$$\hat{X}^{(1)} \cdot \hat{X}^{(2)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} = 0$$

$$\hat{X}^{(1)} \cdot \hat{X}^{(3)} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}} = 0$$

$$\hat{X}^{(2)} \cdot \hat{X}^{(3)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}} = 0$$

right - on the button

Example: Lecture November 5, 7, 9 ? 2001

$$|e_1\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad |e_2\rangle = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad |e_3\rangle = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Step ①

$$\boxed{|e_1'\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}$$

form $|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

we need $\langle e_1'|e_2\rangle = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\langle e_1'|e_2\rangle = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

form $\langle e_1'|e_2\rangle|e_1'\rangle = \frac{4}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix}$

Now form

$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

Normalize

$$\sqrt{(1/3)^2 + (2/3)^2 + (1/3)^2}$$

$$= \sqrt{\frac{1}{9} + 4/9 + 1/9} = \sqrt{\frac{6}{9}}$$

or Normalization = $\dfrac{\sqrt{6}}{3}$

$$\therefore\ |e_2'\rangle = \frac{3}{\sqrt{6}}\begin{bmatrix}-1/3\\ 2/3\\ -1/3\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\end{bmatrix}$$

$$\boxed{|e_2'\rangle = \begin{bmatrix}-\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\end{bmatrix}}$$

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form.

$$|e_3\rangle - \langle e_1'||e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle$$

we need $\langle e_1'|e_3\rangle = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$

$$= \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \boxed{\frac{4}{\sqrt{3}} = \langle e_1'|e_3\rangle}$$

$$\langle e_2' | e_3 \rangle = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2. \end{bmatrix}$$

$$= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} = \boxed{-\frac{1}{\sqrt{6}} = \langle e_2' | e_3 \rangle}$$

Now form:

$$\langle e_1' | e_3 \rangle | e_1' \rangle = \frac{4}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix}$$

and $\langle e_2' | e_3 \rangle | e_2' \rangle = -\frac{1}{\sqrt{6}} \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ -\frac{2}{6} \\ \frac{1}{6} \end{bmatrix}$

→ over

finally form.

$$|e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{4}{3} \\ 4/3 \\ 4/3 \end{bmatrix} - \begin{bmatrix} \frac{1}{6} \\ -\frac{2}{6} \\ \frac{1}{6} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 - 4/3 - \frac{1}{6} \\ 1 - 4/3 + 2/6 \\ 2 - 4/3 - 1/6 \end{bmatrix} = \begin{bmatrix} \frac{6-8-1}{6} \\ \frac{6-8+2}{6} \\ \frac{12-8-1}{6} \end{bmatrix} = \begin{bmatrix} -\frac{3}{6} \\ 0 \\ \frac{3}{6} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Normalization $= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$$

$$\therefore \; \boxed{|e_3'\rangle = \frac{2}{\sqrt{2}}\begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$$

Check all this out!