Mathematical Physics

Lecture: Friday, October 19, 2001 & October 22, 2001

Goal: eigenvalues, eigenvectors, diagonalization of matrices, similarity transformations

That's it!!!

Matrix – a rectangular array of quantities
i.e. numbers (real/complex), fn's, operations

$$A = a_{ij} = \begin{pmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & & & & \\ \vdots & & & & \\ a_{m1} & - & - & - & a_{mn} \end{pmatrix}$$

a_{ij} called element

1st index ⇒ row
2nd index ⇒ column

a_{23} is element 2 row & 3rd column

A above is of order $m \times n$

if $m = n$ matrix is square

we are interested only in square matrices

The main diagonal of a square matrix.

$$(a_{11}, a_{22}, a_{33} -- a_{nn})$$

Vectors: can be written as row or column matrices

$\vec{a} = [a_1, a_2 --- a_n]$ $1 \times n$ matrix

or $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ $m \times 1$ matrix

for matrices of the same order, $A = B$ iff $a_{ij} = b_{ij}$ for all i, j

$$\therefore a_{ij} = b_{ij} \Rightarrow A = B$$

The Null matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 3 \times 3 \text{ Null matrix}$$

Multiplication by a scalar

$$kA = Ak \Rightarrow a_{ij}k = ka_{ij}$$

e.g. $$3\begin{pmatrix} 2 & 1 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 0 & -3i \end{pmatrix}$$

Add | Subtract (same order)

$$C = A \pm B$$
$$c_{ij} = a_{ij} \pm b_{ij}.$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \\ a_{21} & & \\ \vdots & & \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11}, & a_{12} \pm b_{12} & \cdots \\ a_{21} \pm b_{21} & & \\ \vdots & & \end{pmatrix}$$

etc

$A + B = B + A$ Commutes

$(A+B)+C = A + (B+C)$ associative

Matrix Multiplication

Motivation   Set of linear eq.

$$2x - y + 2z = 1$$
$$x + 2y - 4z = 3$$
$$3x - y + z = 0$$

We could write:

$$\begin{pmatrix} 2x & -y & 2z \\ x & 2y & -4z \\ 3x & -y & z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

and maybe $\Downarrow$

$$\begin{pmatrix} 2 & -1 & 2 \\ 1 & 2 & -4 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

How can we achieve above.

dot product of 1st row & 1st column

(1) $(2)(x) - (1)(y) + (2)(z) = 2x - y + 2z$

(2) $(1)(x) + (2)(y) - (4)(z) = x + 2y - 4z$ 2nd row & 1st col

(3) $(3)(x) (-1)(y + (1)(z) = 3x - y + z$ 3rd row & 1st column.

The product AB of two matrices A, B is defined to a matrix C, such that c_{ij} (elements of C) are obtained by summing the products of the i^{th} row of A and the corresponding element of the j^{th} column of B taken in order

$AB = C$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$AB \neq BA$ does not commute in general example

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

e.g

$$E = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$DE \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -3 & 4 \\ 4 & -2 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

on the other hand.

$$ED = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 1 \\ 3 & 0 & 1 \\ 6 & 2 & 2 \end{pmatrix}$$

$$DE \neq ED$$

Division is not uniquely defined.

Derivative $\frac{d}{dx}\begin{bmatrix} x & x^3 \\ e^{-x} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3x^2 \\ -e^{-x} & 0 \end{bmatrix}$

Integral $\int \begin{bmatrix} x & x^3 \\ e^{-x} & 0 \end{bmatrix} dx = \begin{bmatrix} \frac{x^2}{2} & \frac{x^4}{4} \\ -e^{-x} & 0 \end{bmatrix} + \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$

Properties of Arbitrary Matrices

Transpose

Given $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

check notation of source

i.e. we interchange rows and columns.

if $A = a_{ij}$ then $A^T = a_{ji}$

e.g. $A = \begin{pmatrix} -3 & 2 \\ -4 & 5 \end{pmatrix}$ $A^T = \begin{pmatrix} -3 & -4 \\ 2 & 5 \end{pmatrix}$

Complex-Conjugate Matrix

Given A what is $\overline{A}$ or A^*

$A^* = a_{ij}^* = \overline{a_{ij}}$ Take complex conjugate of each element

e.g. $A = \begin{pmatrix} 2+3i & 4-5i \\ 3 & 4i \end{pmatrix}$ $A^* = \overline{A} = \begin{pmatrix} 2-3i & 4+5i \\ 3 & -4i \end{pmatrix}$

$\boxed{\text{Hermitian Congugate}}$ $A^\dagger$ (A dagger

$$A^\dagger = (\overline{A})^T = (A^*)^T$$

Take complex conjugate of A and then take transpose of the complex conjugate matrix

$$A = \begin{pmatrix} 2+3i & 4-5i \\ 3 & 4i \end{pmatrix}$$

$$A^* = \begin{pmatrix} 2-3i & 4+5i \\ 3 & -4i \end{pmatrix}$$

$$A^\dagger = (A^*)^T = \begin{pmatrix} 2-3i & 3 \\ 4+5i & -4i \end{pmatrix}$$

Special Square Matrices

Unit Matrix $I = \delta_{ij}$

$$I = \delta_{ij}$$

where $IA = AI = A$ $\qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

3×3 unit

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonal Matrix

~~$D = \delta_i$~~

$$D = d_i \delta_{ij}$$

eg $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Determinant

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 4 & -4 & 6 \end{pmatrix} = a_{ij}$$

$$\det A = \left| A \right| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 4 & -4 & 6 \end{vmatrix}$$

Expansion by <u>minors</u>.

look at 1st row $(1 \quad -1 \quad 2)$

3 <u>minors</u> of 1st row $\begin{vmatrix} -2 & 3 \\ -4 & 6 \end{vmatrix}$, $\begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix}$, $\begin{vmatrix} 0 & -2 \\ 4 & -4 \end{vmatrix}$ respectively.

Now the cofactor of a_{ij} call it A_{ij} is $(-1)^{i+j}$ times the minor.

Hence, the cofactors of the 3-minors above are, respectively, $A_{11} = (-1)^{1+1} \begin{vmatrix} -2 & 3 \\ -4 & 6 \end{vmatrix}$ $A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix}$

& $A_{13} = (-1)^{1+3} \begin{vmatrix} 0 & -2 \\ 4 & -4 \end{vmatrix}$

$\rightarrow$ over

The importance of cofactors is due to the following

$$\boxed{|A| = \det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} a_{ji} A_{ji}}$$

Example: Using cofactors, find the determinant of A

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Soln Expanding by first row

$$\det A = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 3(-2) - 2(-3) + 1(-2)$$

$$= -6 + 6 - 2 = \underline{\underline{-2}}$$

Expanding by 1st column

$$\begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 3(-2) + 1(4-2) + 2$$

$$= -6 + 2 + 2 = \underline{-2}$$ as required.

Determinants

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 4 & -4 & 6 \end{pmatrix} = a_{ij}$$

$$\det A = |A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 3 \\ 4 & -4 & 6 \end{vmatrix}$$

Expansion by "minors"

look at 1st row $\begin{vmatrix} 1 & -1 & 2 \end{vmatrix}$

3-minors of 1st row $\begin{vmatrix} -2 & 3 \\ -4 & 6 \end{vmatrix}, \begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix}, \begin{vmatrix} 0 & -2 \\ 4 & -4 \end{vmatrix}$

Now the cofactor of a_{ij}, call it A^{ij}, is $\boxed{(-1)^{i+j} \text{ times the minor}}$

Hence, the cofactors of the 3-minors above are

$$A^{11} = (-1)^{1+1}\begin{vmatrix} -2 & 3 \\ -4 & 6 \end{vmatrix}, \quad A^{12} = (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 4 & 6 \end{vmatrix}, \quad A^{13} = (-1)^{1+3}\begin{vmatrix} 0 & -2 \\ 4 & -4 \end{vmatrix}$$

→ over

The importance of cofactors is due to the following

$$|A| = \det A = \sum_{j=1}^{n} a_{ij} A^{ij} = \sum_{j=1}^{n} a_{ji} A^{ji}$$

Example: Using cofactors, find the determinant of A

where $A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Soln

expand by 1^{st} row

$$\det A = \begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 3(-2) - 2(-3) + 1(-2)$$

$$= -6 + 6 - 2 = \underline{\underline{-2}}$$

Expanding by 1^{st} column

$$\begin{vmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 3(-2) + 1(4-2) + 1(2)$$

$$= -6 + 2 + 2 = \underline{\underline{-2}}$$ as required

Cofactor Matrix

$$A^c = A^{ij} \text{ or } A_{ij}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \qquad A^c = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix}$$

where $A^{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ $\qquad A^{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$$A^{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad A^{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A^{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \qquad A^{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A^{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \qquad A^{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$A^{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$\boxed{\text{Adjoint of a Matrix.}}$ $\text{adj}\,A$, or $\tilde{A}$

defined as the "Cofactor matrix transposed"

$$\text{adj}\,A = \tilde{A} = A^{CT}$$

$$A^{CT} = \begin{pmatrix} A^{11} & A^{21} & A^{31} \\ A^{12} & A^{22} & A^{32} \\ A^{13} & A^{23} & A^{33} \end{pmatrix}$$

e.g. $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ $\quad A^{C} = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$

$$A^{CT} = \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix}$$

$\boxed{\text{Self-Adjoint Matrix}}$

if $\text{adj}\,A = A = \tilde{A}$, A is said to be self-adjoint.

e.g. $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad A^{c} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$A^{CT} = \tilde{A} = \text{adj}\,A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = A$$ as required

In general, we may write

$$A(BC) = (AB)C.$$

$$A(B+C) = AB+AC.$$

$$(B+C)A = BA+CA$$

Observe:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

here $AB = 0 \not\Rightarrow$ either A or B is zero

Also, observe

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Shows that although $AB = AC \not\Rightarrow B = C$.

even though $A \neq 0$

Also $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + BA + AB + B^2 \neq A^2 + 2AB + B^2$$

unless A & B commute

Symmetric Matrix

if $A^T = A \Rightarrow A$ is a symmetric matrix.

eg $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_1^T = \sigma_1 \Rightarrow \sigma_1$ is symmetric matrix

Anti-Symmetric Matrix

if $A^T = -A$, $\Rightarrow A$ is an anti-symmetric (skew) matrix

e.g

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_2^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\sigma_2$$

Hermitian Matrix

N.B. In Quantum Mechanics
<u>all</u> Physical <u>observables</u> are represented by Hermitian operators (matrices)

if $A^\dagger = A \Rightarrow A$ is said to be a Hermitian Matrix

e.g.

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_2^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\sigma^\dagger = \left(\sigma_2^*\right)^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$$

$\therefore \sigma_2^\dagger = \sigma_2 \Rightarrow \sigma_2$ is <u>Hermitian</u>

Unitary Matrix

if $AA^{+} = I$, $\Rightarrow$ A is said to be a unitary matrix.

e.g.

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_2^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

$$\sigma_2 \sigma_2^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Orthogonal Matrix

if $AA^{T} = I$, $\Rightarrow$ A is said to be an <u>orthogonal</u> matrix

eq $\quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_1^{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\sigma_1 \sigma_1^{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Note: An orthogonal matrix is a Real Unitary matrix.

Trace of matrix.

def $Tr(A) = \sum_k a_{kk}$

e.g. $A = \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}$ $Tr(A) = 2+7 = 9$

The inverse of a matrix — Square

$A^{-1} \Rightarrow AA^{-1} = A^{-1}A = I$

Definition $A^{-1} = \frac{1}{\det(A)} (\operatorname{adj} A) = \frac{\tilde{A}}{\det(A)}.$

$\det A \neq 0$ here —

Let's find out what this means thru an example

Given: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

find $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ such that.

$AA^{-1} = I$

This defines the problem.

Soln

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a\ b\ c\ d \text{ are known}$$

what does above imply - evidently,

$$aw + by = 1 \qquad ax + bz = 0$$

$$cw + dy = 0 \qquad cx + dz = 1$$

or solving

$$W = \frac{\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d}{ad-bc} \qquad \& \quad y = \frac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c}{ad-bc}.$$

Now look at pair

$$ax + bz = 0$$
$$cx + dz = 0$$

$$x = \frac{\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-b}{ad-bc} \qquad \& \quad z = \frac{\begin{vmatrix} a & 0 \\ c & 1 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{a}{ad-bc}.$$

Evidently,

$$A^{-1} = \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\[2ex] \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Now let us recognize the different parts above

what is $ad-bc$? $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

Now look at $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (A^c)^T = \text{adj}\, A = \tilde{A}$.

check

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ & $(A^c)^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

hence from this example (only), we may infer that

$$A^{-1} = \frac{1}{\det A}\, adj(A) = \frac{1}{\det A}\tilde{A}.$$

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ab \\ cd-cd. & ad-bc. \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example

Given $A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{pmatrix}$ find A^{-1}

Step ① $\det A = \begin{vmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{vmatrix}$

$$= 3(-22) - 2(-4) - 1(-1).$$

$$= -66 + 8 + 1$$

$$= -66 + 9 = -57 \neq 0 \quad A^{-1} \text{ exists.}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A = -\frac{1}{57}\begin{pmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{pmatrix}$$

Check

$$AA^{-1} = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{pmatrix}\begin{pmatrix} \frac{+22}{57} & \frac{1}{57} & \frac{-9}{57} \\ \frac{-4}{57} & \frac{5}{57} & \frac{12}{57} \\ \frac{1}{57} & \frac{13}{57} & \frac{-3}{57} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

right on.

$$\tilde{A} = \operatorname{adj} A = \begin{pmatrix} \begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -22 & 4 & -1 \\ -1 & -5 & -13. \\ 9 & -12 & 3 \end{pmatrix}^T$$

$$\therefore \operatorname{adj} = \tilde{A} = \begin{pmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{pmatrix}$$

Proposition: Any square matrix can be written as the sum of symmetric matrix and a (skew) antisymmetric matrix

Proof: $A = \left(\frac{A}{2} + \frac{A^T}{2}\right) + \left(\frac{A}{2} - \frac{A^T}{2}\right)$

where
$$A_s = \frac{A}{2} + \frac{A^T}{2} \qquad A_{anti} = \frac{A}{2} - \frac{A^T}{2}$$

$$\therefore \; A_s = A_s^T = \left(\frac{A^T}{2} + \frac{A}{2}\right)$$

$$A_{anti} = -A_{anti}^T = \frac{A^T}{2} - \frac{A}{2} = -\left(\frac{A}{2} - \frac{A^T}{2}\right)$$

Example $A = \begin{bmatrix} 2 & 0 & 3 \\ 2 & 0 & 2 \\ -3 & 4 & 2 \end{bmatrix}$ sum symm + anti sy

$$A^T = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & 4 \\ 3 & 2 & 2 \end{bmatrix}$$

$$A_s = \frac{1}{2}\left[A + A^T\right] = \frac{1}{2}\begin{pmatrix} 4 & 2 & 0 \\ 2 & 0 & 6 \\ 0 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{pmatrix}$$

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$$A_{anti} = \tfrac{1}{2}(A - A^T)$$

$$= \tfrac{1}{2}\begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -2 \\ -6 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -1 \\ -3 & 1 & 0 \end{pmatrix}$$

$$A = A_S + A_{anti.} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & -1 \\ -3 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 3 \\ 2 & 0 & 2 \\ -3 & 4 & 2 \end{pmatrix}$$

a) $(AB)^T = B^T A^T$ To be shown.

Proof $(AB)^T_{\cancel{ik}\,ki} = (AB)_{ik} = \sum A_{ij} B_{jk} = \sum_j B^T_{kj} A^T_{ji} = (B^T A^T)_{ki}$

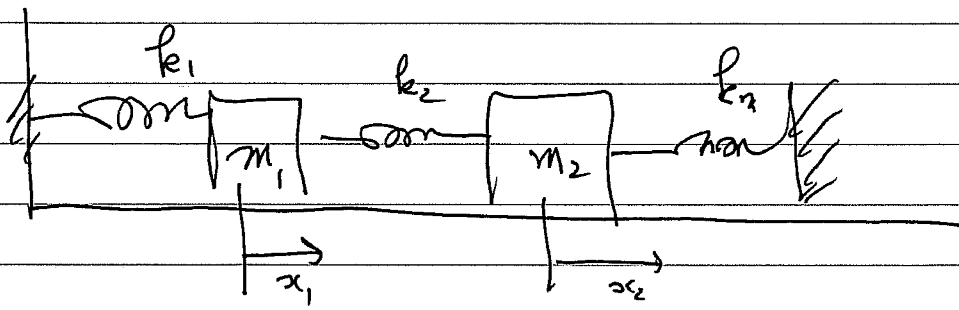
b) $(AB)^{-1} = B^{-1} A^{-1}$ Proof: $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}(I)B$

$= B^{-1}B = I$

c) $(AB)^+ = B^+ A^+$

The Coupled Harmonic Oscillator



$$\sum F_1 = m_1 \ddot{x}_1 = -k_1 x_1 - k_2 x_1 + k_2 x_2 \quad (1)$$

$$\sum F_2 = m_2 \ddot{x}_2 = -k_2 x_2 - k_3 x_2 + k_2 x_1 \quad (2)$$

set $k_1 = k_2 = k_3 = k \qquad m_1 = m_2 = m$

$$\sum F_1 = m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

$$\sum F_2 = m\ddot{x}_2 = -kx_2 + k(x_1 - x_2) = -2kx_2 + kx_1$$

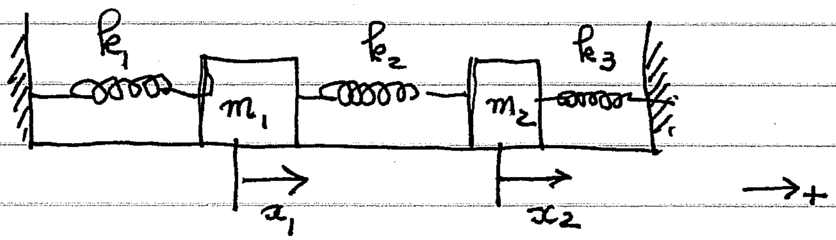
$$\ddot{x}_1 = -\frac{k}{m}x_1 + \frac{k}{m}(x_2 - x_1) = -2\frac{k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = -\frac{k}{m}x_2 + \frac{k}{m}(x_1 - x_2) = \frac{k}{m}x_1 - 2\frac{k}{m}x_2$$

Mathematical Physics -

Lecture, Friday, October 26, 2001

Coupled Oscillators



Look at m_1; hold m_2 fixed move m_1 to the right by x_1 & read off the forces; now allow m_2 to move by x_2

$$\sum F_1 = m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 - k_2x_1 + k_2x_2 \quad (1)$$

$$\sum F_2 = m_2 \frac{d^2x_2}{dt^2} = -k_2x_2 - k_3x_2 + k_2x_1 \quad (2)$$

Now write $\frac{d}{dt} = \cdot \Rightarrow \frac{dx_1}{dt} = \dot{x}_1 \quad \frac{d^2x_1}{dt^2} = \ddot{x}_1$

Now set $k_1 = k_2 = k_3 = k$ & $m_1 = m_2 = m$.

$$(1) \quad \sum F_1 = m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \qquad \frac{1}{m}$$

$$(2) \quad \sum F_2 = m\ddot{x}_2 = -kx_2 + k(x_1 - x_2) \qquad \frac{1}{m}$$

$$\boxed{\begin{aligned} \ddot{x}_1 &= -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \\ \ddot{x}_2 &= \frac{k}{m}x_1 - \frac{2k}{m}x_2 \end{aligned}}$$

we try $\left.\begin{aligned} x_1 &= A_1e^{i\omega t} \\ x_2 &= A_2e^{i\omega t} \end{aligned}\right\}$

we look for soln's in which both masses move with the same frequency.

$$\dot{x}_1 = i\omega A_1 e^{i\omega t}$$
$$\ddot{x}_1 = -\omega^2 A_1 e^{i\omega t} = -\omega^2 x_1$$

$$\ddot{x}_2 = -\omega^2 x_2$$

hence $-\omega^2 x_1 = -2\frac{k}{m} x_1 + \frac{k}{m} x_2$

$$-\omega^2 x_2 = \frac{k}{m} x_1 - 2\frac{k}{m} x_2$$

or we may write above as

$$\begin{pmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A\underline{X} = \lambda \underline{X} \qquad \lambda = -\omega^2$$

the above is called the "eigenvalue" problem

push − sign to lhs & write.

$$\begin{pmatrix} 2\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & 2\frac{k}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{where } \underline{\underline{\lambda = \omega^2}}$$

$$AX = \lambda X \quad \text{"eigenvalue problem"}$$

or $$\begin{bmatrix} \frac{2k}{m}-\lambda & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

∴ Homogeneous set of equations.

A soln' (non-trivial) exists iff

$$\det\begin{pmatrix} \frac{2k}{m}-\lambda & -\frac{k}{m} \\ -k/m & \frac{2k}{m}-\lambda \end{pmatrix} = 0$$

i.e $$\begin{vmatrix} \frac{2k}{m}-\lambda & -\frac{k}{m} \\ -k/m & \frac{2k}{m}-\lambda \end{vmatrix} = 0$$ "characteristic" determinant

evaluating $\left(\frac{2k}{m}-\lambda\right)^2-\left(\frac{k}{m}\right)^2=0$

characteristic equation for λ

or

$$\lambda^2-4\frac{k}{m}\lambda+3\left(\frac{k}{m}\right)^2=0$$

or $\left(\lambda-\frac{k}{m}\right)\left(\lambda-3\frac{k}{m}\right)=0$

or

$\lambda_1 = k/m$ $\quad\lambda_2=\frac{3k}{m}$ [typically ordered: λ_1 smallest

--- λ_n largest.

λ_1, λ_2 called "eigenvalues"

The Calculation of the eigenvectors

for $\lambda_1 = k/m$ return to homogeneous set of equations

$$\begin{pmatrix} \frac{2k}{m} - \lambda & -\frac{k}{m} \\ -k/m & 2k/m - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Now substitute $\lambda_1 = k/m$

$$\begin{pmatrix} \frac{2k}{m} - \frac{k}{m} & -k/m \\ -k/m & \frac{2k}{m} - k/m \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix} = 0.$$

or
$$\left\{ \begin{array}{l} \frac{k}{m} x_1^1 - \frac{k}{m} x_2^1 = 0 \\ -\frac{k}{m} x_1^1 + \frac{k}{m} x_2^1 = 0 . \end{array} \right.$$

these are the <u>same</u>!

$\Rightarrow \frac{k}{m}\left(x_1^1 - x_2^1\right) = 0 \Rightarrow x_1^1 = x_2^1$

set $x_1^1 = 1 \Rightarrow x_2^1 = 1$

$$\underline{X}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

normalize it

$$\boxed{\hat{\underline{X}}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

normalized eigenvector for $\lambda_1 = k/m$

Now substitute $\lambda_2 = 3\frac{k}{m}$ into the homogeneous equations

$$\begin{bmatrix} \frac{2k}{m} - \frac{3k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \frac{3k}{m} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{2} \end{bmatrix} = 0$$

or $-\frac{k}{m} x_1^2 - \frac{k}{m} x_2^2 = 0 \Rightarrow -\frac{k}{m}\left(x_1^2 + x_2^2\right) = 0 \Rightarrow x_1^{(2)} = -x_2^{(2)}$

set $x_1^2 = 1 \Rightarrow x_2^{(2)} = -1$.

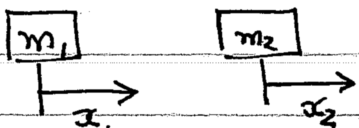
$$X^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boxed{\hat{X}^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ normalized eigenvector for } \lambda_2 = 3k/m}$$

Physically what are $\hat{\underline{X}}^{(1)}, \hat{\underline{X}}^{(2)}$

$\underline{X}^{(1)}$ $\quad \omega_1 = \pm\sqrt{\frac{k}{m}}$ $\quad$ since $\lambda = \omega^2$

$\omega_1 = \sqrt{\frac{k}{m}}$ "positive frequency"

means



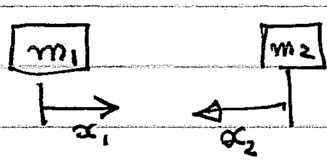
$\Rightarrow x_1, x_2$ are in phase.

Normal mode I $\quad$ low-frequency mode

they move in phase with frequency $\omega_1 = \sqrt{\frac{k}{m}}$.

— — — — —

$\underline{X}^{(2)} \Rightarrow \omega_2 = \sqrt{\frac{3k}{m}}$



$$\boxed{\text{Normal mode II} \quad \omega_2 = \sqrt{\frac{3k}{m}}}$$

the two masses move 180° out of phase with one another ~~at~~ with frequency $\omega_2 = \sqrt{\frac{3k}{m}}$ $\quad$ hi-frequency mode

Note: the following.

If at $t=0$, you start both masses, say in Normal mode I, it will continue to move in that mode forever.

Likewise, if you start up both masses in normal mode II it will continue to move in that mode - forever

We say that Mode I is independent of Mode II

$$\hat{X}^{(1)} \cdot \hat{X}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}} = 0$$

the eigenvectors are normal to each other

Now let us form a matrix from the normalized eigenvectors; call it M:

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Note: M is a symmetric matrix. and an orthogonal matrix

hence $M^{-1} = M^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$AM = \lambda M$$

$$\begin{pmatrix} \frac{2k}{m} & -k/m \\ -k/m & 2k/m \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2k}{m\sqrt{2}} - \frac{k}{m\sqrt{2}} & \frac{2k}{m\sqrt{2}} + \frac{k}{m\sqrt{2}} \\ \frac{-k}{m\sqrt{2}} + \frac{2k}{m\sqrt{2}} & \frac{-k}{m\sqrt{2}} - \frac{2k}{m\sqrt{2}} \end{pmatrix}$$

$$M^{-1} = M^{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad = \begin{pmatrix} \frac{k}{m\sqrt{2}} & \frac{+3k}{m\sqrt{2}} \\ \frac{k}{m\sqrt{2}} & \frac{-3k}{m\sqrt{2}} \end{pmatrix}$$

$$M^{-1}AM = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{k}{\sqrt{2}m} & \frac{3k}{m\sqrt{2}} \\ \frac{k}{m\sqrt{2}} & \frac{-3k}{m\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k}{2m} + \frac{k}{2m} & \frac{3k}{m2} - \frac{3k}{m2} \\ \frac{k}{2m} - \frac{k}{2m} & \frac{3k}{m2} + \frac{3k}{m2} \end{pmatrix}$$

$$= \begin{pmatrix} k/m & 0 \\ 0 & \frac{3k}{m} \end{pmatrix}$$

$$\boxed{M^{-1}AM = M^{-1}\lambda M = \lambda M^{-1}M = \lambda I}$$

Example.

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$ Note it is symmetric.

Obtain eigen values $AX = \lambda X$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda)^2 - 1\,(2-\lambda) = 0$$

or $(2-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

$$2-\lambda\,(4 - 4\lambda + \lambda^2 - 1) = 0$$

$$(2-\lambda)(3 - 4\lambda + \lambda^2) = 0$$

$$(2-\lambda)(1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

eigenvectors

$\lambda_1 = 1$

$$\begin{pmatrix} 2-1 & 0 & -1 \\ 0 & 2-1 & 0 \\ -1 & 0 & 2-1 \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{pmatrix} = 0$$

$x_1^1 + 0x_2^1 - x_3^1 = 0 \Rightarrow x_1^1 = x_3^1$ set $x_1^1 = 1 \Rightarrow x_1^3 = 1$

$$X^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \hat{X}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$

$$\begin{pmatrix} 2-2 & 0 & -1 \\ 0 & 2-2 & 0 \\ -1 & 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix} = 0.$$

$0x_1^2 + 0x_2^2 - x_3^2 = 0 \Rightarrow x_3^2 = 0$

$0x_1^2 + 0x_2^2 + 0x_2^3 = 0$

$\Rightarrow x_2^2$ arbitrary

$-x_1^2 + 0x_2^2 + 0x_2^3 = 0 \Rightarrow x_1^2 = 0$

choose $x_2^2 = 1$

$$X^{(2)} = \hat{X}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 3$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix} = 0 \Rightarrow -x_1^3 + 0x_2^3 - x_3^3 = 0$$

$$\Rightarrow x_1^3 = -x_3^3$$

$$0x_1^3 - x_2^3 + 0x_3^3 = 0 \Rightarrow x_2^3 = 0.$$

$$X^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \hat{X}^{(3)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{X}^{(1)} \cdot \hat{X}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\hat{X}^{(1)} \cdot \hat{X}^{(3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\hat{X}^{(2)} \cdot \hat{X}^{(3)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

orthogonal

form $M = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$ Symmetric

$\therefore \; M^{-1} = M^{T}$.

form $M^{-1} A M = \lambda I$.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Review of some properties of Hermitian, Symmetric, orthogonal and unitary matrices.

Symmetric

Given: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

if A is symmetric $A = A^T$

if A is skew (anti) symmetric $A^T = -A$.

orthogonal $A^T = A^{-1}$

Hermitian $(A^*)^T = A^+ = A$.

skew-Hermitian $(A^*)^T = A^+ = -A$.

Unitary $(A^*)^T = A^+ = A^{-1}$

Note: A symmetric matrix is a <u>real</u> Hermitian matrix.
An orthogonal matrix is a <u>real</u> unitary matrix

N.B.

A

Case: "Underdamped"

$x(0) = x_0$

$v(0) = v_0$

$$x = A e^{-\gamma t} \cos(\omega_1 t + \theta)$$

$$\dot{x} = -\gamma A e^{-\gamma t} \cos(\omega_1 t + \theta) - \omega_1 A e^{-\gamma t} \sin(\omega_1 t + \theta)$$

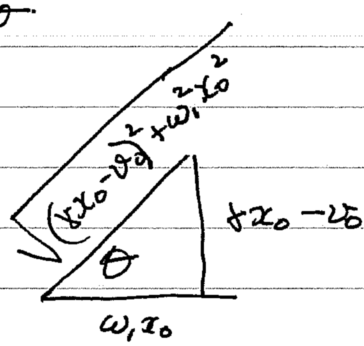
$$x_0 = A \cos\theta \quad \rightarrow \quad A = \frac{x_0}{\cos\theta}$$

$$v_0 = -\gamma A \cos\theta - \omega_1 A \sin\theta$$

$$v_0 = -\gamma \cancel{A} \frac{x_0 \cancel{\cos\theta}}{\cancel{\cos\theta}} - \omega_1 x_0 \frac{\sin\theta}{\cos\theta}$$

$$v_0 = -\gamma x_0 - \omega_1 x_0 \tan\theta$$

$$\boxed{\tan\theta = \frac{\gamma x_0 - v_0}{\omega_1 x_0}}$$



$$\sin\theta = \frac{\gamma x_0 - v_0}{\sqrt{(\gamma x_0 - v_0)^2 + (\omega x_0)^2}}$$

$$\cos\theta = \frac{\omega_1 x_0}{\sqrt{(\gamma x_0 - v_0)^2 + \omega_1^2 x_0^2}}$$

$$\therefore \boxed{A = \frac{x_0 \sqrt{(\gamma x_0 - v_0)^2 + \omega_1^2 x_0^2}}{\omega x_0}}$$

Case "over damped

$t=0 \quad x(0)=x_0$

$v(0)=v_0$

$$x(t)=Ae^{-\gamma_1 t}+Be^{-\gamma_2 t}$$

$$v(t)=-\gamma_1 A e^{-\gamma_1 t}-\gamma_2 B e^{-\gamma_2 t}.$$

$x_0 = A+B.$ or $A+B = x_0$

$v_0 = -\gamma_1 A - \gamma_2 B$ $\quad \gamma_1 A + \gamma_2 B = -v_0$

$$A = \frac{\begin{vmatrix} x_0 & 1 \\ -v_0 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{vmatrix}} = \frac{\gamma_2 x_0 + v_0}{\gamma_2 - \gamma_1}$$

$$B = \frac{\begin{vmatrix} 1 & x_0 \\ \gamma_1 & -v_0 \end{vmatrix}}{\gamma_2 - \gamma_1} = \frac{-v_0 - x_0\gamma_1}{\gamma_2 - \gamma_1}$$

$\therefore$

$$\boxed{x(t) = \frac{\gamma_2 x_0 + v_0}{\gamma_2 - \gamma_1} e^{-\gamma_1 t} - \frac{(v_0 + x_0\gamma_1)}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}}$$

Case "Critical Damping"

$$x(t) = (A + Bt)e^{-\gamma t}$$

$$v(t) = -\gamma(A+Bt)e^{-\gamma t} + Be^{-\gamma t}$$

$$\begin{cases} x_0 = A \\ v_0 = -\gamma A + B. \end{cases}$$

or $v_0 = -\gamma x_0 + B \Rightarrow B = (\gamma x_0 + v_0)$

$$x(t) = (x_0 + (\gamma x_0 + v_0)t)e^{-\gamma t}$$