Mathematical Physics

Lecture, Monday, November 12, 2001

we were sort of half way through studying

$$T = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$$

we showed $T = T^{\dagger}$ Hermitian.

we obtained the eigenvalues

$$\lambda_1 = 0 \qquad \lambda_{2,3} = 3$$

for $\lambda_1 = 0$ the

eigenvector $\hat{X}^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}$

However, for $\lambda_2 = \lambda_3 = 3$

$$\hat{X}_0^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \hat{X}_0^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

but we observed $\hat{X}_0^{(2)} \cdot \hat{X}_0^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \neq 0$

these two vectors are <u>not</u> orthogonal.

Let us make them orthogonal.

$\boxed{\text{Gram-Schmidt}}$

$$X_0^{(2)} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad X_0^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

set $|e_1\rangle = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad |e_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Step 1 $|e_1'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$

Step 2 $\quad |e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

$$\langle e_1'|e_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i & 0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}.$$

$$\langle e_1'|e_2\rangle|e_1'\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1\\-i\\0\end{pmatrix}$$

Now form
$$|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$$

$$\begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\ \frac{-i}{2}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ i/2\\ 1\end{bmatrix}$$

Now normalize $\quad \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2\right]^{1/2} = \left(\frac{1}{4} + \frac{1}{4} + 1\right)^{1/2}$

$$= \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$$

$$\therefore |e_2'\rangle = \frac{2}{\sqrt{6}}\begin{bmatrix}\frac{1}{2}\\ i/2\\ 1\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\ i\\ 2\end{bmatrix}$$

put this all together

$$\hat{X}^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \qquad \hat{X}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = |e_1'\rangle$$

$$\hat{X}^{(3)} = |e_3'\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix}$$

Now check orthogonality

$$\hat{X}^{(1)} \cdot \hat{X}^{(2)} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -i & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}} = 0.$$

$$\hat{X}^{(1)} \cdot \hat{X}^{(3)} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -i & -1 \end{bmatrix}\begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix}\frac{1}{\sqrt{6}} = 0$$

finally $\hat{X}^{(2)} \cdot \hat{X}^{(3)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix} \frac{1}{\sqrt{6}} = 0$

right on the button

Now form S

$$S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

S is unitary $\Rightarrow S^{-1} = S^{+}$

$$\therefore S^{-1} = S^{+} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

to simplify the arithmetic, pull out $\frac{1}{\sqrt{6}}$

$$S = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2}\,i & -i\sqrt{3} & i \\ -\sqrt{2} & 0 & 2 \end{pmatrix}$$

$$S^{-1} = S^{\dagger} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -\sqrt{2}\,i & -\sqrt{2} \\ \sqrt{3} & i\sqrt{3} & 0 \\ 1 & -i & 2 \end{pmatrix}$$

Now form

$$S^{-1}TS = \frac{1}{6}\begin{pmatrix} \sqrt{2} & -i\sqrt{2} & -\sqrt{2} \\ \sqrt{3} & i\sqrt{3} & 0 \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2}i & -i\sqrt{3} & i \\ -\sqrt{2} & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2}i & -i\sqrt{3} & i \\ -\sqrt{2} & 0 & 2 \end{pmatrix} = ($$

$$= \begin{pmatrix} 2\sqrt{2}-\sqrt{2}-\sqrt{2} & 2\sqrt{3}+\sqrt{3} & 2-1+2 \\ -i\sqrt{2}+2\sqrt{2}i-i\sqrt{2} & -i\sqrt{3}-2i\sqrt{3} & -i+2i+2i \\ \sqrt{2}+\sqrt{2}-2\sqrt{2} & \sqrt{3}-\sqrt{3}+0 & 1+1+4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3\sqrt{3} & 3 \\ 0 & -3i\sqrt{3} & 3i \\ 0 & 0 & 6 \end{pmatrix}$$

$$S^{-1}\left(\quad\right) = \frac{1}{6}\begin{pmatrix} \sqrt{2} & -i\sqrt{2} & -\sqrt{2} \\ \sqrt{3} & i\sqrt{3} & 0 \\ 1 & -i & 2 \end{pmatrix}\begin{pmatrix} 0 & 3\sqrt{3} & 3 \\ 0 & -3i\sqrt{3} & 3i \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 0 & \cancel{}\,3\sqrt{6} - 3\sqrt{6} & 0 \\ 0 & 18 & 3\sqrt{3} - 3\sqrt{3} + 0 \\ 0 & 3\sqrt{3} - 3\sqrt{3} & 18 \end{pmatrix}$$

$$S^{-1}TS = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ as required

Final Version

Mathematical Physics

Lecture, Monday/Wednesday, November 12, 14 2001

The Similarity transformation – What does it mean

Consider the followin set of Equations:

This is an example: –

$$\boxed{\begin{aligned} X &= 5x - 2y \\ Y &= -2x + 2y. \end{aligned}}$$

or

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \qquad \boxed{\vec{R} = M\vec{r}}$$

what does this mean?

M operating on $\vec{r}$ produces $\vec{R}$

Note: M is symmetric

observation

$\Rightarrow$ eigenvectors will be orthogonal

diagonalize M.

$$\begin{vmatrix} (5-\lambda) & -2 \\ -2 & (2-\lambda) \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$\lambda_1 = 1 \quad \lambda_2 = 6$ eigenvalues –

$\boxed{\text{Obtain eigenvectors}}$. a_1, a_2 dummies.

$\boxed{\lambda_1 = 1}$

$$\begin{pmatrix} 5-1 & -2 \\ -2 & 2-1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow 4a_1 - 2a_2 = 0 \Rightarrow 2a_1 = a_2 \qquad a_1 = 1, \quad a_2 = 2$$

$$\boxed{\hat{X}^{(1)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\boxed{\lambda_2 = 6}$

$$\begin{pmatrix} 5-6 & -2 \\ -2 & 2-6 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \Rightarrow -a_1 - 2a_2 = 0 \Rightarrow a_1 = -2a_2. \qquad a_2 = 1, \quad a_1 = -2.$$

$$\boxed{\hat{X}^{(2)} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$$\therefore \hat{X}^{(1)} \cdot \hat{X}^{(2)} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{5}} = 0$$

N.B. $\longrightarrow$

implied by symmetric matrix orthogonal as they should be form the matrix $\boxed{C = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}$

Note: <u>C is orthogonal</u> $\Rightarrow C^{-1} = C^{T}$ obvious because M is symmetric.

$$\therefore \boxed{C^{-1} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}$$

check $C^{-1}C = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{+2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

Suppose we had formed <u>C^{-1}</u> from <u>unnormalized</u> eigenvectors

Say $C' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ $\quad C'^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$$C'C^{-1\prime} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

<u>not</u> the unit matrix

but if we divide by $\frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}} = \frac{1}{5}$ (normalization)

then $C' C'^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\boxed{\text{Notice}}$ $MC = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -12 \\ 2 & 6 \end{pmatrix}$

$$\underline{\underline{MC}} = \boxed{\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-12}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}} \end{pmatrix}}$$

Now

$$\boxed{C^{-1}MC = D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}}$$

Now form $CD = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-12}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{6}{\sqrt{5}} \end{pmatrix}$

hence $\boxed{MC = CD}$ This is important.

& of course $C^{-1}MC = C^{-1}CD$

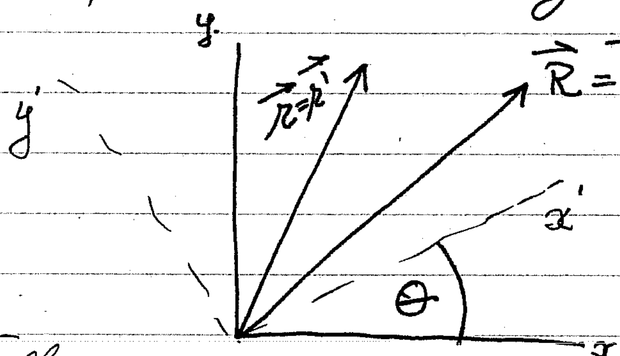
or $\boxed{\underline{D = C^{-1}MC}}$

what we have done is diagonalized simultaneously

$\boxed{M}$ & M' where $\boxed{M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}}$ Simultaneous diagonalization.

& $\boxed{M' = C^{-1}MC = D}$

What do we mean by $C^{-1}MC = D$?



Tell class to observe $\boxed{\text{fig ①, fig ② – hand-out.}}$

The (x, y) and (x', y') coordinates of one-point

(or components of one vector $\vec{r} = \vec{r}'$ are related

as follows

→ over

$$\boxed{\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}}$$

or

$$x = \frac{\begin{vmatrix} x' & \sin\theta \\ y' & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} = x'\cos\theta - y'\sin\theta$$

& of course

$$y = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = y'\cos\theta + x'\sin\theta$$

or we may write.

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$\boxed{\vec{r} = C\vec{r}\,'} \qquad |r\rangle = C|r'\rangle$$

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{r} = C\vec{r}\,'$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}.$$

$\boxed{\text{N.B.}}$ the statement

$$\vec{r} = C\vec{r}\,' \quad \text{with} \quad C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

this is an important result! Stress this!

is true for $\boxed{\text{any}}$ single vector with components given in the two systems.

Now suppose we consider another vector $-\ \vec{R}$

where $\vec{R} = \vec{R}'$

with X, Y, & X', Y' components. fig 2

Once again we may write.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix}$$

$$\vec{R} = C\vec{R}' \qquad |R\rangle = C|R'\rangle$$

Now let M be a matrix (transformation) which describes a deformation of the plane in the (x, y) - system.

in old system

Then, the equation $\vec{R} = M\vec{r}$ $\left(|R\rangle = M|r\rangle\right)$ says that the vector $\vec{r}$ $(|r\rangle)$ becomes the vector $\vec{R}$ $[|R\rangle]$ after the deformation, both vectors being given (described) relative to the (x, y) axes original (bases).

∴ old

We now ask, how can we describe the deformation in the (x', y') system (new basis)?

i.e. what matrix carries $|r'\rangle \to |R'\rangle$.

i.e. $\vec{r}\,'$ into $\vec{R}\,'$??? $\quad \vec{R}\,' = (\,?\,)\vec{r}\,'$

To see this, we write.

① $\vec{r} = C\vec{r}\,' \qquad |r\rangle = C|r'\rangle$

② $\vec{R} = C\vec{R}\,' \qquad |R\rangle = C|R'\rangle$

③ $\vec{R} = M\vec{r} \qquad \Rightarrow |R\rangle = M|r\rangle$

Substitute ①, ② into ③

$$C\vec{R}\,' = MC\vec{r}\,'$$

Now operate with C^{-1} as follows:

$$C^{-1}\left[C\vec{R}\,' = MC\vec{r}\,'\right]$$

$$C^{-1}C\vec{R}\,' = C^{-1}MC\vec{r}\,'$$

or $\vec{R}\,' = C^{-1}MC\,\vec{r}\,' \qquad$ since $C^{-1}C = I$

Now $D = C^{-1}MC$ is the matrix which describes in the (x', y') system (new basis), the same deformation that M describes in the (x, y) system (old basis).

Next, we want to show that if the matrix C

is chosen to make $D = C^{-1}MC$ a diagonal matrix,

then the new axes (x', y') (new bases) are along

the directions of the eigen vectors of M.

Recall from our example that the columns of

$\underline{C}$ are the components of the unit eigenvectors (normalized eigenvectors)

as they are in this example, then the new axes (x', y') new bases

along the eigen vector direction are a set of

perpendicular axes (orthogonal) rotated

from (x, y) old bases by some angle θ

Recall our matrix (our example)

$$C = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad \boxed{C \text{ diagonalizes } M} \quad !!$$

the columns of C are the components of the unit eigenvectors (normalized eigenvectors) as per the example, ~~we~~, then the new axes (x', y') (new basis) ~~are~~ (along the eigenvector directions) are a set of perpendicular axes rotated from (x, y) old basis by some angle θ.

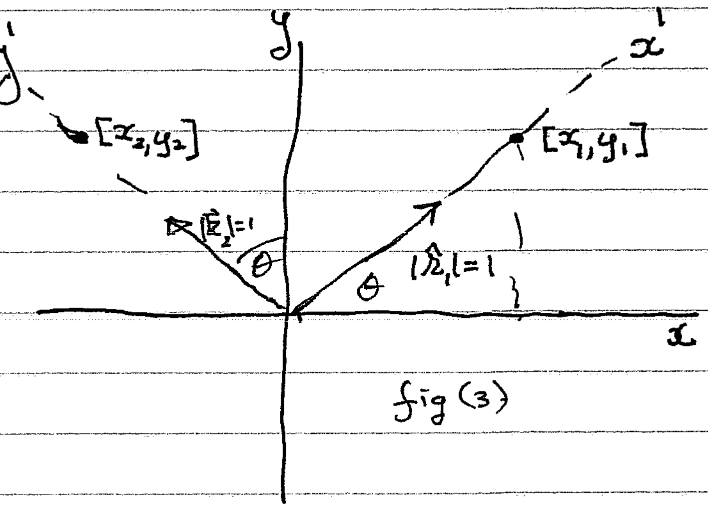


fig (3)

$$x_1 = |\hat{r}_1| \cos\theta = \cos\theta$$

$$y_1 = |\vec{r}_1| \sin\theta = \sin\theta$$

$$x_2 = -|\vec{r}_2| \sin\theta = -\sin\theta$$

$$y_2 = |\vec{r}_2| \cos\theta = \cos\theta$$

or $$C = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Thus, the matrix C which diagonalizes M is the $\longrightarrow$

rotation matrix

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

when the (x', y') axes (new bases) are along the directions of the eigen vectors of M!

Relative, to these new axes (new bases), the diagonal matrix D describes the deformation

for example, we have

$$\boxed{\vec{R}' = D\vec{r}'}$$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

or $X' = x'$

A $Y' = 6y'$

In words (A) says that [in the (x', y') system - new basis]

each point (x', y') new basis has its x' coordinate

unchanged by the deformation, and its y' coordinate multiplied by 6; that is, the deformation is simply a <u>stretch</u> in the y'-direction

N.B: this is a <u>simpler</u> description of the deformation and is physically clearer than

the description $\vec{R} = M\vec{r}$ in the old basis (x, y)

i.e. $$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{\vec{R} = M\vec{r}}$$

old basis

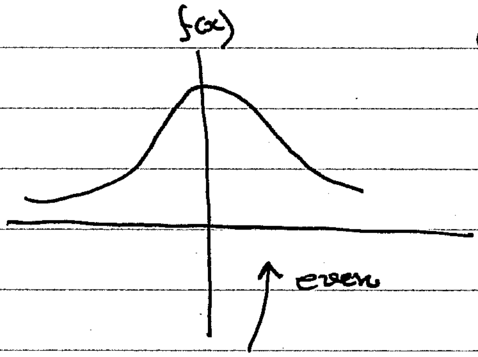
Mathematical Physics

Lecture, Monday, November 19, 2001

Fourier Series — preliminaries.

Even/odd - periodic functions.

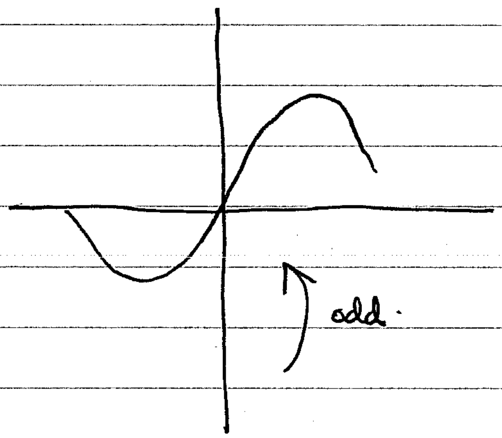
Let $f(x)$ be defined on an x interval, finite/infinite centered on $x=0$



We say that $f(x)$ is an even fn

if $f(-x) = f(x)$

Note: graph symmetric about ~~x=0~~ vertical axis



We say that $f(x)$ is an odd fn.

if $f(-x) = -f(x)$

Note: graph is symmetric about the origin.

Examples: even fn's: x^2, $3x^4$, $\cos x$, $\sin|x|$, e^{-x^2}

odd fn's: x, $3x^3$, $2x^5$, $\sin x$, $x\cos x$.

Use ful properties:

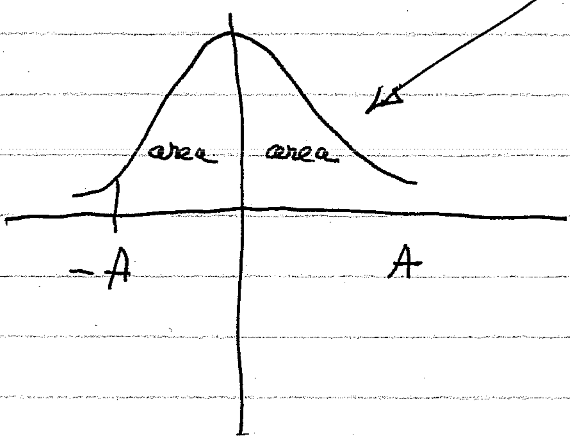
even + even = even.	(even)(even) = even.
odd + odd = odd.	(odd)(odd) = even.
	(even)(odd) = odd

Two useful integral properties

If $f(x)$ is even

then $$\int_{-A}^{A} f(x)\,dx = 2\int_{0}^{A} f(x)\,dx \quad \left(f(x) \text{ even}\right)$$

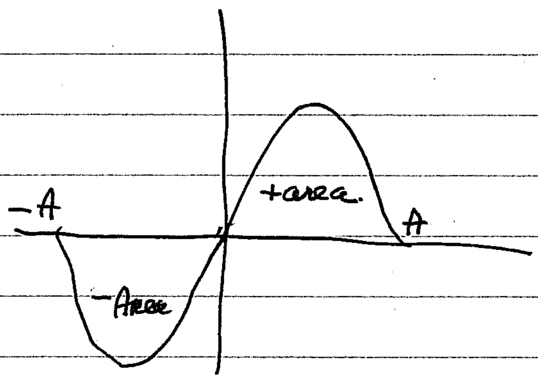
Note: $-A \to A$
symmetric limits



If, $f(x)$ is odd, then

$$\int_{-A}^{A} f(x)\,dx = 0 \qquad \left(f(x) \text{ odd} \right)$$

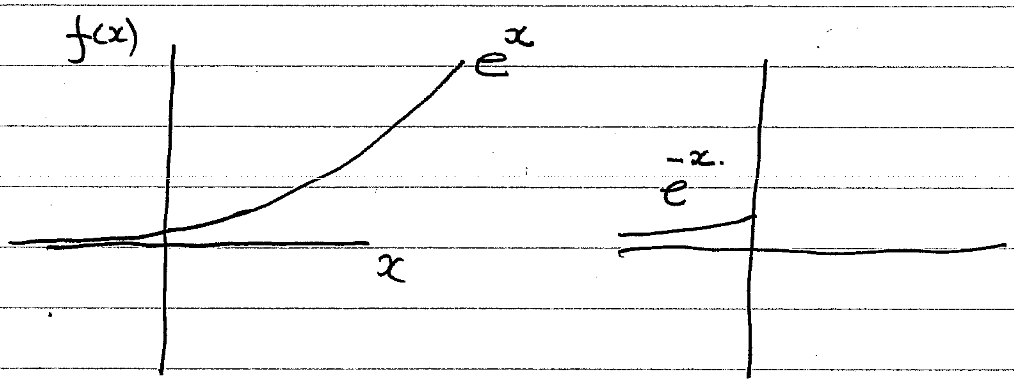
$-A \to A$ symmetric limits.



$\sum$ area $= 0$

Note: a given fn is not necessarily even or odd, it may be both even and odd; or it may be neither

Example: $f(x) = e^{x}$ is neither even or odd.



Useful Procedure: Every fn' can be uniquely decomposed into a sum of an even fn, say f_e, and an odd fn, say f_o, as the following implies:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= f_e(x) + f_o(x).$$

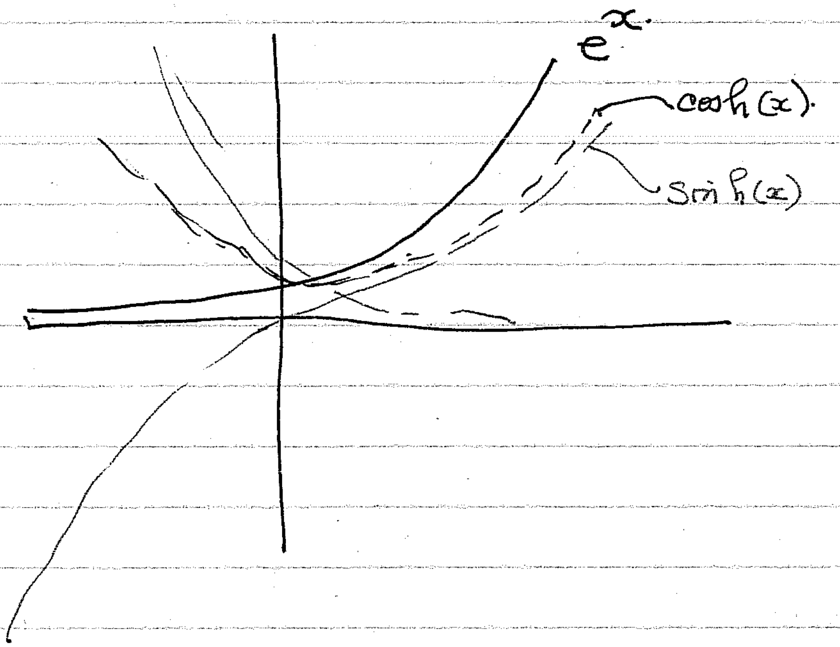
example

Let $f(x) = e^x$, $f(-x) = e^{-x} = \frac{1}{e}$

$$f_e(x) = \frac{e^x + e^{-x}}{2} \qquad f_o(x) = \frac{e^x - e^{-x}}{2}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\cosh(x) \qquad\qquad \sinh(x)$$



Periodic Functions

Suppose for a given fn $f(x)$ there exists a positive constant T such that.

$$f(x+T) = f(x)$$

for every x in the domain of $f(x)$.

Then we say that $f(x)$ is a periodic fn of x with period T — sometimes, we say that $f(x)$ is T periodic

Example: $f(x) = \sin x$ is periodic with period 2π

because $f(x+2\pi) = \sin(x+2\pi) = \sin x \cos 2\pi + \cos x \sin 2\pi = \sin x$.

for all x

Examples:

1) $f(x) = \cos x$

$$\cos(x+2\pi m) = \cos x \cos 2\pi m - \sin x \sin 2\pi m$$

$$= \cos x \quad \text{since } \cos 2\pi m = 1 \ \& \ \sin 2\pi m = 0$$

for any integer m. hence $T = 2\pi m$, $m=1$ gives smallest value $T = 2\pi$

$\sin x$.

$\sin(x+2\pi m) = \sin x \cos 2\pi m + \cos x \sin 2\pi m$

$\therefore \sin(x+2\pi m) = \sin x$ for any integer m.

for $m=1$ $T = 2\pi$ smallest

$\cos 2x$: we want $\cos 2(x+T) = \cos 2x$

$\therefore \cos(2x+2T) = \cos 2T$ $\quad \cos 2x \cos 2T - \sin 2x \sin 2T$

or

$2T = 2\pi m$ $\quad m=1$ smallest.

$\therefore \underline{\underline{T = \pi}}$ $\quad \cos 2T = 1$

$2T = 2\pi m$.

$\sin 2x$: $\quad \sin(2x+2T) = \sin 2x$.

$2T = 2\pi m$ $\quad m$ smallest

$T = \pi$

$\cos \pi x$ $\quad \cos \pi(x+T) = \cos \pi x$ $\quad \cos \pi x \cos \pi T - \sin \pi x \sin \pi T$

$\cos(\pi x + \pi T) = \cos \pi x$. $\quad \cos \pi T = 1$

$\pi T = 2\pi m$ $\quad \pi T = 2\pi m$ $\quad m=1$

$T = 2$ smallest $\quad T = 2$

$\sin \pi x$ gives same $T = 2$

$\cos 2\pi x$: $\cos(2\pi x + 2\pi T) = \cos 2\pi x$.

$2\pi T = 2\pi m$ $\qquad m = 1$

$\underline{\underline{T = 1}}$

example

$f(x) = \sin(\omega x + \phi)$

$\sin(\omega[x + T] + \phi) = \sin(\omega x + \phi)$.

$\sin(\underline{\omega x + \phi} + \underline{\omega T}) = \sin(\omega x + \phi)$

$\sin(\omega x + \phi)\cos \omega T + \cos(\omega x + \phi)\sin \omega T = \sin(\omega x + \phi)$.

$\therefore \cos \omega T = 1$

$\omega t = \cos^{-1}(1) = 2n\pi$ $\qquad n = 1$ lowest.

$T = \frac{2n\pi}{\omega}$, $\quad \underline{\underline{T = \frac{2\pi}{\omega}}}$

$f(x) = \cos 6x$

$\cos 6(x + T) = \cos 6x$.

$\cos 6x \cos 6T - \sin 6x \sin 6T = \cos 6x$.

$\cos 6T = 1$

$6T = \cos^{-1}(1) = 2\pi n$,

$T = \frac{2\pi n}{6} = \pi/3\, n$

$n = 1$ smalles

$T = \pi/3$

$\tan x$ — period.

$$\tan(x+T) = \tan x$$

$$\tan(x+T) = \frac{\sin(x+T)}{\cos(x+T)} = \frac{\sin x\cos T + \cos x \sin T}{\cos x \cos T - \sin x \sin T} = \frac{\sin x}{\cos x}$$

or $\cos x\left[\sin x\cos T + \cos x \sin T\right] = \sin x\left[\cos x \cos T - \sin x \sin T\right]$

$$\cos x \cancel{\sin x \cos T} + \cos^2 x \sin T = \sin x \cancel{\cos x} \cos T - \sin^2 x \sin T$$

$$(\cos^2 x + \sin^2 x)\sin T = 0 \Rightarrow \sin T = 0$$

Smallest <u>non-zero</u> root

$\sin \pi = 0$ $\quad n\pi$

$\underline{\underline{T = \pi}}$ $\quad = n\pi \quad n = 1$

$\boxed{\text{Some important and useful Integrals}}$

Claim: $$\int_{-l}^{l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0 \end{cases}$$

m, n integers.

take case $m \neq n$.

use $2\cos A\cos B = \cos(A-B) + \cos(A+B)$.

$$\therefore \int_{-l}^{l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\left[\int_{-l}^{l}\cos\frac{\pi}{l}(m-n)x\,dx + \int_{-l}^{l}\cos\frac{\pi}{l}(m+n)x\,dx\right]$$

$$= \frac{1}{2}\left[\frac{l}{\pi(m-n)}\sin\frac{\pi}{l}(m-n)x\Big|_{-l}^{l} + \frac{l}{\pi(m+n)}\sin\frac{\pi}{l}(m+n)x\Big|_{-l}^{l}\right]$$

$$= \frac{1}{2}\left[\frac{l}{\pi(m-n)}\left[\sin\pi(m-n)\right]2 + \frac{l}{\pi(m+n)}\left[\sin\pi(m+n)\right]2\right]$$

set $m-n = k$ an integer

$m+n = h$ an integer

$\Rightarrow \quad \sin k\pi = 0 \qquad \sin h\pi = 0$

$$\therefore \int_{-l}^{l}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = 0 \qquad m \neq n$$

Now for $m=n$ we get

$$\int_{-l}^{l}\cos^2\frac{m\pi x}{l}\,dx \qquad\qquad \int\cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= \frac{x}{2} + \cancel{\frac{\sin\frac{2m\pi x}{l}}{\frac{4m\pi}{l}}}^{\,0}\Bigg|_{-l}^{l}$$

$$= \frac{1}{2}(l+l) = l$$

$$\therefore \int_{-l}^{l}\cos^2\frac{m\pi x}{l}\,dx = l$$

hence
$$\boxed{\int_{-l}^{l}\cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m\neq n \\ l & m=n\end{cases}}$$

2. Show $$\int_{-l}^{l} \sin\frac{m\pi x}{l}\,\sin\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0 \end{cases}$$

use ~~1/2~~ $$\sin A \sin B = \tfrac{1}{2}\left\{\cos(A-B) - \cos(A+B)\right\}$$

$$\therefore \int_{-l}^{l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\left[\int_{-l}^{l} \cos\frac{\pi}{l}(m-n)x\,dx - \int_{-l}^{l} \cos\frac{\pi}{l}(m+n)x\,dx\right]$$

$$= \frac{1}{2}\left[\left\{\frac{l}{\pi(m-n)}\sin\frac{\pi}{l}(m-n)x\right\}\Bigg|_{-l}^{l} - \left\{\frac{l}{\pi(m+n)}\sin\frac{\pi}{l}(m+n)x\right\}\Bigg|_{-l}^{l}\right]$$

$$= \frac{1}{2}\left\{\frac{2}{\pi}\frac{l}{(m-n)}\sin\pi(m-n) - \frac{2l}{\pi(m+n)}\sin\pi(m+n)\right\}$$

$k = m-n \qquad h = m+n.$

$\sin k\pi = 0 \qquad \sin h\pi = 0$

$$\therefore \int_{-l}^{l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = 0 \qquad m \neq n.$$

finally, $m = n$ $$\int_{-l}^{l} \sin^2\frac{m\pi x}{l}\,dx = \frac{x}{2} - \xcancel{\frac{\sin\frac{2m\pi x}{l}}{\frac{4m\pi}{l}}}^{\,0}\Bigg|_{-l}^{l}$$

$$= \frac{l}{2} + \frac{l}{2} = l.$$

$$\therefore \int_{-l}^{l} \sin^2\frac{m\pi x}{l}\,dx = l$$

&. $$\boxed{\int_{-l}^{l} \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m \neq n \\ l & m = n \\ & m = n \neq 0 \end{cases}}$$

3. $\displaystyle\int_{-l}^{l} \cos\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx.$ use S.I. $\displaystyle\int \sin px \cos qx\,dx$

$$= -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$$

$p = n$
$q = m.$

$$\therefore \int_{-l}^{l} \sin\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = -\frac{\cos\frac{\pi}{l}(n-m)x}{2(n-m)}\Bigg|_{-l}^{l} - \frac{\cos\frac{\pi}{l}(n+m)x}{2(n+m)}\Bigg|_{-l}^{l} = 0$$

odd fn - symmetric limits.

Now suppose $n=m.$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-l}^{l} \sin 2\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{2}\,\frac{l}{2n\pi}\left(-\cos\frac{2n\pi x}{l}\right)\Bigg|_{-l}^{l}$$

$$= \frac{-l}{4\pi n}\left[\cos 2n\pi - \cos(-2n\pi)\right] = 0$$

even fn -

finally $\displaystyle\int_{-l}^{l} \cos\frac{n\pi x}{l}\,dx = \frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big|_{-l}^{l} = \frac{l}{n\pi}\Big[\sin n\pi - \sin(-n\pi)\Big] = 0$

$$\int_{-l}^{l} \sin\frac{m\pi x}{l}\,dx = 0$$

odd fn / symmetric limits.

We have looked at integrals of the form.

$$\int_{-l}^{l} \cos\frac{m\pi x}{l}\cos\frac{n\pi x}{l}\,dx = \begin{cases} 0 & m \neq n \\ l & m = n \neq 0. \end{cases}$$

What is the period of $\cos\frac{m\pi x}{l}$?

$$\cos\frac{m\pi}{l}\left(x+T\right) = \cos\frac{m\pi x}{l}$$

$$\cos\frac{m\pi x}{l}\underbrace{\left[\cos\frac{m\pi T}{l}\right]}_{=1} - \sin\frac{m\pi x}{l}\sin\frac{m\pi T}{l} = \cos\frac{m\pi x}{l}$$

$\therefore$ we demand $\cos\frac{m\pi T}{l} = 1$

or $\frac{m\pi T}{l} = \cos^{-1}(1) = 2m\pi$

hence we can get results for $T = 2\pi$ by setting $l = \pi$

$\therefore \boxed{T = 2l}$

Hence we must conclude

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases} \quad \delta_{mn}\pi$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases} \quad \delta_{mn}\pi$$

also $\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$ for any m, n.

and $\int_{-\pi}^{\pi} \cos mx \, dx = 0$ for $m \neq 0$

& for $m=0$ $\int_{-\pi}^{\pi} dx = 2\pi$

We are now ready for a Fourier Series Expansion.

The Dirichlet's Theorem

→ over

Dirichlet's Theorem.

If, for the interval $[-\pi, \pi]$, the fn $f(x)$ is:

1) single-valued
2) bounded
3) has at most a finite # of maxima/minima.
4) has only a finite # of discontinuities.

   i.e, is piecewise continuous.

5) $f(x+2\pi) = f(x)$ for values of x outside of $[-\pi, \pi]$

Then $\boxed{S_p(x) = a_0/2 + \sum_{n=1}^{p} [a_n \cos nx + b_n \sin nx]}$

converges to $f(x)$ as $p \longrightarrow \infty$ at values of x for which $f(x)$ is continuous;

and to $\boxed{\frac{1}{2}\left[f(x+0) + f(x-0)\right]}$

at points of discontinuity

N.B. a fn' $f(x)$ is said to be <u>bounded</u> if

$$|f(x)| \le M$$

for some constant M for <u>all</u> values of x.

Now in the event of change of period

from $(-\pi, \pi)$ or period 2π.

Say
We want to change to $[-L, L]$ with period $2L$ —

How do we do it?

Write: $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\phi x + b_n \sin\phi x$

where
ϕ is to be determined.

hence $f(x+2L) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\phi(x+2L) + b_n \sin\phi(x+2L)$

$$= a_0/2 + \sum_{n=1}^{\infty} a_n \{\cos\phi x \cos\phi 2L - \sin\phi x \sin\phi 2L\}$$
$$+ b_n \{\sin\phi x \cos\phi 2L + \cos\phi x \sin\phi 2L\} = f(x)$$

We will succeed if we set $\left.\begin{array}{l}\cos\phi 2L = 1 \\ \sin\phi 2L = 0\end{array}\right\}$

i.e.

$$f(x+2L) = f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\phi x + b_n \sin\phi x.$$

$\left.\begin{array}{l}\cos\phi 2L = 1 \\ \sin\phi 2L = 0\end{array}\right\}$ both can be satisfied.

if we set $\phi 2L = n2\pi$

$$\Rightarrow \phi = \frac{2\pi n}{2L} = \frac{n\pi}{L}.$$

$[-L, L]$ $2L$ = period

$$\therefore f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}.$$

Mathematical Physics.

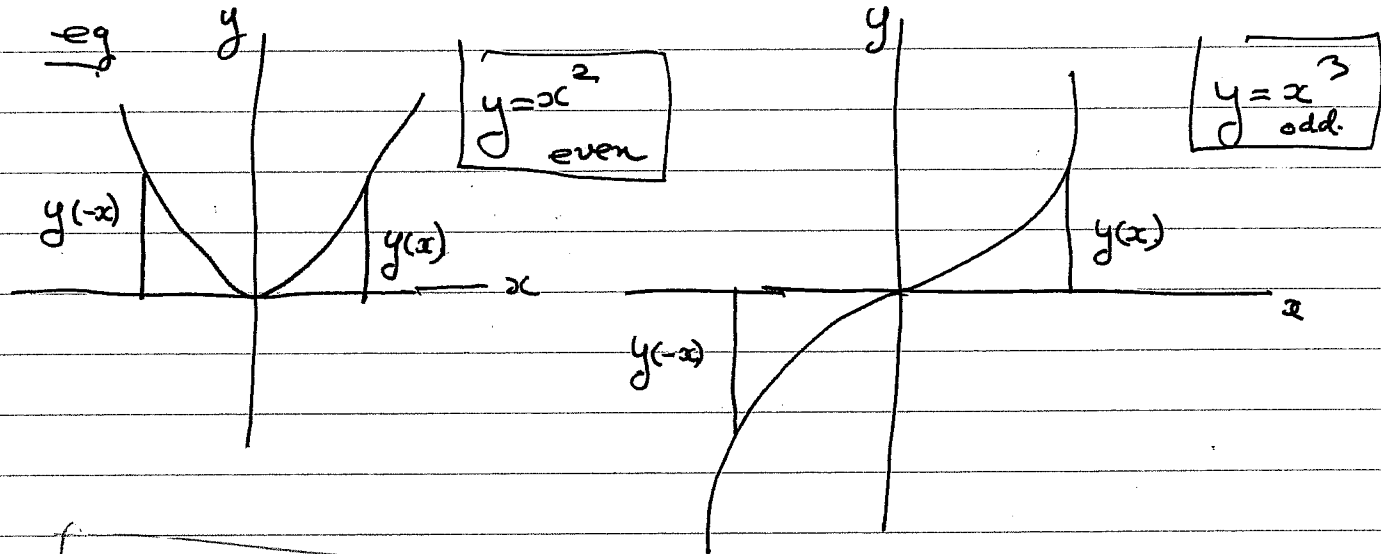
Lecture, Monday, November 26, 2001

Recall even/odd fn's

Even fn': $g(-x) = g(x)$.

Odd fn: $g(-x) = -g(x)$

eg



Also, recall $\int_{-L}^{L} g(x)\,dx = 0$ for $g(x)$ odd

& $\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$ for $g(x)$ even

Problem of a non-symmetric expansion interval

Suppose we are interested in $f(x)$ defined on a non-symmetrical interval: say $0 < x < L$

How do we calculate the Fourier Series for such a fn?

[N.B. In practice this is what ~~is~~ is needed]

Answer: We extend the fn' so that it is defined over the symmetrical interval $-L < x < L$. But,

How do we define the extension for $f(x)$, on $-L < x < 0$?

The Answer: we can define the extension to be any fn. over $-L < x < 0$ we choose as long as the extension and its derivatives fit the Dirichlet conditions — are piecewise continuous a la Dirichlet —

No matter what piecewise continuous fn. we define as the extension over $-L < x < 0$.

The Fourier series expansion will equal the given fn, $f(x)$, at all points of continuity over $\boxed{0 < x < L}$ (also the appropriate values at pts of discontinuity. (See Dirichlet –)

Of course, the Fourier series will also converge to whatever extension we have chosen for $-L < x < 0$.

There are two special extensions that are especially useful – whose Fourier coefficients are easy to calculate –

These are the so-called even/odd extensions of $f(x)$.

$\longrightarrow$ over

Even Extension $\Longleftrightarrow$ Cosine Series

Suppose $f(x)$ is specified on the

interval $0 < x < L$.

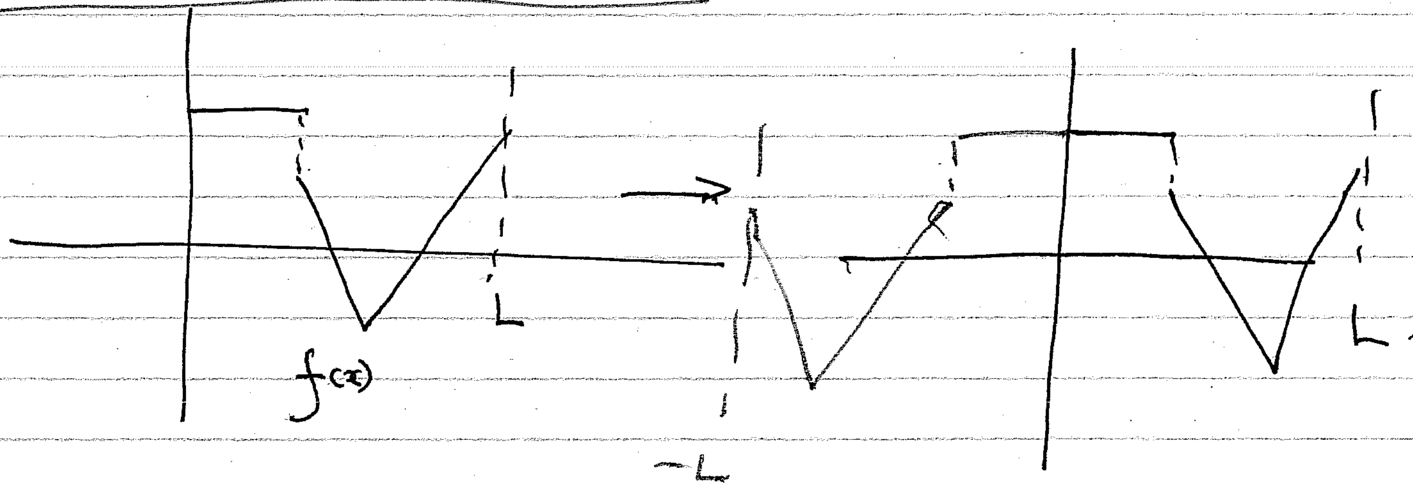
We define the even extension of $f(x)$,

by requiring that $f(-x) = f(x)$

on $-L < x < L$

Class: hand-out pg 4

No B_0



for this discussion.

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx.$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{f(x) \sin \frac{n\pi x}{L}}_{\substack{\text{odd} \\ \text{integrand}}} dx = 0$$

Hence the Fourier series for $f(x)$ becomes,

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

Because the Fourier Coefficients b_n are all zero, no sine terms appear in the Fourier expansion; the series is called the $\boxed{\text{Fourier Cosine Series}}$ of $f(x)$

N.B. It converges to the original fn $f(x)$ over the interval $0 < x < L$ & to its **Even** extension over the interval $-L < x < 0$.

Hence, we write:

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where $a_0 = \frac{(2)}{L} \int_0^L f(x)\,dx$.

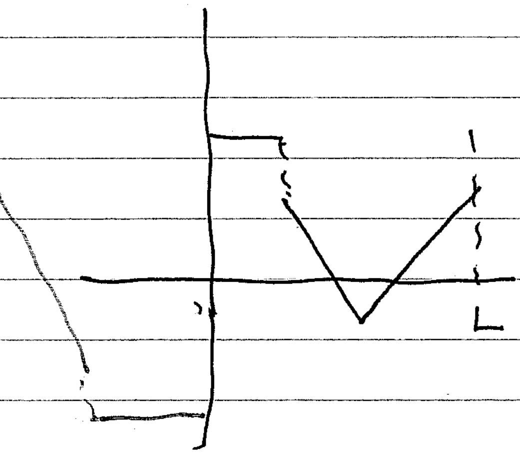
& $a_n = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{L}\,dx$

Called a Fourier Cosine Series

Over ⟶

Odd Extension: Fourier Sine Series

Once, again, consider the fn' $f(x)$ specified on the interval $0 < x < L$. We define the odd extension by requiring $f(-x) = -f(x)$ on $-L < x < L$.



$$f(x) = a_0/2 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

here however

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = 0 \qquad f(x) \text{ is odd on } -L < x < L$$

$$a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{f(x) \cos \frac{n\pi x}{L}}_{\text{odd integrand}} dx = 0 \qquad \text{on } -L < x < L$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{f(x) \sin \frac{n\pi x}{L}}_{\text{even}} dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\,dx$$

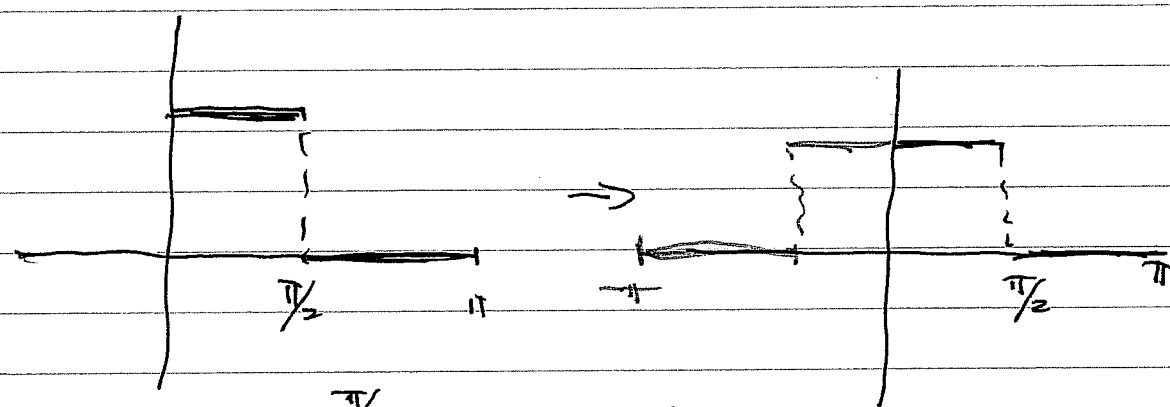
hence in this case $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

This series contains only sin terms; hence it is called a <u>Fourier Sine Series</u>.

Example:
Given

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$

Obtain even extension, i.e. a Cosine series expansion



Then $a_0 = \frac{2}{\pi}\int_0^{\pi/2} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2}(1)\,dx.$

$$= \frac{2}{\pi}\, x\Big|_0^{\pi/2} = 1$$

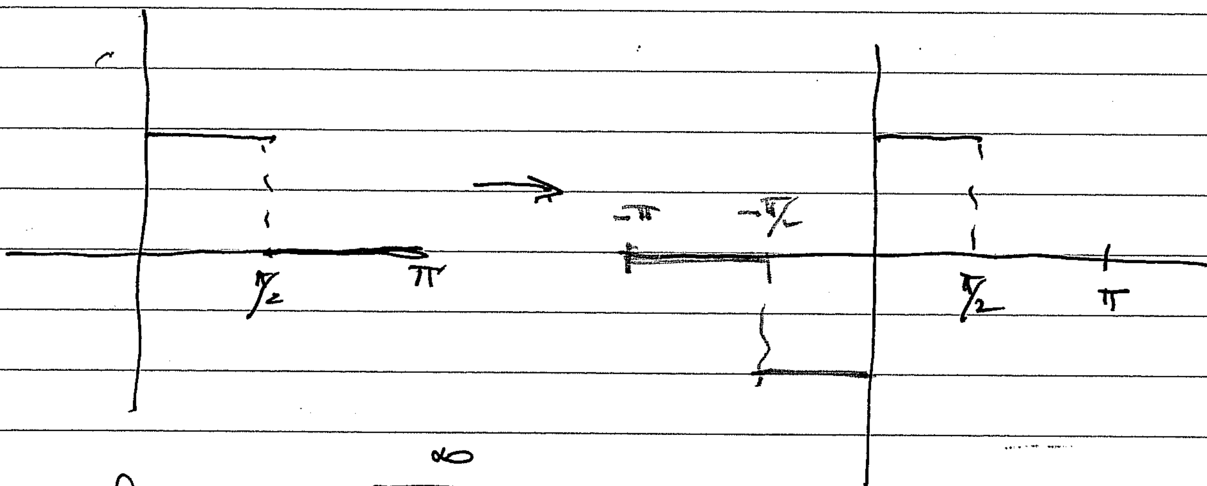
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{n\pi x}{\pi} = \frac{2}{\pi}\int_0^{\pi/2}(1)\cos nx\,dx = \frac{2}{\pi}\,\frac{\sin nx}{n}\Big|_0^{\pi/2}$$

$$= \frac{2}{\pi n}\sin n\pi/2$$

$$\therefore f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= 1/2 + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n} \sin n\pi/2\right) \cos nx.$$

Now lets get the odd extension Sine Series



here $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$ $\quad L = \pi$.

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin \frac{n\pi x}{\pi} \, dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi/2} (1) \sin nx \, dx = \frac{-2}{\pi n} \cos nx \Big|_0^{\pi/2}$$

$$= \frac{-2}{\pi n} \left(\cos n\pi/2 - 1\right)$$

$$= \frac{2}{\pi n} \left(1 - \cos n\pi/2\right)$$

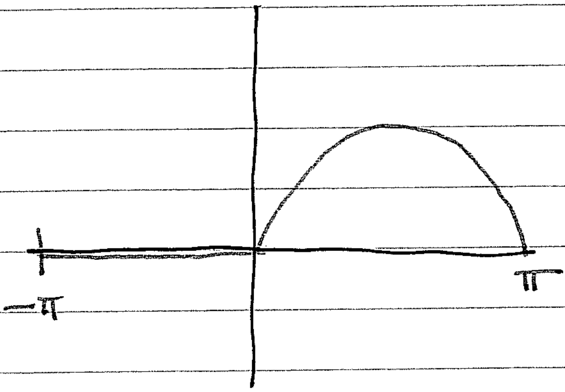
$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n} \left[1 - \cos \frac{n\pi}{2}\right] \sin nx$$

Mathematical Physics

Lecture, Mon/Wed Nov. 26/28, 2001

Examples

Given $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$.



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx \qquad a_0 = 2/\pi$$

$$= \frac{-\cos x}{\pi}\Big|_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

SI.

$$\int \sin px \cos qx\,dx = -\frac{\cos(p-q)x}{2(p-q)} - \frac{\cos(p+q)x}{2(p+q)}$$

in our case $a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = -\frac{1}{\pi}\left[\frac{\cos(1-n)x}{2(1-n)} + \frac{\cos(1+n)x}{2(1+n)}\right]_0^{\pi}$

We must exclude case $n=1$.

for $n \geq 2$

$$a_n = -\frac{1}{\pi}\left\{\frac{1+\cos n\pi}{n^2-1}\right\}$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \frac{\sin 2x \, dx}{2} = -\frac{1}{\pi}\frac{\cos 2x}{4}\Bigg|_0^{\pi}$$

$= 0$ as required

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

S.I.

$$\int \sin px \sin qx \, dx = \frac{\sin(p-q)x}{2(p-q)} - \frac{\sin(p+q)x}{2(p+q)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = \frac{1}{\pi}\left[\frac{\sin(1-n)x}{2(1-n)} - \frac{\sin(1+n)x}{2(1+n)}\right]\Bigg|_0^{\pi} = 0$$

for $n \neq 1$

Now look at $n=1$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi}\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\right] = 1/2 \Rightarrow b_1 = 1/2$$

put it together

$$\therefore \quad f(x) = \frac{1}{\pi} - \frac{1}{\pi} \sum_{n=2} \left(\frac{1 + \cos n\pi}{n^2 - 1} \right) \cos nx \quad + \frac{1}{2} \sin x.$$

$$= \frac{1}{\pi} - \frac{2}{\pi} \left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \cdots \right) + \frac{1}{2} \sin x.$$