Mathematical Physics –

Friday, November 23, 2001 $\qquad [-L, L] \qquad T = 2L.$

Complex form of Fourier Series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$= a_0/2 + \sum_{n=1}^{\infty} a_n\left\{\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2}\right\} + b_n\left\{\frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i}\right\}$$

$$= a_0/2 + \sum_{n=1}^{\infty}\left(\frac{a_n - b_n i}{2}\right)e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty}\frac{(a_n + ib_n)}{2}e^{-\frac{in\pi x}{L}}$$

define $c_0 = a_0/2$

$$c_n = \tfrac{1}{2}(a_n - ib_n)$$

$$c_{-n} = \tfrac{1}{2}(a_n + ib_n)$$

Claim: we may write $\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}}$ (I)

$$f(x) = \cancel{a_0/2} + \sum_{n=0}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{n=-1}^{-\infty} c_{-n} e^{-\frac{in\pi x}{L}}$$

$n=0$

how do we get c_n ?

multiply (I) by $e^{\frac{-im\pi x}{L}}$ and integrate from $[-L, L]$

$$\int_{-L}^{L} f(x) e^{\frac{-im\pi x}{L}} dx = \sum_{n=-\infty}^{\infty} C_n \int_{-L}^{L} e^{\frac{ix(n-m)\pi}{L}} dx.$$

Now let us look at $\int_{-L}^{L} e^{\frac{ix(n-m)\pi}{L}} dx.$

$$= \left. \frac{e^{\frac{ix(n-m)\pi}{L}}}{\frac{i\pi(n-m)}{L}} \right|_{-L}^{L}$$

$$= \frac{L}{i\pi(n-m)} \left[e^{i(n-m)\pi} - e^{-i(n-m)\pi} \right]$$

$$= \frac{L}{\pi(n-m)} 2 \left[\frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{2i} \right]$$

$$= \frac{2L}{\pi(n-m)} \sin\pi(n-m) = 2L \frac{\sin\pi(n-m)}{\pi(n-m)}$$

above is 0 if $n \neq m$.
1 if $n = m$.

$$\therefore \int_{-L}^{L} e^{\frac{ix(n-m)\pi}{L}} dx = 2L\delta_{mn}$$

$$\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \right) = 1$$

$$\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}{\theta}$$

(2-a)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i n\pi x/l}$$

$n = 0$

$$= \sum_{n=-\infty}^{\infty} c_0 = \sum_{n=-\infty}^{\infty} = \tfrac{1}{2} a_0$$

C $\quad n = 2.$

$$c_2 = \tfrac{1}{2}(a_2 - i b_2) e^{i\frac{2\pi x}{l}}$$

$$= \tfrac{1}{2}(a_2 - i b_2) \cos\frac{2\pi x}{L} + i \sin\frac{2\pi x}{l}$$

$$= \tfrac{1}{2}\left[a_2 \cos\frac{2\pi x}{L} + i a_2 \sin\frac{2\pi x}{l} - i b_2 \cos\frac{2\pi x}{L} + b_2 \sin\frac{2\pi x}{l}\right]$$

$$c_{-2} = \tfrac{1}{2}(a_2 + i b_2)\left(\cos\frac{2\pi x}{L} - i \sin\frac{2\pi x}{l}\right)$$

$$= \tfrac{1}{2}\left[a_2 \cos\frac{2\pi x}{L} + i b_2 \cos\frac{2\pi x}{L} - i a_2 \sin\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right]$$

$$C_2 + C_{-2} = \tfrac{1}{2}\left[a_2 \cos\frac{2\pi x}{L} + \cancel{i a_2 \sin\frac{2\pi x}{L}} - \cancel{i b_2 \cos\frac{2\pi x}{L}} + b_2 \sin\frac{2\pi x}{L}\right]$$

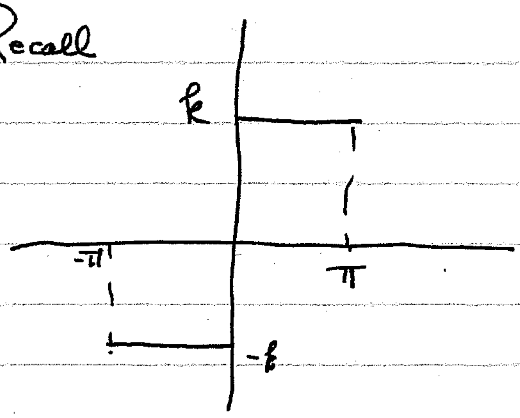
$$+ \tfrac{1}{2}\left[a_2 \cos\frac{2\pi x}{L} + \cancel{i b_2 \cos\frac{2\pi x}{L}} - \cancel{i a_2 \sin\frac{2\pi x}{l}} + b_2 \sin\frac{2\pi x}{L}\right]$$

$$= a_2 \cos\frac{2\pi x}{L} + b_2 \sin\frac{2\pi x}{L}$$

hence $$\int_{-L}^{L} f(x)\, e^{-i\frac{m\pi x}{L}}\, dx = C_m\, \delta_{mn}\, 2L$$

or $$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx}$$

Recall



$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

we obtained:

$$f(x) = \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{[1-\cos n\pi]}{n} \sin nx.$$

Let us try to expand the above fn in an exponential series.

$[-\pi, \pi]$, $T = 2\pi$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$\text{or } c_n = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k e^{-inx}\, dx + \int_{0}^{\pi} k e^{-inx}\, dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[-k \frac{e^{-inx}}{-in}\Big|_{-\pi}^{0} + k\frac{e^{-inx}}{-in}\Big|_{0}^{\pi}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{-k\left[\frac{e^{0}-e^{in\pi}}{-in}\right] + k\left[\frac{e^{-in\pi}-e^{0}}{-in}\right]\right\}$$

$$= \frac{1}{2\pi}\left[\frac{1}{-in}\left[-k\left(1-e^{i\pi n}\right) + k\left(e^{-i\pi n}-1\right)\right]\right]$$

$$= \frac{1}{2\pi(-in)}\left[-k + ke^{i\pi n} + ke^{-i\pi n} - k\right]$$

$$= \frac{1}{2\pi(-in)}\left[-2k + 2k\cos n\pi\right]$$

$$= \frac{2k}{2\pi(in)}\left[1 - \cos n\pi\right]$$

or finally $\boxed{C_n = \frac{k}{\pi i n}\left[1 - \cos n\pi\right]}$

$$f(x) = \frac{k}{\pi i}\sum_{n=-\infty}^{\infty}\left(\frac{1-\cos n\pi}{n}\right)e^{inx}.$$

Check out this result!

$a_0 = 2C_0 =$

$$C_n = \tfrac{1}{2}(a_n - i b_n)$$

$$C_{-n} = \tfrac{1}{2}(a_n + i b_n)$$

$$+ \quad \boxed{C_n + C_{-n} = a_n}$$

hence $a_n = \frac{k}{\pi i n}\left[1 - \cos n\pi\right] + \frac{k}{\pi i (-)n}\left[1 - \cos(-n\pi)\right] = 0$

Also $C_n = \tfrac{1}{2}(a_n - i b_n)$

$$C_{-n} = \tfrac{1}{2}(a_n + i b_n)$$

$$-$$

$$C_n - C_{-n} = -i b_n$$

or $\boxed{b_n = -\frac{(C_n - C_{-n})}{i} = i(C_n - C_{-n})}$

$$b_n = i\,(c_n - c_{-n}).$$

$$= i\left[\left(\frac{k}{\pi i n}[1-\cos n\pi] - \frac{k}{-\pi i n}[1-\cos(-n\pi)\right)\right]$$

$$= i\left[\frac{2k}{\pi i n}\{1-\cos n\pi\}\right]$$

$$= \frac{2k}{\pi}\left\{\frac{1-\cos n\pi}{n}\right\}$$

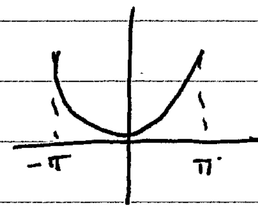
& hence, we again conclude

$$f(x \; \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{[1-\cos n\pi]}{n} \sin nx.$$

right on the button

Example. on $[-\pi, \pi]$ $T = 2\pi$

$$f(x) = x^2 \quad \text{even fn.}$$



$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\frac{x^3}{3}\Bigg|_0^{\pi} = 2/3\,\frac{\pi^3}{\pi} = 2/3\,\pi^2$$

even fn

$$\therefore \boxed{a_0 = 2/3\,\pi^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2/\pi \int_0^{\pi} x^2 \cos nx\,dx$$

even fn S.I.

S.I. $$\int x^2 \cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

Hence $$a_n = 2/\pi \left[\frac{2x}{n^2}\cos nx + \left[\frac{x^2}{n} - \frac{2}{n^3}\right]\sin nx \right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] = \frac{4\pi}{\pi n^2}\cos n\pi$$

$$= \frac{4}{n^2}\cos n\pi$$

$$\therefore \boxed{a_n = \frac{4}{n^2}\cos n\pi}$$

finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0$
odd integrand

$$\therefore f(x) = \tfrac{2}{3}\frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4\cos n\pi}{n^2}\cos nx.$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}\cos nx.$$

$$= \frac{\pi^2}{3} + 4\left\{\frac{(-1)^1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{\cos 3x}{3^2} + \cdots\right\}$$

$\boxed{\text{Try Complex expansion}}$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

where $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$

$$\therefore C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{x^2 e^{-inx}}_{\text{S.I}}\, dx.$$

$$\text{S.I} \int x^2 e^{ax}\, dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$$

$a = -in$

$$\therefore C_n = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\left[x^2 - \frac{2x}{-in} + \frac{2}{(-in)^2}\right]\right]_{-\pi}^{\pi}$$

$$C_n = \frac{1}{2\pi}\left\{ \frac{e^{-in\pi}}{-in}\left[\pi^2 - \frac{2\pi}{-in} + \frac{2}{(-in)^2}\right] - \frac{e^{in\pi}}{-in}\left[\pi^2 - \frac{2(-\pi)}{-in} + \frac{2}{(-in)^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{ e^{-in\pi}\left[\frac{\pi^2}{-in} - \frac{2\pi}{(-in)^2} + \frac{2}{(-in)^3}\right] - e^{in\pi}\left[\frac{\pi^2}{-in} + \frac{2\pi}{(-in)^2} + \frac{2}{(-in)^3}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{-2\pi}{(-in)^2} e^{-in\pi} - \frac{2\pi}{(-in)^2} e^{in\pi} + e^{-in\pi}\left[\frac{\pi^2}{-in} + \frac{2}{(-in)^3}\right] - e^{in\pi}\left[\frac{\pi^2}{-in} + \frac{2}{(-in)^3}\right]\right.$$

$$= \frac{1}{2\pi}\left\{ \frac{-2\pi}{(-in)^2}\left[e^{in\pi} + e^{-in\pi}\right] + \left[\frac{\pi^2}{-in} + \frac{2}{(-in)^3}\right]\left[-\left[e^{in\pi} - e^{-in\pi}\right]\right]\right.$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-2\pi}{(-in)^2}(2)\cos n\pi\right] - \left[\frac{\pi^2}{-in} + \frac{2}{(-in)^3}\right]\frac{2i}{2i}\left[e^{in\pi} - e^{-in\pi}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{-4\pi\cos n\pi}{(-in)^2}\right] - 2i\,\underbrace{\left[\frac{\pi^2}{(-in)} + \frac{2}{(-in)^3}\right]\sin n\pi}_{\to 0}\right\}$$

$$= \frac{-4\pi}{2\pi}\,\frac{\cos n\pi}{-n^2}$$

$$C_n = \frac{2\cos n\pi}{n^2}$$

$$\therefore\quad \boxed{f(x) = x^2 = \sum_{n=-\infty}^{\infty} \frac{2\cos n\pi}{n^2} e^{inx}.}$$

$n \neq 0$

check $\quad a_0 = 2c_0$

$$C_n = \frac{2\cos n\pi}{n^2}$$

$$C_n = \frac{2}{n^2}\left(1 - \frac{(n\pi)^2}{2!} + \frac{(n\pi)^4}{4!} - \cdots\right)$$

$$= \frac{2}{n^2} - \frac{2}{2!\,n^2} n^2\pi^2$$

$$c_n = \frac{2}{n^2}\left(1 - \frac{n^2\pi^2}{2}\right)$$

Now we must investigate C_0 separately.

$$C_n = \frac{1}{2\pi}\left[\frac{-4\pi\cos n\pi}{(-in)^2} - 2i\left[\frac{\pi^2}{(-in)} + \frac{2}{(-in)^3}\right]\sin n\pi\right.$$

$$2\pi C_n = \frac{-4\pi\cos n\pi}{-n^2} - 2i\left[\frac{\pi^2}{-in} + \frac{2}{+in^3}\right]\sin n\pi .$$

$$2\pi C_n = \frac{4\pi\cos n\pi}{n^2} + \frac{2\pi^2\sin n\pi}{n} - \frac{4}{n^3}\sin n\pi$$

l'Hopital Rule L'Hôpital's Rule

$$2\pi C_n = \frac{4\pi\frac{d}{dn}(\cos n\pi)}{\frac{d}{dn}(n^2)} + 2\pi^3\frac{\sin n\pi}{n\pi} - \frac{4}{n^3}\sin n\pi$$

$$= \frac{-4\pi^2\sin n\pi}{2n} + 2\pi^3\frac{\sin n\pi}{n\pi} - \frac{4}{n^3}\sin n\pi .$$

$$= \frac{-4\pi^3}{2}\frac{\sin n\pi}{n\pi} + 2\pi^3\frac{\sin n\pi}{n\pi} - \frac{4\frac{d}{dn}\sin n\pi}{\frac{d}{dn}n^3} .$$

cancel

$$= \frac{-4\pi\cos n\pi}{3n^2}$$

$$2\pi C_n = -\frac{4}{3}\pi\frac{\frac{d}{dn}\cos n\pi}{\frac{d}{d3}n^2} = -\frac{4}{3}\pi\frac{(-\pi)\sin n\pi}{2n}$$

$$= \frac{4}{3}\pi^2\frac{\sin n\pi}{2n}$$

$$2\pi C_n = \left(\frac{4}{3}\pi^3\right)\frac{1}{2}\frac{\sin n\pi}{n\pi}$$

$\therefore$ for $n=0$

$$2\pi C_0 = \frac{2\pi^3}{3} \Big/ \lim \left(\frac{\sin n\pi}{n\pi}\right) \quad 1$$

$$C_0 = \frac{2\pi^3}{3\,2\pi} = \frac{\pi^2}{3}.$$

& $a_0 = 2C_0 = \dfrac{2\pi^2}{3}$

& $$f(x) = x^2 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2\cos n\pi}{n^2} e^{inx}$$

$$C_n = \frac{2\cos n\pi}{n^2}$$

&

$$C_{-n} = \frac{2\cos n\pi}{n^2}$$

$$a_n = C_n + C_{-n} = \frac{2\cos n\pi}{n^2} + \frac{2\cos n\pi}{n^2}.$$

$$b_n = i(C_n - C_{-n})$$

$$a_n = \frac{4\cos n\pi}{n^2}$$

$$b_n = i(C_n - C_{-n}) = i\left(\frac{2\cos n\pi}{n^2} - \frac{2\cos n\pi}{n^2}\right) = 0$$

$\therefore$

finally $a_0 = 2C_0 \Rightarrow a_0/2 = C_0.$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos n\pi}{n^2} \cos nx. \qquad \text{Same}$$

if $C_n = \dfrac{2\cos n\pi}{n^2}$

$C_0 = \dfrac{2(1)}{0^2}$???

Recall

$$2\pi C_n = \left[\frac{-4\pi\cos n\pi}{(-in)^2}\right] - 2i\left[\frac{\pi^2}{(-in)} + \frac{2}{(-in)^3}\right]\sin n\pi$$

$$= \frac{4\pi\cos n\pi}{n^2} + \left[\frac{2\pi^2}{n} \quad \frac{-4i}{(-)(i)n^3}\right]\sin n\pi.$$

$$2\pi C_n = \frac{4\pi\cos n\pi}{n^2} + \frac{2\pi^2\sin n\pi}{n} - \frac{4\sin n\pi}{n^3}$$

L'Hopital Rule

$$2\pi C_n = 4\pi\frac{\frac{d}{dn}\cos n\pi}{\frac{d}{dn}(n^2)} + 2\pi^2\frac{\sin n\pi}{n} - 4\frac{\frac{d}{dn}\sin n\pi}{\frac{d}{dn}n^3}$$

$$= -4\pi\frac{2\,\cancel{\sin n\pi}}{2\,\cancel{n}} + \cancel{2\pi^2\frac{\sin n\pi}{n}} - 4\frac{\frac{d}{dn}(\sin n\pi)}{\frac{d}{dn}n^3}$$

$$= -4\frac{\pi\cos n\pi}{3n^2}$$

$$= -4\pi\frac{\frac{d}{dn}(\cos n\pi)}{3\frac{d}{dn}n^2} = \frac{+4\pi^2\sin n\pi}{(3)(2)n}$$

$$= \frac{4\pi^3}{(3)(2)}\frac{\sin n\pi}{n\pi} \quad n=0$$

1

$$2\pi C_n = \frac{2\pi^3}{3}$$

$$\boxed{C_n = \frac{\pi^2}{3} = a_0/2}$$

Mathematical Physics

Lecture, Wednesday, November 28, 2001

Finish up extensions – even/odd –

Parseval's Identity

Assume that the Fourier Series corresponding to $f(x)$ converges (a la Dirichlet) on $[-L, L]$

Problem: Show

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Soln:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

multiply by $f(x)$ & integrate term by term from $-L \to L$.

$$\int_{-L}^{L}\{f(x)\}^2 dx = a_0/2\int_{-L}^{L} f(x)dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\right\}$$

but $\int_{-L}^{L} f(x)dx = La_0$; $\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n L$

& $\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = b_n L$

hence, we must conclude that,

$$\int_{-L}^{L} \{f(x)\}^2 dx = \frac{a_0^2 L}{2} + \sum_{n=1}^{\infty} a_n^2 L + b_n^2 L.$$

or

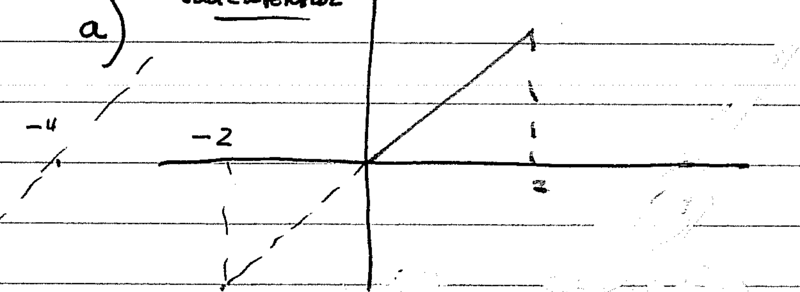
$$\boxed{\frac{1}{L}\int_{-\infty}^{\infty} \{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$ Parseval's Identity.

Some half-range extensions

Spiegel 2.12 pg 32

Expand $f(x) = x$, on $0 < x < 2$ in a half-range
a) sine series b) cosine series.

a) odd extension



$\boxed{\begin{array}{c} L = 2 \\ 2L = 4 \end{array}}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = 0$$
odd fn.

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x \cos\frac{n\pi x}{L}}_{\text{odd fn}} dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L} dx = \frac{1}{2}\int_{-2}^{2} x \sin\frac{n\pi x}{2} dx = \frac{2}{2}\int_{0}^{2} x \sin\frac{n\pi x}{2} dx.$$

hence $b_n = \displaystyle\int_0^2 x \sin\frac{n\pi x}{2}\,dx$

S.I.

One finds

$$\int x\sin ax\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

$$\therefore\ b_n = \int_0^2 x\sin\frac{n\pi}{2}x\,dx = \left[\frac{\sin\frac{n\pi}{2}x}{\left(\frac{n\pi}{2}\right)^2} - \frac{x\cos\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]\Bigg|_0^2$$

$$= \left[x\left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]\Bigg|_0^2$$

$$= 2\left(\frac{-2}{n\pi}\right)\cos n\pi + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - 0$$

$$= \frac{-4}{n\pi}\cos n\pi$$

$$\therefore\ \boxed{f(x) = \sum_{n=1}^{\infty}\Big(\frac{-4}{n\pi}\underbrace{\cos n\pi}_{}\Big)\sin\frac{n\pi x}{2}}$$

$(-1)^{n+1}$

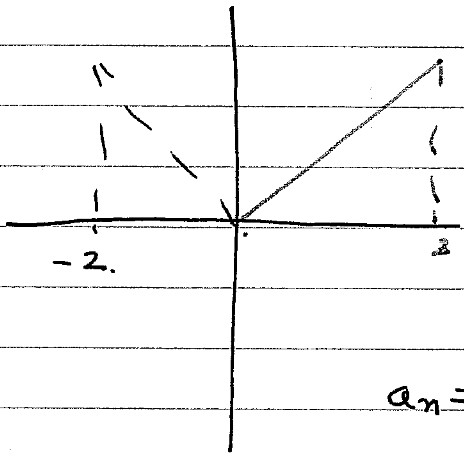
odd-extension or sine expansion

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

$\boxed{\text{even extension}}$

4

b)



$L = 2.$

$$a_0 = \frac{2}{2}\int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = 2.$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

in our case

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx.$$

S.I

$$\boxed{\text{S.I}\int x\cos ax\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}}$$

Hence

$$a_n = \frac{2}{2}\left[\frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2} + \frac{x\sin\frac{n\pi}{2}x}{\frac{n\pi}{2}}\right]_0^2 \qquad \underline{\underline{n \neq 0}}$$

$$= \left[\frac{\cos n\pi}{\left(\frac{n\pi}{2}\right)^2} + 2\,\frac{\sin n\pi}{n\pi/2} - \frac{1}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \left[\frac{4}{n^2\pi^2}\left(\cos n\pi - 1\right)\right] \qquad n \neq 0$$

Now look at $n=0$ $\quad a_0 = \frac{2}{2}\int_0^2 x\,dx = 2$

So $f(x) = a_0/2 + \sum_{n=1} a_n \cos\frac{n\pi x}{2}$

$$= 1 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left(\cos n\pi - 1\right) \cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

Now do Parseval's Identity.

$\boxed{\text{Parseval's Identity}}$ Problem

Recall even extension of $f(x) = x \quad 0 < x < 2.$

we obtained .

$$f(x) = 1 + \sum \frac{4}{\pi^2 n^2}\left(\cos n\pi - 1\right) \cos\frac{n\pi x}{2}$$

Write Parseval's Identity for above expansion.

$L = 2;\quad a_0 = 2;\ a_n = \frac{4}{\pi^2 n^2}\left(\cos n\pi - 1\right) \quad n \neq 0,\ b_n = 0$

$$\therefore \frac{1}{2}\int_{-2}^{2}\{f(x)\}^2 dx = \frac{1}{2}\int_{-2}^{2} x^2 dx = \frac{(2)^2}{2} + \sum_{n=1}^{\infty}\frac{16}{\pi^4 n^4}\left(\cos n\pi - 1\right)^2$$

or

Parseval's Identity Spiegel pg 33.

Recall <u>even</u> extension of $f(x) = x$ on $0 < x < 2$

we obtained.

Problem:

$$f(x) = 1 + \sum_{n=1} \frac{4}{\pi^2 n^2}\left(\cos n\pi - 1\right)\cos\frac{n\pi x}{2}$$

a) Write Parseval's identity for the above expansion

$L = 2 \quad a_0 = 2; \quad a_n = \frac{4}{\pi^2}\left(\frac{\cos n\pi - 1}{n^2}\right), \; n \neq 0; \quad b_n = 0$

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2 dx = \frac{1}{2}\int_{-2}^{2} x^2 dx = \frac{(2)^2}{2} + \sum_{n=1}^{\infty}\frac{16}{\pi^4 n^4}\left(\cos n\pi - 1\right)^2$$

$\cancel{(\cos n\pi - 1)^2 = \cos^2 n\pi - 2\cos n\pi + 1}$

$$\frac{1}{2}\left.\frac{x^3}{3}\right|_{-2}^{2} = \frac{16}{(2)(3)} = \frac{8}{3} = 2 + \frac{16}{\pi^4}\left[\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \cdots\right]$$

$$\text{or } \frac{8}{3} = 2 + \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$$

$\frac{8}{3} = \frac{2\pi^4 + [illegible]}{\pi^2}$

$$8/3 - \frac{6}{3} = \frac{64}{\pi^4}\left[\quad\right]$$

$$\left(\frac{2}{3}\right)\frac{\pi^4}{64} = \left[\quad\right] \Rightarrow \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

b) Determine from above result the Sum S

of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \frac{1}{n^4}$

Let S = Sum

So Sum $= \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right).$

S

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right)$$

$$= \frac{\pi^4}{96} + \frac{S}{16}$$

or $S - \frac{S}{16} = \frac{\pi^4}{96}$

$$S\left(1 - \frac{1}{16}\right) = \frac{\pi^4}{96}$$

$$S\left(\frac{15}{16}\right) = \frac{\pi^4}{96} \Rightarrow \boxed{S = \left(\frac{\pi^4}{96}\right)\left(\frac{16}{15}\right) = \frac{\pi^4}{90}}$$

Problem: Obtain a) cosine expansion b) sin expansion

for $f(x) = x(\pi - x)$ on $0 < x < \pi$

Cosine expansion

a) $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$

$= \frac{2}{\pi}\int_0^\pi x(\pi - x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\,dx$

$= \frac{2}{\pi}\left(\frac{\pi x^2}{2} - \frac{x^3}{3}\right)\Big|_0^\pi = \frac{2}{\pi}\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right)$

$= \frac{2}{\pi}\left(\frac{\pi^3}{6}\right) = \frac{\pi^2}{3}$

$\therefore \boxed{a_0 = \frac{\pi^2}{3}}$

$a_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^\pi (\pi x\cos nx - x^2\cos nx)\,dx.$

$\int x^2\cos ax = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\pi x\cos nx\,dx}_{\text{odd } fn} - \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx$ SI

$= -\frac{2}{\pi}\left[\frac{2x\cos nx}{n^2} + \left(\frac{x^2}{n} - \frac{2}{n^3}\right)\underset{0}{\sin nx}\right]\Bigg|_0^\pi$

$a_n = \frac{-4}{\pi n^2}\pi\cos n\pi = \frac{-4\cos n\pi}{n^2}$

$\therefore f(x) = x(\pi - x) = \frac{\pi^2}{6} - 4\sum_1 \frac{\cos n\pi}{n^2}\cos nx$

No!

$$a_n = \frac{2}{\pi}\left[\int_0^\pi \left[\pi x \cos nx - x^2 \cos nx\right] dx\right]$$

$$= \frac{2}{\pi}\left\{\pi \frac{\cos nx}{n^2} + \underbrace{x\frac{\sin nx}{n}}_{=0} - \left[\frac{2x}{n^2}\cos nx + \underbrace{\left(\frac{x^2}{n} - \frac{2}{n^3}\right)\sin nx}_{=0}\right]\right\}\Bigg|_0^\pi$$

$$= \frac{2}{\pi}\left\{\pi\left(\frac{\cos n\pi - 1}{n^2}\right\} - \frac{\cancel{2\pi}}{n^2}\left[2\pi \cos n\pi\right.\right.$$

$$= \frac{2}{\pi}\left\{\frac{\pi \cos n\cancel{\pi} - 2\cancel{\pi}\cos n\pi - \pi}{n^2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi\cos n\pi - \pi}{n^2}\right\} = 2\left\{\frac{-\cos n\pi - 1}{n^2}\right\} = -2\left\{\frac{\cos n\pi + 1}{n^2}\right\}$$

$\therefore$

$$f(x) = \cancel{\pi(x-}$$

$$= x(\pi - x) = \frac{\pi^2}{6} - 2\sum_{n=1}\left\{\frac{\cos n\pi + 1}{n^2}\right\}\cos nx$$

$$= \frac{\pi^2}{6} - 2\left\{\frac{2}{2^2}\cos 2x + \frac{2}{4^2}\cos 4x + \frac{2}{6^2}\cos 6x + \cdots\right\}$$

$$= \frac{\pi^2}{6} - 4\left\{\frac{\cos 2x}{4} + \frac{\cos 4x}{16} + \frac{\cos 6x}{36}\right.$$

$$= \frac{\pi^2}{6} - \frac{4}{4}\left\{\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right.$$

Let us now obtain the odd extension, i.e. sine series representation

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\Big(\underbrace{\pi x\sin nx}_{S.I} - \underbrace{x^2\sin nx}_{S.I}\Big)dx$$

$$= \frac{2}{\pi}\left[\pi\left\{\frac{\sin nx}{n^2} - \frac{x\cos nx}{n}\right\} - \left\{\frac{2x}{n^2}\sin nx + \left(\frac{2}{n^3} - \frac{x^2}{n}\right)\cos nx\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\left\{0 - \frac{\pi\cos n\pi}{n}\right\} - \left\{0 + \left(\frac{2}{n^3}\cos \pi n - \frac{\pi^2}{n}\cos n\pi - \left(\frac{2}{n^3}\right)\right)\right\}\right]$$

$$= \frac{2}{\pi}\left[\cancel{-\frac{\pi^2\cos n\pi}{n}} - \frac{2\cos n\pi}{n^3} + \cancel{\frac{\pi^2\cos n\pi}{n}} + \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}\left[1 - \cos n\pi\right]\right]$$

$$\boxed{b_n = \frac{4}{\pi n^3}\left[1 - \cos n\pi\right]}$$

right on the button

Hence we must conclude

$$f(x) = x(\pi - x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(1-\cos n\pi)}{n^3} \sin nx.$$

or

$$f(x) = x(\pi - x) = \frac{4}{\pi}\left\{ \frac{2}{1^3}\sin x + \frac{2\sin 3x}{3^3} + \frac{2\sin 5x}{5^3} + \cdots \right\}$$

$$= \frac{8}{\pi}\left\{ \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots \right\}$$

as required

Problem: By using the soln's to the above problem & Parseval's Identity show that

a) $\sum_{n=1} \frac{1}{n^4} = \frac{\pi^4}{90}$

b) $\sum_{n=1} \frac{1}{n^6} = \frac{\pi^6}{945}$

Soln:

We have shown that on $0<x<\pi$ (even extension)

of $f(x) = x(\pi - x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{\cos n\pi + 1}{n^2}\right]\cos nx$

Parseval Conditions

$L = \pi, \quad a_0 = \frac{\pi^2}{3} \qquad a_n = -2\frac{(\cos n\pi + 1)}{n^2}.$

$$\therefore \frac{2}{\pi}\int_0^{\pi}[x(\pi-x)]^2\,dx = \frac{\left(\frac{\pi^2}{3}\right)^2}{2} + \sum_{n=1}^{\infty} 4\frac{(\cos n\pi + 1)^2}{n^4}$$

$$= \frac{\pi^4}{18} + 4\left[\frac{2^2}{2^4} + \frac{2^2}{4^4} + \frac{2^2}{6^4} + \frac{2^2}{8^4} + \cdots\right]$$

$$= \frac{\pi^4}{18} + \frac{16}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \frac{1}{n^4}\cdots\right]$$

~~Now look at~~

$$\cancel{\frac{2}{\pi}\int_0^{\pi}[x(\pi-x)]^2dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}}$$

$$\cancel{= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] = \pi^2}$$

No!

$$\frac{2}{\pi}\int_0^{\pi}[x(\pi-x)]^2dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)^2dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2x^2 - 2\pi x^3 + x^4)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi^2x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^5}{3} - \frac{2\pi^5}{4} + \frac{\pi^5}{5}\right] = \frac{2}{\pi}\left[\frac{20\pi^5 - 30\pi^5 + 12\pi^5}{60}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi^5}{60}\right] = \frac{2}{\pi}\left[\frac{\pi^5}{30}\right]$$

or $\dfrac{\pi^4}{15} = \dfrac{\pi^4}{18} + \dfrac{16}{2^4}\left[\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots \dfrac{1}{n^4}\right]$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} = \left[\sum \frac{1}{n^4}\right]$$

$$\frac{18\pi^4 - 15\pi^4}{(15)(18)} = \sum \frac{1}{n^4}$$

$$\left[\frac{3\pi^4}{(15)(18)}\right] = \sum \frac{1}{n^4}$$

or $\dfrac{\pi^4}{(5)(18)} = \sum \dfrac{1}{n^4}$

or $\boxed{\dfrac{\pi^4}{90} = \sum \dfrac{1}{n^4}}$ as required.

b) from the odd extension of $f(x) = x(\pi - x)$

$$f(x) = x(\pi - x) = \frac{4}{\pi} \sum_{n=1} \frac{(1-\cos n\pi)}{n^3} \sin nx.$$

+ Parseval's Identity we have

$L = \pi \quad a_0 = 0 \quad b_n = \dfrac{4}{\pi} \dfrac{(1-\cos n\pi)}{n^3}$

$$\therefore \frac{2}{\pi}\int_0^{\pi} [x(\pi - x)]^2 dx = \sum_{n=1} b_n^2$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{30}\right] = \sum \frac{16}{\pi^2}\left(\frac{1-\cos n\pi}{n^3}\right)^2$$

$$= \frac{16}{\pi^2} \sum \frac{(1-\cos n\pi)^2}{n^6}$$

$$= \frac{16}{\pi^2}\left[\frac{2^2}{1^6} + \frac{2^2}{3^6} + \frac{2^2}{5^6} + \frac{2^2}{7^6} + \cdots\right]$$

hence $\frac{2}{\pi}\left[\frac{\pi^5}{30}\right] = \frac{64}{\pi^2}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots\right]$

We seek $S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots$

but $S = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} \cdots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \cdots\right)$

$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots\right) + \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots\right)$

$$S = \left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots\right] + \frac{1}{2^6} S$$

or $S\left(1 - \frac{1}{2^6}\right) = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots\right)$

but $\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots\right] = \frac{2}{\pi}\left[\frac{\pi^5}{30}\right]\frac{\pi^2}{64}$

hence $S\left(1 - \frac{1}{2^6}\right) = \frac{2}{\pi}\left[\frac{\pi^5}{30}\right]\left[\frac{\pi^2}{64}\right]$

$$S\left(\frac{64-1}{64}\right) = \frac{2}{\pi}\left[\frac{\pi^5}{30}\right]\left[\frac{\pi^2}{64}\right]$$

$$= \frac{\pi^6}{(15)(63)} = \frac{\pi^6}{945}$$

$\boxed{\therefore S = \sum \frac{1}{n^6} = \frac{\pi^6}{945}}$ as required

Mathematical Physics

Lecture, Friday November 30, 2001

Finish up $f(x) = x(\pi - x)$ even/odd extensions.

a) for the $\boxed{\text{even}}$ extension we obtained

$$f(x) = x(\pi - x) = \pi^2/6 - 2\sum_{n=1}\left\{\frac{\cos n\pi + 1}{n^2}\right\}\cos nx$$

$$= \pi^2/6 - \left\{\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right\}$$

b) for the odd extension, we will obtain

$$f(x) = x(\pi - x) = \frac{4}{\pi}\sum_{n=1}\frac{(1 - \cos n\pi)}{n^3}\sin nx$$

do this in class - see Nov 28, 01 notes.

$\boxed{\text{Problem:}}$ By using above problem $f(x) = x(\pi - x)$ even/odd show

i) $\displaystyle\sum_{n=1}\frac{1}{n^4} = \frac{\pi^4}{90}$ use even extension

ii) $\displaystyle\sum_{n=1}\frac{1}{n^6} = \frac{\pi^6}{945}$ use odd extension

See notes Nov 28, 01 <u>notes</u>

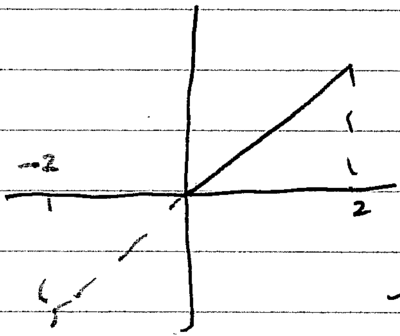
Integration / differentiation of Fourier Series

Problem:

Recall

$$f(x) = x \quad \text{on} \quad 0 < x < 2.$$

the sine series yielded



$$f(x) = -\frac{4}{\pi}\sum_{n=1} \frac{\cos n\pi}{n} \sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right).$$

Now by integration find.

the Fourier series for $f(x) = x^2$, $0 < x < 2$

$$\therefore x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2}\right.$$

$$\int_0^x x\,dx = \frac{4}{\pi}\left[\int_0^x \sin\frac{\pi x}{2}dx - \frac{1}{2}\int_0^x \sin\frac{2\pi x}{2}dx + \frac{1}{3}\int_0^x \sin\frac{3\pi x}{2}dx - \cdots\right.$$

$$\frac{x^2}{2} = \frac{4}{\pi}\left[-\frac{\cos\frac{\pi x}{2}}{\frac{\pi}{2}} + \frac{1}{2}\frac{\cos\frac{2\pi x}{2}}{\frac{2\pi}{2}} - \frac{1}{3}\frac{\cos\frac{3\pi x}{2}}{\frac{3\pi}{2}} + \cdots\right]\Bigg|_0^x$$

$$x^2 = \frac{8}{\pi}\left[-\frac{\cos\frac{\pi x}{2}}{\pi/2} + \frac{1}{2}\frac{\cos\frac{2\pi x}{2}}{\frac{2\pi}{2}} - \frac{1}{3}\frac{\cos\frac{3\pi x}{2}}{\frac{3\pi}{2}}\right.$$

$$= \frac{16}{\pi^2}\left[-\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\frac{2\pi x}{2} - \frac{1}{3^2}\cos\frac{3\pi x}{2}\right] + C$$

$$= -\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right)$$

for the lower limit 0

$$C = -\frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

but

$$x^2 = C - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3x\pi}{2}\right.$$

C must be $\frac{a_0}{2}$ for $f(x) = x^2$ on $0 < x < 2$

$L = 2$

$$C = \frac{a_0}{2}$$

$$a_0 = \frac{2}{2}\int_0^2 x^2\,dx. = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}.$$

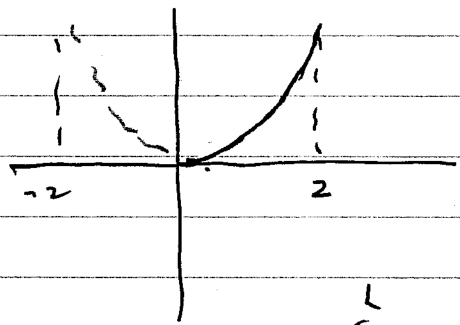
$$C = \frac{a_0}{2} = \frac{1}{2}\left(\frac{8}{3}\right) = 4/3$$

$$\therefore \boxed{x^2 = 4/3 - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right)}$$

on $0 < x < 2$

$\boxed{\text{do it directly.}}$

$f(x) = x^2 \qquad 0 < x < 2 \qquad$ even extention.



$L = 2$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 x^2\,dx = \frac{8}{3}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2}\int_0^2 x^2 \cos\frac{n\pi x}{2}\,dx$$

S.I

$$\int x^2 \cos ax\,dx = \frac{2x}{a^2}\cos ax + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin ax$$

$$a_n = \int_0^2 x^2 \cos\frac{n\pi x}{2}\,dx = \frac{2x}{\left(\frac{n\pi}{2}\right)^2}\cos\frac{n\pi}{2}x + \left(\frac{x^2}{\frac{n\pi}{2}} - \frac{2}{\left(\frac{n\pi}{2}\right)^3}\right)\underbrace{\sin\frac{n\pi x}{2}}_{0}$$

$$= \frac{2(2)}{\left(\frac{n\pi}{2}\right)^2}\cos n\pi$$

$$\boxed{a_n = \frac{16}{\pi^2}\,\frac{\cos n\pi}{n^2}}$$

$\therefore$ on $0 < x < 2$

$$f(x) = x^2 = a_0/2 + \sum_1 a_n \cos\frac{n\pi x}{2}$$

$$= (8/3)\frac{1}{2} + \sum_1 \frac{16}{\pi^2}\frac{\cos n\pi}{n^2}\cos\frac{n\pi x}{2}$$

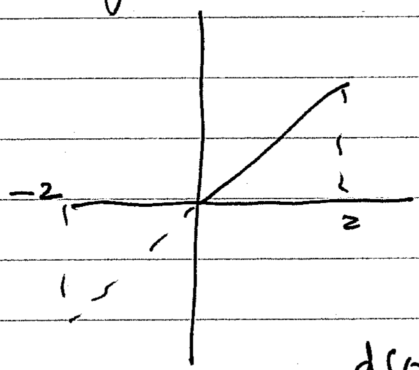
$$= 4/3 + \frac{16}{\pi^2}\left[-\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{2\pi x}{2}}{2^2} - \frac{\cos\frac{3\pi x}{2}}{3^2}\right]$$

$$= 4/3 - \frac{16}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right]$$

Same as required

Differentiation:

for odd extention of $f(x) = x$ on $0 < x < 2$



$$f(x) = x = \sum \frac{-4}{n\pi} \cos n\pi \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} \right.$$

$$\frac{df(x)}{dx} = 1 = \frac{4}{\pi}\left(\frac{\pi}{2} \cos \frac{\pi x}{2} - \frac{1}{2}\frac{2\pi}{2} \cos \frac{2\pi x}{2} + \frac{1}{3}\frac{3\pi}{2} \cos \frac{3\pi x}{2} \right.$$

$$= 2\left(\cos \frac{\pi x}{2} - \cos \frac{2\pi x}{2} + \cos \frac{3\pi x}{2} \right)$$

does not converge —

n^{th} term does not approach 0

Mathematical Physics

Lecture, Friday, November 30, 2001

$\boxed{\text{The Fourier Integral Theorem}}$ heuristic Treatment

on $[-L, L]$ $T = 2L$

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

use dummy variable for integration

$$\therefore f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\left\{\cos\frac{n\pi x}{L}\cos\frac{n\pi t}{L} + \sin\frac{n\pi x}{L}\sin\frac{n\pi t}{L}\right\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\left[\frac{n\pi}{L}(x-t)\right]dt$$

no big deal here

Set $u = \frac{n\pi}{L} \Rightarrow \Delta u = \frac{\pi}{L}\Delta n$ but $\Delta n = 1$

$\Delta u = \frac{\pi}{L} \Rightarrow \frac{1}{L} = \frac{\Delta u}{\pi}$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{\pi}\sum_{\substack{n=1 \\ u=\frac{\pi}{L}}} \Delta u \int_{-L}^{L} f(t)\cos\Big(\underbrace{\frac{n\pi}{L}}_{u}(x-t)\Big)dt$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{\pi}\sum_{\substack{n=1 \\ \text{or } u=\frac{\pi}{L}}} \Delta u \int_{-L}^{L} f(t)\cos(u(x-t))\,dt$$

Now if $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges $\Rightarrow \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} f(t)\,dt \to 0$.

and

$$\sum_{u=\frac{\pi}{L}} \Delta u \to \int_0^{\infty} du \quad \text{as } L \to \infty$$

hence, we must conclude -

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty} du \int_{-\infty}^{\infty} f(t)\cos(u[x-t])\,dt}$$

Fourier Integral Theorem.

Mathematical Physics.

Friday, November 30, 2001 Fourier Integral Theorem

The Fourier Series representation on $[-L, L]$

$T = 2L$

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

u = dummy

$$a_n = \frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{n\pi u}{L}\,du \qquad a_0 = \frac{1}{L}\int_{-L}^{L} f(u)\,du$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(u)\sin\frac{n\pi u}{L}\,du$$

Substitute into series representation

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)\,du + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\left\{\cos\frac{n\pi u}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi u}{L}\sin\frac{n\pi x}{L}\right\}du$$

$$= \frac{1}{2L}\int_{-L}^{L} f(u)\,du + \frac{1}{L}\sum_{n=1}^{\infty} f(u)\cos\frac{n\pi}{L}(x-u)\,du$$

Now big deal here $L \longrightarrow \infty$

If we assume $\int_{-\infty}^{\infty} |f(u)|\,du$ converges then $\frac{1}{2L}\int_{-L}^{L} f(u)\,du \longrightarrow 0$

set $\alpha = \frac{n\pi}{L} \Rightarrow \Delta\alpha = \pi/L \, \Delta n$ but $\Delta n = 1$

$$\therefore \Delta\alpha = \pi/L \Rightarrow \frac{1}{L} = \frac{\Delta\alpha}{\pi}$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)\,du + \frac{1}{\pi}\sum_{\substack{n=1\\ \alpha=\frac{\pi}{L}}} \Delta\alpha \int_{-L}^{L} f(u)\cos\Big(\underbrace{\frac{n\pi}{L}}_{\alpha}(x-u)\Big)\,du$$

$$= \frac{1}{2L}\int_{-L}^{L} f(u)\,du + \frac{1}{\pi}\sum_{\substack{n=1\\ \alpha=\pi/L}} \Delta\alpha \int_{-L}^{L} f(u)\cos(\alpha[x-u])\,du$$

Now let $L \to \infty$.

then $\frac{1}{2L}\int_{-L}^{L} f(u)\,du \to 0$ $\quad \sum_{\alpha=\pi/L} \Delta\alpha \longrightarrow \int_0^\infty d\alpha$

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} f(u)\cos(\alpha[x-u])\,du.}$$

Fourier Integral Theorem

Problem. Show:

$$f(x) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha.$$

where $A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

Soln

Start $f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha$

We may write:

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\left[\cos\alpha x\cos\alpha u + \sin\alpha x\sin\alpha u\right]du\,d\alpha.$$

or

$$f(x) = \int_{\alpha=0}^{\infty}\{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\,d\alpha$$

where $A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du$$

Problem Show

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\, du\, d\alpha.$$

Soln.

Start with

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\, du\, d\alpha$$

$\cos\alpha(x-u)$

$\boxed{\text{is an even fn of } \alpha}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\, du\, dx. \qquad \text{I}$$

Now use the fact that $\sin\alpha(x-u)$ is an <u>odd</u> fn of α

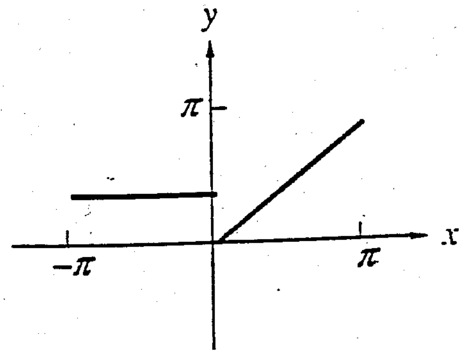
Then

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\alpha(x-u)\, du\, dx \qquad \text{II}$$

multiply II by i and add to I

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\left[\cos\alpha(x-u) + i\sin(x-u)\right] du\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\, du\, d\alpha$$



$$f(x) = \begin{cases} 1 & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$

FIGURE

Piecewise continuous function

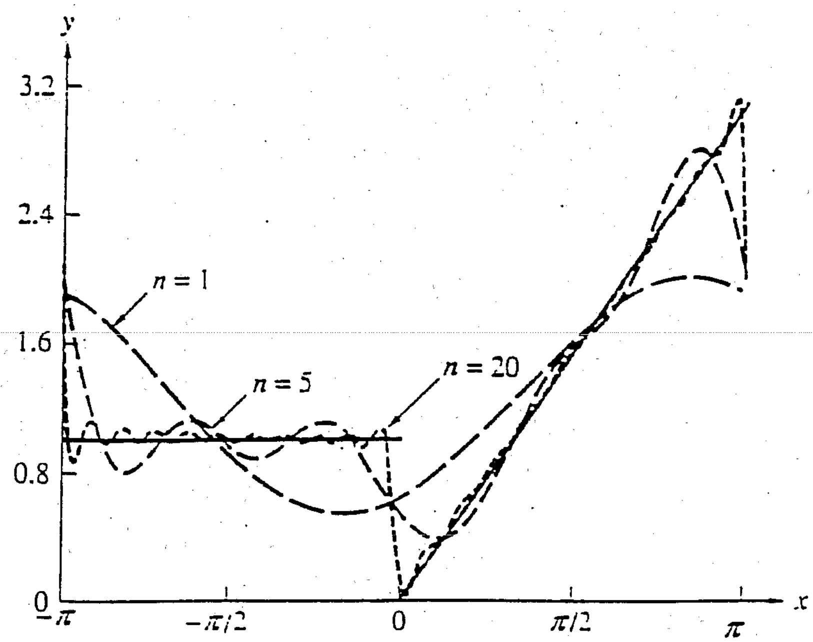
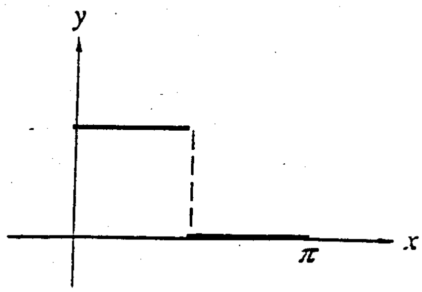


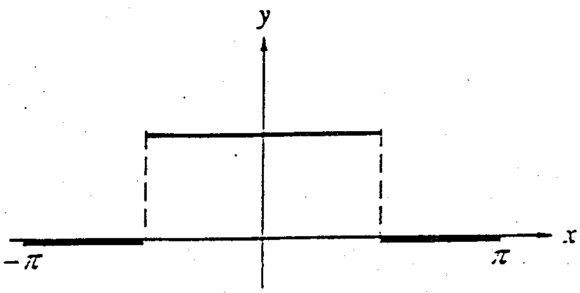
FIGURE
Fourier series approximations of function (   ) as n varies up to one, five, and 20 terms in the infinite series. As n increases, the Fourier approximations approach the actual $f(x)$ values.



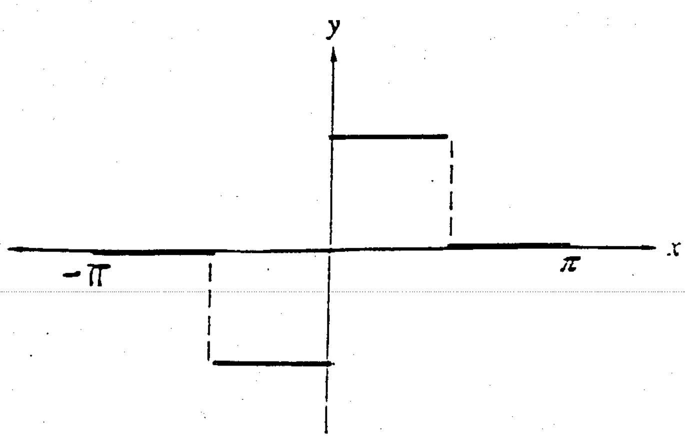
FIGURE

A

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi/2 \\ 0 & \pi/2 \le x \le \pi \end{cases}$$



even extension



odd extension

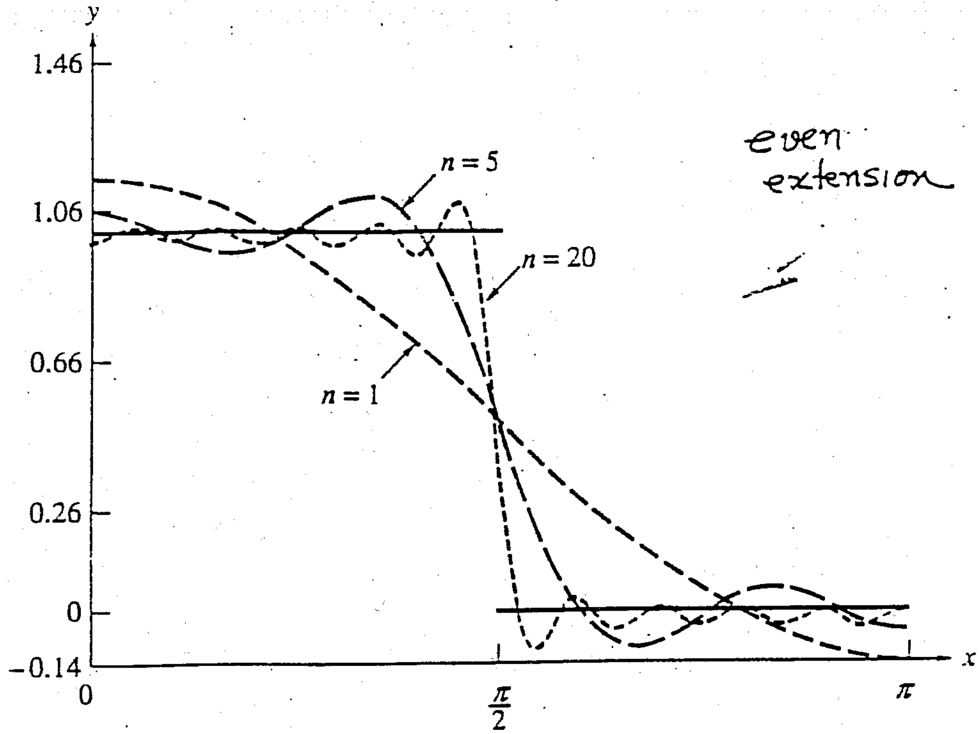


FIGURE
Fourier cosine series approximations of function (A) as n varies up to one, five, and 20 terms in the infinite series. As n increases, the Fourier cosine approximations approach the actual values of function $f(x)$. Note that each Fourier cosine approximation passes through the value $y = 0.5$, the midvalue of the jump, at the point of discontinuity $x = \pi/2$.

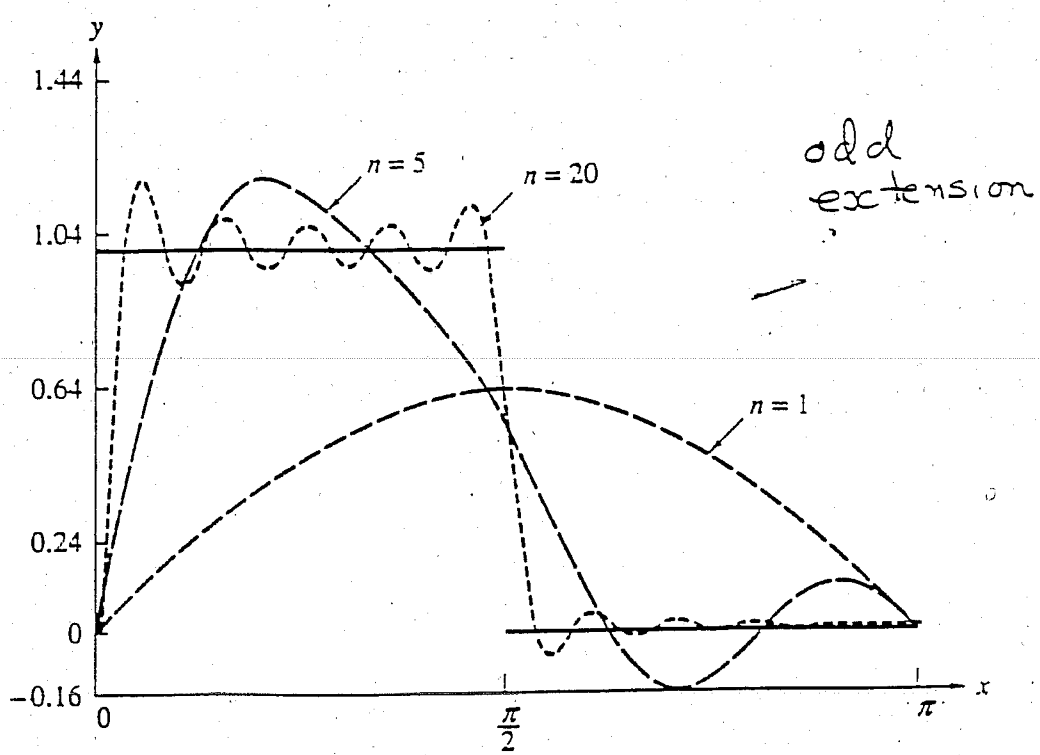


FIGURE
Fourier sine approximations of the function A as n varies up to one, five, and 20 terms in the infinite series. As n increases, the Fourier sine approximations approach the actual function values of $f(x)$. The Fourier sine approximations converge to the midpoint of the jump at the point

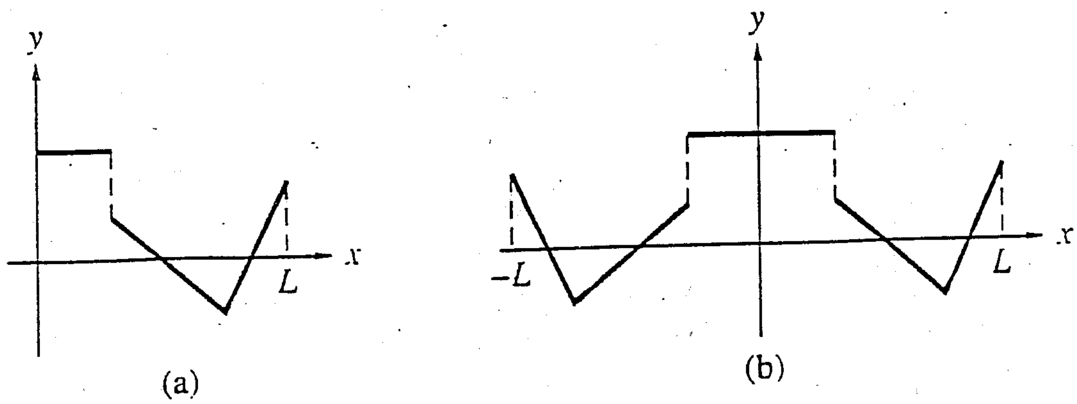


FIGURE
(a) Original piecewise continuous function f defined over $0 < x < L$. (b) The even extension of f over $-L < x < L$.

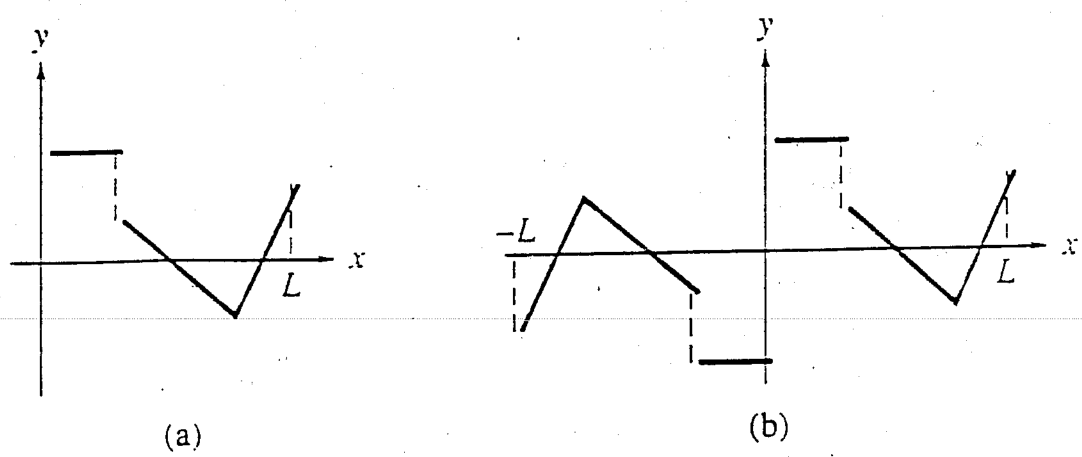


FIGURE
(a) Original piecewise continuous function f defined over $0 < x < L$. (b) Odd extension of f over $-L < x < L$.