Mathematical Physics

$\boxed{\text{Lecture, Friday, January 4, 2002}}$

Fourier Transforms

x, k; x, α; t, ω — Fourier Transform variables

Recall:

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du$$

$$\text{or } g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

u, x are dummies.

&

$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha \\ g(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \end{aligned}}$$

$\frac{1}{2\pi}$, appear in front of either one or $\frac{1}{\sqrt{2\pi}}$ in front of each one

also $\cancel{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\left[\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx\right] d\alpha}$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\left[\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du\right] d\alpha$$

to be compared to

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}}\, dx \\ f(x) &= \cancel{\tfrac{1}{2\pi}} \int_{-\infty}^{\infty} g(\alpha)\, e^{i\alpha x}\, d\alpha \qquad g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx. \end{aligned}}$$

Even/odd, cosine/sine transforms.

We showed that if $f(x)$ is odd, then $g(\alpha)$ is odd also.

Likewise, if $f(x)$ is even, then $g(\alpha)$ is even also

hence, we write.

$$\boxed{\begin{cases} f_s(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty g_s(\alpha) \sin \alpha x \, d\alpha \\ g_s(\alpha) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f_s(x) \sin \alpha x \, dx \end{cases}}$$

t, ω

x, k

& likewise

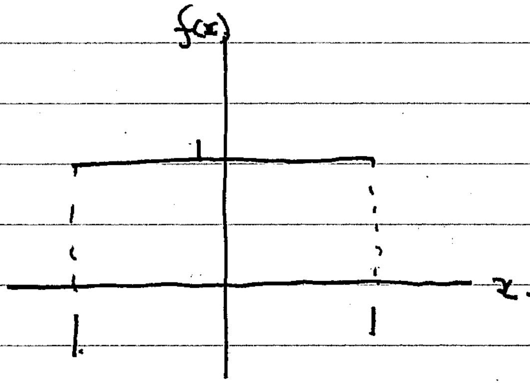
$$\boxed{\begin{aligned} f_c(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\alpha) \cos \alpha x \, d\alpha \\ g_c(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(x) \cos \alpha x \, dx \end{aligned}}$$

Lecture, Friday, January 4, 2002 — Fourier Transforms

Let us represent a non-periodic fn as a Fourier Integral

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| > 1 \end{cases}$$

See pg 252 hand-out



This fn might represent an impulse in mechanics {i.e. a force applied only over a short time such as a bat hitting a baseball}, or a sudden short surge of current in electricity, or a short pulse of sound or light which is not repeated.

Since the fn is not periodic, it cannot be expanded in a Fourier series, since a Fourier series always represents a periodic fn.

Instead, we write $f(x)$ as a Fourier integral as follows:-

find $g(\alpha)$: this is like finding the c_n's for a Fourier series

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{2\pi}\int_{-1}^{1} (1)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{2\pi} \left. \frac{e^{-i\alpha x}}{-i\alpha} \right|_{-1}^{1}$$

$$= \frac{1}{\pi\alpha}\left(\frac{e^{-i\alpha} - e^{i\alpha}}{-2i}\right)$$

→ over

or $g(\alpha) = \frac{1}{\pi\alpha}\left(\frac{e^{i\alpha}-e^{-i\alpha}}{2i}\right)$

$$g(\alpha) = \frac{\sin\alpha}{\pi\alpha}$$

Now substitute this back for $f(x)$ [this is like substituting the evaluated coefficients into a Fourier Series

i.e. $f(x) = \int_{-\infty}^{\infty} \frac{\sin\alpha}{\pi\alpha} e^{i\alpha x}\, d\alpha$

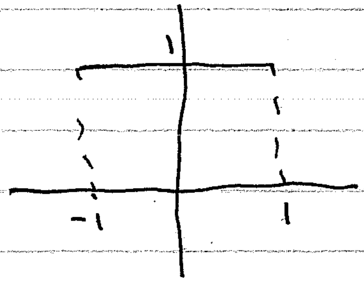
$$= \int_{-\infty}^{\infty} \frac{\sin\alpha}{\pi\alpha}\left(\cos\alpha x + i\sin\alpha x\right) d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin\alpha\cos\alpha x}{\alpha} + \underbrace{i\,\frac{\sin\alpha\sin\alpha x}{\alpha}}_{0 \text{ odd fn}}\right] d\alpha$$

but $\frac{\sin\alpha}{\alpha}$ is an even fn

Hence $f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\alpha\cos\alpha x}{\alpha}\, d\alpha$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\alpha\cos\alpha x}{\alpha}\, d\alpha \text{ represents the fn}$$



$\Longleftrightarrow \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\alpha\cos\alpha x}{\alpha}\, d\alpha = f(x)$

We can use $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha \cos\alpha x \, d\alpha}{\alpha}$

to evaluate a definite integral.

$$\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ \frac{\pi}{4} & |x| = 1 \\ 0 & |x| > 1 \end{cases}$$

N.B. we have used the fact that the Fourier integral represents the midpoint of the jump in $f(x)$ at $|x| = 1$. If we let $x = 0$

we get $\int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha = \frac{\pi}{2}$ Important result

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Example: pg. 83 Spiegel 5.3

this is problem 1 in assignment

a) Find the Fourier transform of

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

b) Graph $f(x)$ and its Fourier transform for $a = 3$

drop $\frac{1}{2\pi}$ in $g(\alpha)$

a) $g(\alpha) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$

$$= \int_{-\infty}^{\infty} (1) e^{-i\alpha x} dx = \int_{-a}^{a} (1) e^{-i\alpha x} dx = \left. \frac{e^{-i\alpha x}}{-i\alpha} \right|_{-a}^{a}$$

$$= \frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} = 2\frac{\sin\alpha a}{\alpha} \qquad \alpha \neq 0$$

for $\alpha=0$ $\quad g(\alpha)=2a$

because $\quad \dfrac{2\sin\alpha a}{\alpha} = 2\left\{ \dfrac{\alpha a}{\alpha} - \dfrac{\frac{(\alpha a)^3}{3!}}{\alpha} + \cdots \right\}$.

$= 2a \quad$ if $\alpha = 0$

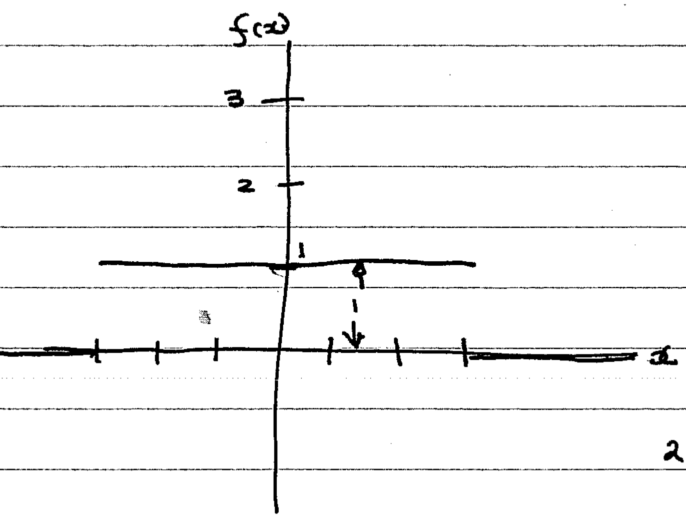
or L Hopital.

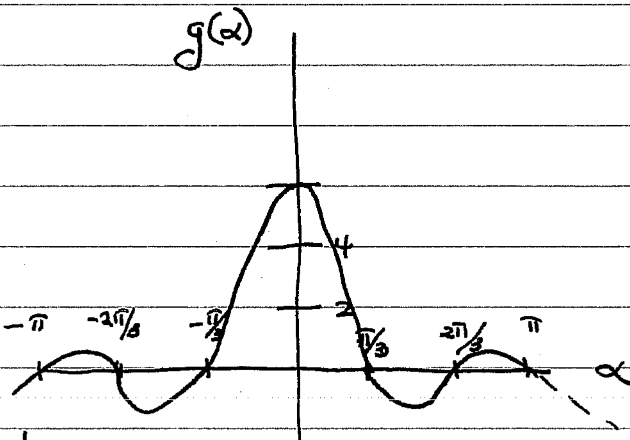
$$\frac{d}{d\alpha}\left(\frac{2\sin\alpha a}{\alpha}\right) = 2\left\{\frac{d}{d\alpha}\left(\alpha^{-1}\sin\alpha a\right)\right\}$$

$$= 2\left\{-\alpha^{-2}\sin\alpha a + \alpha^{-1}a\cos\alpha a\right\}$$

check this out at another time.

b) The graphs of $f(x)$ and $g(\alpha)$ for $a=3$ are shown





$$\left.\frac{2\sin\alpha a}{\alpha}\right|_{\alpha=\pi/3}$$

$$\frac{2\sin\frac{\pi}{3}a}{\frac{\pi}{3}} = 0$$

Examples: Spiegel pg 94

5.26 $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$

a) Sine transform
b) cosine transform.

a) $g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(x) \sin \alpha x \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (1) \sin \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^1 (1) \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} (-) \left[\frac{\cos \alpha x}{\alpha} \right] \Big|_0^1$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{\cos \alpha}{\alpha} - 1 \right]$$

$$\therefore \quad g_s(\alpha) = \sqrt{\frac{2}{\pi}} \left[1 - \frac{\cos \alpha}{\alpha} \right]$$

$$\& \quad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \sin \alpha x \, d\alpha$$

$$f(x) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty \left(1 - \frac{\cos \alpha}{\alpha} \right) \sin \alpha x \, d\alpha = 1$$

$$\Rightarrow \int_0^\infty \left(1 - \frac{\cos \alpha}{\alpha} \right) \sin \alpha x \, d\alpha = \frac{\pi}{2}$$

b) $g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(x) \cos \alpha x \, dx.$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1) \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin \alpha x}{\alpha} \right|_0^1$$

or $g_c(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\sin\alpha}{\alpha}$

hence $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\alpha) \cos\alpha x \, d\alpha.$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\alpha}{\alpha} \cos\alpha x \, d\alpha = 1$$

$$\int_0^\infty \frac{\sin\alpha \cos\alpha x \, d\alpha}{\alpha} = \frac{\pi}{2}$$

let $x=0$ $\quad \int_0^\infty \frac{\sin\alpha}{\alpha} d\alpha = \pi/2$ $\quad$ as before

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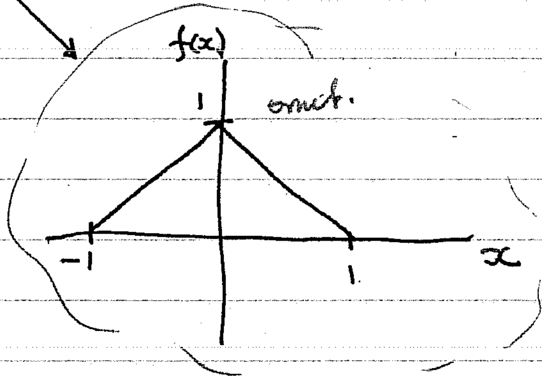
$\boxed{\text{Fletcher pg 262}}$

39. $f(x) = \begin{cases} 1-x & 0 \le x \le 1 \\ 1+x & -1 \le x \le 0 \end{cases}$

or $f(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ 1+t & -1 \le t \le 0 \end{cases}$



$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (1+x) e^{-i\alpha x} dx + \int_0^1 (1-x) e^{-i\alpha x} dx \right.$$

Let $x \to -x$ in the first integral. Save yourself some work

& obtain $\quad g(\alpha) = \frac{1}{\sqrt{2\pi}} \left[\int_0^1 (1-x) e^{i\alpha x} dx + \int_0^1 (1-x) e^{-i\alpha x} dx \right]$

$$\text{or } g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^1 (1-x)\left[e^{i\alpha x} + e^{-i\alpha x}\right] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x) \cos\alpha x \, dx$$

S.I.

$$\int x \cos ax \, dx = \frac{\cos ax}{a^2} + \frac{x}{a} \sin ax.$$

$$g(\alpha) = \sqrt{\frac{2}{\pi}} \left[\frac{\sin\alpha x}{\alpha} - \left\{ \frac{\cos\alpha x}{\alpha^2} + \frac{x}{\alpha} \sin\alpha x \right\} \right]\Bigg|_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\alpha}{\alpha} - \left[\left\{ \frac{\cos\alpha}{\alpha^2} + \frac{\sin\alpha}{\alpha} \right\} - \frac{1}{\alpha^2} \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\cancel{\frac{\sin\alpha}{\alpha}} - \frac{\cos\alpha}{\alpha^2} - \cancel{\frac{\sin\alpha}{\alpha}} + \frac{1}{\alpha^2} \right]$$

$$g(\alpha) = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos\alpha}{\alpha^2} \right)$$

As required.

Example: given $f(x) = \dfrac{x}{x^2+b^2}$ find sine transform

$$\boxed{g(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{x^2+b^2} \sin \alpha x \, dx}$$

lookup definite integral.

$$\boxed{\int_0^\infty \frac{x \sin mx \, dx}{x^2+a^2} = \pi/2 \; e^{-ma}}$$

complex variable.

$\therefore$

$$g(\alpha) = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} e^{-\alpha b} \right]$$

Example $f(x) = e^{-x} \cos x$ — find sine transform.

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos x \sin \alpha x \, dx$$

$$\sin \alpha x \cos x = \frac{1}{2} \left[\sin(\alpha-1)x + \sin(\alpha+1)x \right]$$

$\therefore$

$$g_s(\alpha) = \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty \left[e^{-x} \sin(\alpha-1)x + e^{-x} \sin(\alpha+1)x \right] dx$$

S.I. $\displaystyle \int e^{ax} \sin bx \, dx = e^{ax} \left(\frac{a \sin bx - b \cos bx}{a^2+b^2} \right)$

set $a = -1$

$b = \alpha - 1$

$$\int_0^\infty e^{-x}\sin(\alpha-1)x\,dx = e^{-x}\left(\frac{-\sin(\alpha-1)x-(\alpha-1)\cos(\alpha-1)x}{1+(\alpha-1)^2}\right)\Bigg|_0^\infty$$

$$= \frac{+(\alpha-1)}{1+(\alpha-1)^2}$$

likewise

$$\int_0^\infty e^{-x}\sin(\alpha+1)x\,dx = \frac{\alpha+1}{1+(\alpha+1)^2}$$

or

$$g_s(\alpha) = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\alpha-1}{1+(\alpha-1)^2} + \frac{(\alpha+1)}{1+(\alpha+1)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\;\frac{2\cancel{k}\alpha^3}{\alpha^4+4}$$

$$\boxed{g_s(\alpha) = \sqrt{\frac{2}{\pi}}\;\frac{\alpha^3}{\alpha^4+4}}$$

as required.

$$g(k) = e^{-\frac{k^2a^2}{4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\left(x+i\frac{ka^2}{2}\right)^2/a^2}$$

set

$$u^2 = \left(x + i\frac{ka^2}{2}\right)^2$$

we need integral:

$$I_1 = \int_0^{\infty} e^{-ax^2} dx \Rightarrow I_1 = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{2} I$$

even

Look at

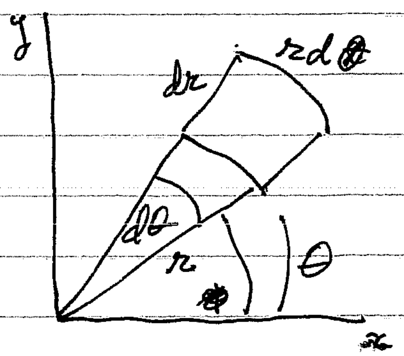
$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx\,dy.$$

polar coordinates $\quad x^2+y^2 = r^2 \qquad x = r\cos\theta$

$y = r\sin\theta$



$d\,\text{area} = r\,dr\,d\theta$

$x, y \quad -\infty \rightarrow \infty$

$r \rightarrow \infty \qquad \theta = 2\pi$

$$I^2 = \int_0^\infty r$$

$\cancel{dx = r\cos\theta\, d\theta + dr}$

$dx = -r\sin\theta\, d\theta + dr\cos\theta$

$dy = r\cos\theta\, d\theta$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dx\,dy$$

$$= \int_0^\infty e^{-ar^2} r \int_0^{2\pi} d\theta.$$

$$= 2\pi \int_0^\infty e^{-ar^2} r\,dr.$$

$u = ar^2$

$du = 2ar\,dr.$

$\Rightarrow r\,dr = \dfrac{du}{2a}.$

$$I^2 = 2\pi \int_0^\infty e^{-u} \frac{du}{2a}.$$

$$= \frac{2\pi}{2a} \int_0^\infty e^{-u} du = \frac{2\pi}{2a} \left.\frac{e^{-u}}{-1}\right|_0^\infty,$$

$$= -\frac{\pi}{a}\left(e^{-\infty} - e^{0}\right) = \frac{\pi}{a}$$

$$\therefore \quad I = \sqrt{I^2} = \sqrt{\frac{\pi}{a}}.$$

back to pg

Example. $\boxed{\text{pg 251 - hand-out - Fletcher}}$ use x, k here

Find Fourier Transform $f(x) = e^{-\frac{x^2}{a^2}}$.

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2} - ikx}\, dx.$$

Complete the square in exponent as follows.

$$\frac{x^2}{a^2} + ikx = \frac{x^2}{a^2} + ikx - \frac{k^2a^2}{4} + \frac{k^2a^2}{4}$$

$$= \left(\frac{x}{a} + \frac{ika}{2}\right)^2 + \frac{k^2a^2}{4}$$

check this out! $= \frac{x^2}{a^2} - \frac{k^2a^2}{4} + \frac{2xkai}{2a} + \frac{k^2a^2}{4}$

$$= \frac{x^2}{a^2} - \frac{k^2a^2}{4} + ixk + \frac{k^2a^2}{4}.$$

Hence we replace $\frac{x^2}{a^2} + ikx = \left(\frac{x}{a} + i\frac{ka}{2}\right)^2 + \frac{k^2a^2}{4}$

or

$$\boxed{\frac{x^2}{a^2} + ikx = \frac{1}{a^2}\left(x + i\frac{ka^2}{2}\right)^2 + \frac{k^2a^2}{4}.}$$

$$\therefore \quad g(k) = e^{-\frac{k^2a^2}{4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{\left(x+i\frac{ka^2}{2}\right)^2}{a^2}}$$

$$u = x + \frac{ika^2}{2}$$

$$du = dx.$$

$$\therefore \quad g(k) = e^{-\frac{k^2a^2}{4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{a^2}}.$$

$$= e^{-\frac{k^2a^2}{4}} \frac{1}{\sqrt{2\pi}} (2) \int_{0}^{\infty} du\, e^{-\frac{u^2}{a^2}}$$

$$\cancel{I = \int_0^\infty du\, e^{-u^2}} \qquad \int_0^\infty du\, e^{-\frac{u^2}{a^2}} = \frac{\sqrt{\pi a^2}}{2}$$

$$w = \frac{u^2}{a^2}$$

Mathematical Physics

Lecture: Monday, January 7, 2002

Finish up Fourier Transforms — do Gaussian as an example.

$\boxed{\text{First evaluate:}}$

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$$

square above

$$I^2 = \int_{-\infty}^{\infty} e^{-ax^2}\,dx \int_{-\infty}^{\infty} e^{-ay^2}\,dy. = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,dx\,dy.$$

$\Rightarrow$ integration over entire x-y plane.

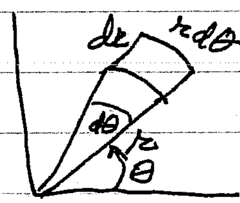
N.B.

transform to polar coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad r^2 = x^2+y^2$

$dx\,dy = d(\text{area})$

in polar $d\,\text{area} = r\,d\theta\,dr = r\,dr\,d\theta.$



hence $I^2 = \int_0^{\infty} r e^{-ar^2}\,dr \int_0^{2\pi} d\theta$

$$I^2 = 2\pi \int_0^{\infty} r e^{-ar^2}\,dr$$

Now write $w = ar^2 \Rightarrow dw = 2ar\,dr$

or

$$r\,dr = \frac{dw}{2a}$$

hence

$$I^2 = 2\pi \int_0^\infty e^{-w} \frac{dw}{2a} = \frac{2\pi}{2a} \int_0^\infty e^{-w} dw$$

$$= \frac{\pi}{a} \left. \frac{e^{-w}}{-1} \right|_0^\infty$$

$$= -\frac{\pi}{a}\left(e^{-\infty} - e^{-0}\right) = \frac{\pi}{a}.$$

$$\therefore I^2 = \pi/a \Rightarrow I = \sqrt{\frac{\pi}{a}}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.} \quad \text{or} \quad \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Problem: Now given $f(x) = e^{-\frac{x^2}{a^2}}$, find Fourier transform

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{a^2}} e^{-ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\left(\frac{x^2}{a^2} + ikx\right)}$$

we can put this into recognizable form as follows:

Look at exponent and complete the square

$$\frac{x^2}{a^2}+ikx = \frac{x^2}{a^2}+ikx - \frac{k^2a^2}{4} + \frac{k^2a^2}{4}$$

$$= \left(\frac{x}{a} + i\frac{ka}{2}\right)^2 + \frac{k^2a^2}{4}$$

or finally $= \frac{1}{a^2}\left(x+i\frac{ka^2}{2}\right)^2 + \frac{k^2a^2}{4}$

put back into integral

$$g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\left\{\left[\frac{1}{a^2}\left(x+i\frac{ka^2}{2}\right)\right]^2\right\}} e^{-\frac{k^2a^2}{4}}$$

$$= e^{-\frac{k^2a^2}{4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\left[x+i\frac{ka^2}{2}\right]^2/a^2}$$

Now set $u = x+i\frac{ka^2}{2}$ $\quad du = dx$

$$g(k) = e^{-\frac{k^2a^2}{4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{a^2}}$$

Now look at $\int_{-\infty}^{\infty} du\, e^{-\frac{u^2}{a^2}} = \int_{-\infty}^{\infty} du\, e^{-\frac{au^2}{a^3}}$ $\quad$ set $v = \frac{u}{\sqrt{a^3}}$

$dv = \frac{du}{\sqrt{a^3}}$

Hence $\int_{-\infty}^{\infty} du e^{-\frac{au^2}{a^3}} = \int_{-\infty}^{\infty} \sqrt{a^3}dv e^{-v^2a} = \sqrt{a^3}\int_{-\infty}^{\infty} dv e^{-v^2a}$ $\quad du = \sqrt{a^3}\,dv$

$$= \sqrt{a^3}\sqrt{\frac{\pi}{a}} = \sqrt{\pi a^2} = a\sqrt{\pi}$$

$\longrightarrow$ over

put it all together & write:

$$g(k) = e^{-\frac{k^2a^2}{4}} \frac{1}{\sqrt{2\pi}}\left(a\sqrt{\pi}\right)$$

$$\boxed{g(k) = \frac{a}{\sqrt{2}} e^{-\frac{k^2a^2}{4}}} \checkmark$$

Now get the inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{a}{\sqrt{2}} e^{-\frac{k^2a^2}{4}} e^{ikx}\, dk.$$

$$= \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{k^2a^2}{4}\right)+ikx}\, dk = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{k^2a^2}{4} - ikx\right]}\, dk$$

Complete the square in exponent

$$\cancel{+\frac{k^2a^2}{4} - ikx + \frac{x^2}{a^2} - \frac{x^2}{a^2} = \left(\frac{ka}{2} - \frac{ix}{a}\right)^2 + \frac{x^2}{a^2}.}$$

$$\frac{k^2a^2}{4} - ikx = \frac{k^2a^2}{4} - ikx + \frac{x^2}{a^2} - \frac{x^2}{a^2}.$$

$$= \left(\frac{ka}{2} - \frac{ix}{a}\right)^2 + \frac{x^2}{a^2}$$

or $f(x) = \dfrac{a}{2\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \displaystyle\int_{-\infty}^{\infty} e^{-\left(\frac{ka}{2} - \frac{ix}{2}\right)^2} dk$

set $k' = \dfrac{ka}{2} - \dfrac{ix}{2}$.

$dk' = dk \dfrac{a}{2}$

$\therefore dk = \dfrac{2}{a} dk'$

look at

$$\int_{-\infty}^{\infty} e^{-\left(\frac{ka}{2} - \frac{ix}{2}\right)^2} dk. = \int_{-\infty} \frac{2}{a} dk' \, e^{-k'^2}$$

$$= \frac{2}{a} \int_{-\infty}^{\infty} dk' \, e^{-a\frac{k'^2}{a}}$$

set $v = \dfrac{k'}{\sqrt{a}}$

$dv = \dfrac{dk'}{\sqrt{a}}$

$dk' = \sqrt{a}\, dv.$

$$= \frac{2}{a} \int_{-\infty}^{\infty} \sqrt{a}\, dv \, e^{-av^2}$$

$$= \frac{2\sqrt{a}}{a} \sqrt{\frac{\pi}{a}} = \frac{2}{a} \sqrt{\frac{\pi a}{a}} = \frac{2}{a} \sqrt{\pi}$$

$\therefore$

$$f(x) = \frac{a}{2\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \left[\frac{2}{a} \sqrt{\pi} \right] = e^{-\frac{x^2}{a^2}}$$

as required

Mathematical Physics.

Lecture: Begin hopefully on Monday, Jan 7, 2002

$\boxed{\text{2}^{nd}\text{ order Linear differential equations}}$

The General form for a 2^{nd} order linear diff eq $\boxed{\text{homogeneous.}}$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

[linear – dependent variable & all derivatives are raised to power 1 – only]

$\boxed{\text{Inhomogeneous}}$

$$\boxed{\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = R(x).}$$

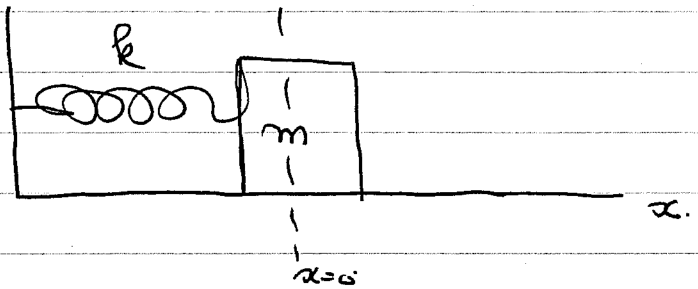
$\boxed{\text{Soln-properties for homogeneous equation}}$

1. If $f(x)$ is a soln, then $Bf(x)$ is also a soln. $B = \text{const.}$

2. If $f(x)$ and $g(x)$ are soln then:

$$Af(x) + Bg(x) \text{ is also a soln}$$

SHO



$$\vec{F} = -kx\hat{i}$$

$$\overrightarrow{\Sigma F} = m\frac{d^2x\hat{i}}{dt^2} \Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

$$\text{or } \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\text{set } \omega^2 = \frac{k}{x} \quad \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

Soln try $x(t) = e^{pt}$ p to be determined.

$$\frac{dx}{dt} = \dot{x} = pe^{pt}$$

$$\frac{d^2x}{dt^2} = \ddot{x} = p^2 e^{pt}$$

Substitute $p^2 e^{pt} + \omega^2 e^{pt} = 0$

$$e^{pt}(p^2 + \omega^2) = 0 \qquad e^{pt} \neq 0 \text{ for all time.}$$

$$\therefore p^2 + \omega^2 = 0 \Rightarrow p = \pm\sqrt{-\omega^2} = \pm i\omega$$

$$\therefore x(t) = ae^{i\omega t} + be^{-i\omega t}$$ but $x(t) =$ is real

$$\text{set } a = \frac{A}{2}e^{i\theta}$$

$$b = \frac{A}{2}e^{-i\theta}$$

→ over

hence

$$x(t) = \frac{A}{2} e^{i(\omega t+\theta)} + \frac{A}{2} e^{-i(\omega t+\theta)}$$

$$= A\left[\frac{e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta)}}{2}\right]$$

$$\boxed{x(t) = A\cos(\omega t+\theta)}$$

Note: 2 constants of integration A, θ

how to get A, θ

of course we may re-write above as follows

$$x(t) = A\{\cos\omega t\cos\theta - \sin\omega t\sin\theta\}$$

$$= (A\cos\theta)\cos\omega t + (-A\sin\theta)\sin\omega t$$

Set $A' = A\cos\theta$
$B' = -A\sin\theta$

$\therefore \boxed{x(t) = A'\cos\omega t + B'\sin\omega t}$

how to get A', B'

Let us do the physics of the S.H.O.

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

The system under consideration is Conservative.

$\therefore$ $K + U = $ Constant. Conservation of Mechanical Energy

$\therefore$ $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E = $ constant

or $\boxed{\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx = E}$

$\rightarrow$ over

before doing Energy —

how to get A, θ, A', B'

$$\boxed{\begin{array}{l} A, \theta \\ \text{Soln } x(t) = A\cos(\omega t + \theta). \\ \text{Given at } t=0 \quad x(0) = x_0 \\ \qquad\qquad\qquad\quad v(0) = v_0 \end{array}}$$

Initial Conditions

$$x = A\cos(\omega t + \theta)$$

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \theta)$$

$t=0$ $\quad x(0) = x_0 = A\cos\theta \qquad (1)$

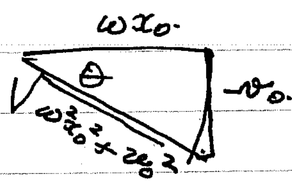
$$v(0) = -\omega A \sin\theta = v_0 . \quad (2)$$

(2) $\quad \sin\theta = -\dfrac{v_0}{\omega A}$

(1) $\quad \cos\theta = \dfrac{x_0}{A}$

$\therefore$

$$\tan\theta = \frac{\frac{-v_0}{\omega A}}{\frac{x_0}{A}} = \frac{-v_0}{\omega \cancel{A}} \cdot \frac{A}{x_0} = \frac{-v_0}{\omega x_0}$$



$$\boxed{\tan\theta = \frac{-v_0}{\omega x_0}}$$

from (1) $\quad A = \dfrac{x_0}{\cos\theta}$

$$= \frac{x_0 \sqrt{\omega^2 x_0^2 + v_0^2}}{\omega x_0}$$

$$A = \frac{\sqrt{\omega^2 x_0^2 + v_0^2}}{\omega}$$

$$\cos\theta = \frac{\omega x_0}{\sqrt{\omega^2 x_0^2 + v_0^2}}$$

$$\sin\theta = \frac{-v_0}{\sqrt{\omega x_0^2 + v_0^2}}$$

Now write

$$x(t) = A'\cos\omega t + B'\sin\omega t.$$

$t=0 \quad x(0) = x_0 = A'$

$$v(t) = -A'\omega\sin\omega t + B'\omega\cos\omega t.$$

$v(0) = v_0 = B'\omega \quad \Rightarrow B' = v_0/\omega$

$$\therefore \boxed{x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t.}$$

but $A' = A\cos\theta = \sqrt{\frac{\omega x_0^2 + v_0^2}{\omega}} \frac{\omega x_0}{\sqrt{\omega x_0^2 + v_0^2}} = x_0$ ✓

$B' = -A\sin\theta = \left(-\sqrt{\frac{\omega x_0^2 + v_0^2}{\omega}}\right)\left(\frac{-v_0}{\sqrt{\omega x_0^2 + v_0^2}}\right) = \frac{v_0}{\omega}.$ ✓

checks out.

The Energy Equation

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E.$$

$$\tfrac{1}{2}mv^2 = E - \tfrac{1}{2}kx^2$$

$$v^2 = \frac{2E}{m} - \frac{\cancel{2}kx^2}{\cancel{2}m}$$

$$=$$

$$v^2 = \frac{2}{m}\left(E - \tfrac{1}{2}kx^2\right)$$

$$= \frac{2E}{m} - \frac{k}{m}x^2.$$

$$v^2 = \frac{2E}{m} - \omega^2 x^2 =$$

$$v = \pm\omega\sqrt{\frac{2E}{m\omega^2} - x^2}$$

$$v = \pm\omega\sqrt{\frac{2E}{m\omega^2} - x^2}.$$

$$\frac{dx}{dt} = \pm\omega\sqrt{\frac{2E}{m\omega^2} - x^2}$$

$$\frac{dx}{\pm\sqrt{\frac{2E}{m\omega^2} - x^2}} = \omega\, dt$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2E}{m\omega^2} - x^2}} = \omega\int_0^t dt$$

$t>0$

choose + sign

SI

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}.$$

$$\therefore \quad \sin^{-1}\frac{x}{a}\Big|_{x_0}^{x} = \omega t$$

$$\sin^{-1}\frac{x}{a} - \sin^{-1}\frac{x_0}{a} = \omega t.$$

$$\phi = \sin^{-1}\left(\frac{x_0}{a}\right)$$

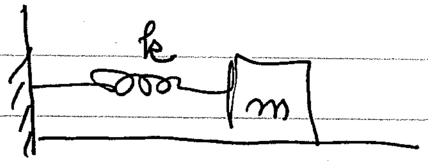
$$\sin^{-1}\frac{x}{a} = \omega t + \sin^{-1}\frac{x_0}{a}$$

$$\therefore \quad \frac{x}{a} = \sin(\omega t + \phi)$$

$$a = \sqrt{\frac{2E}{m\omega^2}}$$

$$\boxed{x = a \sin(\omega t + \phi)}$$

Damping



one-dimen

$F = -kx - bv$ $\qquad$ $F_{friction} \cong \cdot v$ simplest

$$m\frac{d^2x}{dt^2} + bv + kx = 0$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0} \qquad \gamma = \frac{b}{2m}.$$

Assume soln $x = e^{pt}$.

Lecture, Wednesday, January 9, 2002

SHO via Energy & comparison with same initial conditions.

$\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = E$ Conservative System.

Solve $v^2 = \frac{2}{m}\left(E - \frac{1}{2} k x^2\right)$ $\omega^2 = \frac{k}{m}$

$= \frac{2E}{m} - \frac{k}{m} x^2$

$\boxed{\begin{array}{l} \text{at } t=0 \quad x(0) = x_0. \\ v(0) = v_0 \end{array}}$

$= \frac{2E}{m} - \omega^2 x^2$

$v^2 = \omega^2\left[\frac{2E}{m\omega^2} - x^2\right]$

or

$v = \pm\,\omega\sqrt{\frac{2E}{m\omega^2} - x^2}$ let $a = \sqrt{\frac{2E}{m\omega^2}}$.

$v = \pm\,\omega\sqrt{a^2 - x^2}$

but $v = \frac{dx}{dt} = \pm\,\omega\sqrt{a^2 - x^2}$.

or

$\frac{dx}{\pm\sqrt{a^2 - x^2}} = \omega\, dt$

but $t \geq 0$. drop − sign lhs

& write

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \omega \int dt + \phi$

constant of integration

S.I

(I) (2)

$$S.I = \int \frac{dx}{\sqrt{b^2 - x^2}} = \sin^{-1}\left(\frac{x}{b}\right)$$

in our case $b = a$

hence $\sin^{-1}\left(\frac{x}{a}\right) = \omega t + \phi$

Constant of integration

$$x = a\sin(\omega t + \phi).$$

at $t = 0$ $x(0) = x_0$

$v(0) = v_0$

$$x(0) = x_0 = a\sin\phi \quad (1)$$

$$v(t) = \omega a\cos(\omega t + \phi)$$

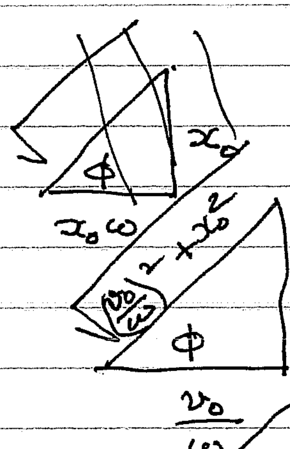
$$v(0) = v_0 = \omega a\cos\phi \quad (2)$$

$$a\sin\phi = x_0.$$

$$a\cos\phi = v_0/\omega.$$

$$\boxed{\tan\phi = \frac{x_0\omega}{v_0}}$$

$$\tan\phi = \frac{x_0}{\frac{v_0}{\omega}}$$



$$\sin\phi = \frac{x_0}{\sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2}}$$

$$\cos\phi = \frac{v_0/\omega}{\sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2}}$$

Look at 4

$$a = \frac{x_0}{\sin\phi} = \frac{x_0}{\dfrac{x_0}{\sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^2}}}$$

$$\boxed{\boxed{a = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}.} \quad \boxed{\phi = \tan^{-1}\left(\frac{x_0}{\frac{v_0}{\omega}}\right)}}$$

hence $\boxed{x = a\sin(\omega t + \phi)}$

we should compare this to soln $x = A\cos(\omega t + \theta)$

where $A = \sqrt{\dfrac{\omega^2 x_0^2 + v_0^2}{\omega^2}}$

$$A = \frac{\sqrt{\omega^2 x_0^2 + v_0^2}}{\omega}.$$

$$\tan\theta = \frac{-v_0}{\omega x_0}$$

Compare $a = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2} = \frac{\sqrt{x_0^2\omega^2 + v_0^2}}{\omega}$

$$A = \frac{\sqrt{x_0^2\omega^2 + v_0^2}}{\omega}$$ ✓ checks out.

but $\boxed{\tan\theta = \frac{-v_0}{\omega x_0}}$

$\boxed{\tan\phi = \frac{x_0}{\frac{v_0}{\omega}} = \frac{x_0\omega}{v_0}}$

these still have to check out.

Recall.

$$x = A\cos(\omega t + \theta)$$

$$A = \sqrt{\frac{\omega^2 x_0^2 + v_0^2}{\omega^2}} = \frac{\sqrt{\omega^2 x_0^2 + v_0^2}}{\omega}$$

$$\tan\theta = \frac{-v_0}{\omega x_0}$$

Now we have found

$$x = a\sin(\omega t + \phi)$$

$$a = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2} = \frac{\sqrt{v_0^2 + x_0^2\omega^2}}{\omega^2}$$

$$\tan\phi = \frac{x_0}{\frac{v_0}{\omega}} = \frac{x_0\omega}{v_0}$$

we find $a = A$ okay!

but $\tan\theta = \dfrac{-v_0}{\omega x_0}$

$\tan\phi = \dfrac{x_0 \omega}{v_0}$ } check these out.

Look at $x = a\sin(\omega t + \phi)$

$x = A\cos(\omega t + \theta)$

but $\sin\left(\alpha + \pi/2\right) = \sin\alpha\cos\pi/2 + \cos\alpha\sin\pi/2 = \cos\alpha.$

Hence $x = a\sin\left(\omega t + \phi + \pi/2\right) = A$

$$x = A\cos(\omega t + \theta) = a\sin\left(\omega t + \phi + \pi/2\right).$$

$$\text{or } \theta = \phi + \pi/2.$$

$$\tan\theta = \frac{\sin(\phi + \pi/2)}{\cos(\phi + \pi/2)} = \frac{\sin\phi\cos\pi/2 + \cos\phi\sin\pi/2}{\cos\phi\cos\pi/2 - \sin\phi\sin\pi/2}$$

$$\tan\theta = -\cot\phi$$

$$\Downarrow$$

$$\frac{-v_0}{\omega x_0} = -\frac{1}{\dfrac{x_0\omega}{v_0}} = -\frac{v_0}{x_0\omega}$$

right on the button

Also

$$a = \sqrt{\frac{2E}{m\omega^2}}$$

$$E = \tfrac{1}{2} m v_0^2 + \tfrac{1}{2} k x_0^2 .$$

$$a = \sqrt{\frac{2\left(\tfrac{1}{2} m v_0^2 + \tfrac{1}{2} k x_0^2\right)}{m\omega^2}}$$

$$= \sqrt{\frac{m v_0^2 + k x_0^2}{m\omega^2}}$$

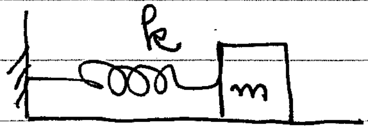
$$= \sqrt{\frac{v_0^2}{\omega^2} + \frac{k x_0^2}{m\omega^2}} = \sqrt{\frac{v_0^2}{\omega^2} + \frac{\cancel{\omega^2} x_0^2}{\cancel{\omega^2}}}$$

$$= \sqrt{\frac{v_0^2}{\omega^2} + x_0^2}$$

right on the button

Damped harmonic Oscillator

done Friday
Jan 11, 2002



$f_k \propto v$

$f_k = -bv$ opposes motion

Hence

$$\sum F = m\frac{d^2x}{dt^2} = -kx - bv$$

or $m\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + kx = 0$ write it as follows

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \gamma = \frac{b}{2m}, \quad \omega_0^2 = k/m.$$

hence we must solve

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0.$$

or $\boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0}$ 2^{nd} order linear const coeff — homogeneous —

try $x = e^{pt}$ $\dot{x} = pe^{pt}$
$\ddot{x} = p^2 e^{pt}$.

or $p^2 e^{pt} + 2\gamma p e^{pt} + \omega_0^2 e^{pt} = 0.$

$$(p^2 + 2\gamma p + \omega_0^2)e^{pt} = 0 \qquad e^{pt} \neq 0 \text{ for all } t -$$

hence $\boxed{p^2 + 2\gamma p + \omega_0^2 = 0}$ Characteristic equation

$$p = \frac{-2\gamma \pm \sqrt{(2\gamma)^2 - 4\omega_0^2}}{2}$$

$$= -\gamma \pm \sqrt{\frac{4\gamma^2}{4} - \frac{4}{4}\omega_0^2}$$

$$\boxed{p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}}$$

$p_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$

$p_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$

$$\boxed{x(t) = A e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t} + B e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t}}$$

That's fine — there is no feel for result

Look at quadratic soln — & we see three possibilities

Case I: $\boxed{\gamma^2 > \omega_0^2 \Rightarrow \text{radical real}}$ overdamped

$$p_1 = -\gamma_1 = -\gamma - (\gamma^2 - \omega_0^2)^{1/2}$$

$$p_2 = -\gamma_2 = -\gamma + (\gamma^2 - \omega_0^2)^{1/2}$$

then $\boxed{x(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}}$

Case II $\boxed{\gamma^2 < \omega_0^2 \Rightarrow \text{radical imaginary}}$ underdamped

$$p = -\gamma \pm \sqrt{-(\omega_0^2 - \gamma^2)}$$

$$= -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} \qquad \text{Set } \sqrt{\omega_0^2 - \gamma^2} = \omega_1$$

$$p = -\gamma \pm i\omega_1$$

$$\therefore \; x(t) = C_1 e^{(-\gamma + i\omega_1)t} + C_2 e^{(-\gamma - i\omega_1)t}$$

$$= e^{-\gamma t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$$

$$C_1 = \frac{A}{2} e^{i\theta} \quad C_2 = A/2\, e^{-i\theta}$$

$$\boxed{x(t) = e^{-\gamma t} A \left\{ \frac{e^{i(\omega_1 t + \theta)} + e^{-i(\omega_1 t + \theta)}}{2} \right\} = e^{-\gamma t} A \cos(\omega_1 t + \theta)}$$

Case III: $\gamma^2 = \omega_0^2$ the "critical damping"

only "one" solution $p_1 = -\gamma$

$$\therefore \quad x(t) = A e^{-\gamma t}$$

need another soln –

try $x = t e^{-\gamma t}$

$$\dot{x} = e^{-\gamma t} - \gamma t e^{-\gamma t}$$

$$\ddot{x} = -\gamma e^{-\gamma t} - \gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t}$$
$$= -2\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t}.$$

hence substitute

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$$

$$-2\gamma e^{-\gamma t} + \gamma^2 t e^{-\gamma t} + 2\gamma\left(e^{-\gamma t} - \gamma t e^{-\gamma t}\right) + \omega_0^2 t e^{-\gamma t} = 0.$$

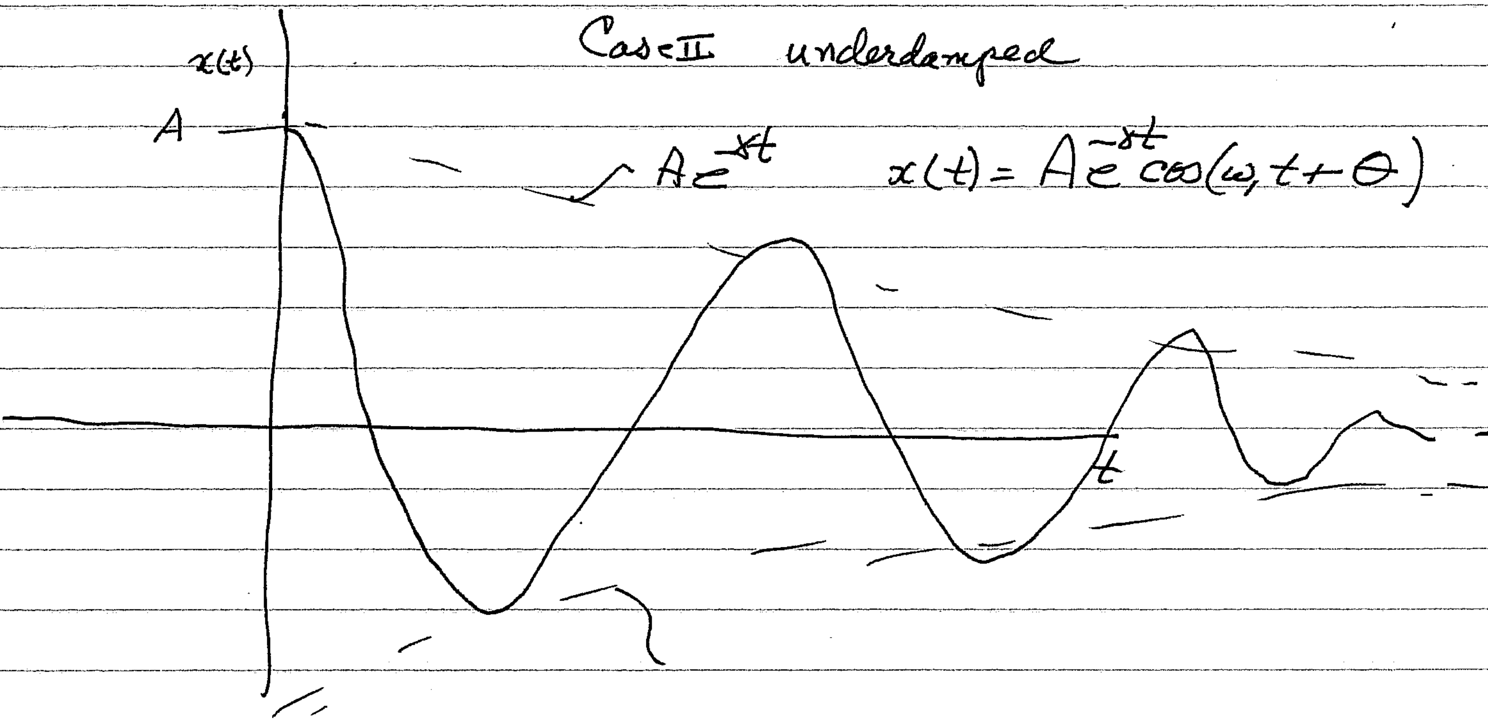
$$\cancel{-2\gamma e^{-\gamma t}} + \gamma^2 t e^{-\gamma t} \cancel{+2\gamma e^{-\gamma t}} - 2\gamma^2 t e^{-\gamma t} + \omega_0^2 t e^{-\gamma t} = 0.$$

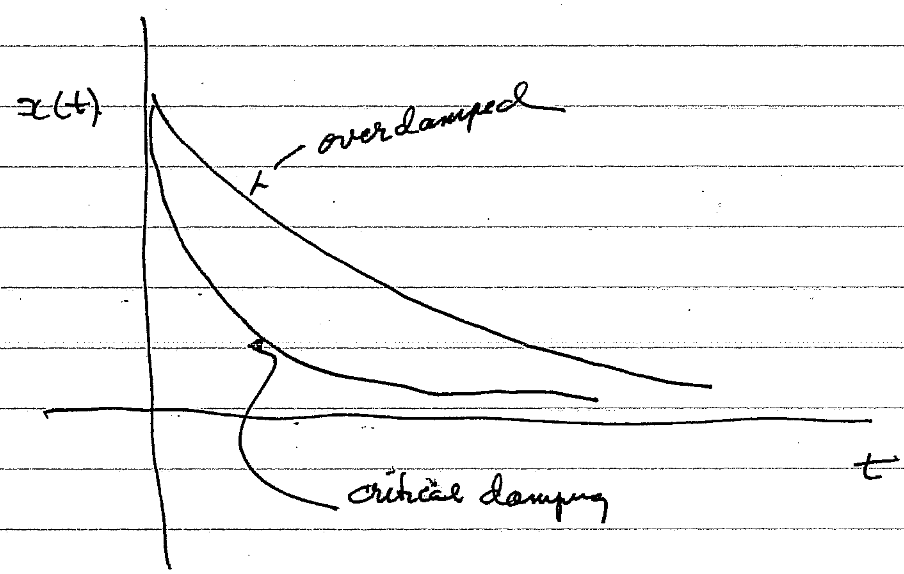
$$t e^{-t\gamma}\left[\omega_0^2 - \gamma^2\right] = 0 \qquad t e^{-\gamma t} \neq 0 \text{ for all } t.$$

$$\Rightarrow \omega_0^2 = \gamma^2 \text{ is condition}$$

right on the button

Plot $x(t)$ vs t


Case II underdamped
$x(t)$
A
$Ae^{-\gamma t}$
$x(t) = Ae^{-\gamma t}\cos(\omega_1 t + \theta)$
t


$x(t)$
overdamped
critical damping
t

January 2002

Kydon

Legendre's Differential Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$$

for the time being, set $l(l+1) = \alpha$ for simplicity

hence $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha y = 0$$

Try
$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

then $$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Substitute:
$$(1-x^2)\sum_{n=2} a_n n(n-1)x^{n-2} - 2x\sum_{n=1} a_n n x^{n-1} + \alpha \sum_{n=0} a_n x^n = 0$$

or
$$\sum_{n=2} a_n n(n-1)x^{n-2} - \sum_{n=2} a_n n(n-1)x^n - 2\sum_{n=1} a_n n x^n + \alpha\sum_{n=0} a_n x^n = 0$$

shift index in first term above & write

$$\sum_{n=0} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2} a_n n(n-1)x^n - 2\sum_{n=1} a_n n x^n + \alpha \sum_{n=0} a_n x^n = 0$$

Now take out separately the $n=0$ & $n=1$ contributions from this expression

$n=0$ $\quad 2a_2 + \alpha a_0 \quad \Rightarrow \quad (2a_2+\alpha a_0)x^0 = 0$

$n=1$ $\quad 6a_3 x - 2a_1 x + \alpha a_1 x \Rightarrow \left[6a_3 - (2-\alpha)a_1\right]x = 0$

& also

$$\sum_{n=2}\left[a_{n+2}(n+2)(n+1) - a_n n(n-1) - 2n a_n + \alpha a_n\right]x^n = 0$$

$$\Rightarrow \quad a_{n+2}(n+2)(n+1) - \left[n^2+n-\alpha\right]a_n = 0$$

$$\text{or } a_{n+2} = \frac{\left[n^2+n-\alpha\right]a_n}{(n+2)(n+1)} \qquad n=2,3,4,----$$

There are two arbitrary constants, a_0, a_1

Now put back $\alpha = \ell(\ell+1)$

We can write: $a_2 = \dfrac{-\ell(\ell+1)a_0}{2\cdot 1}$, $\quad a_3 = \dfrac{\left[2-\ell(\ell+1)\right]a_1}{3\cdot 2\cdot 1}$

& $a_{n+2} = \dfrac{\left[n^2+n-\ell(\ell+1)\right]a_n}{(n+2)(n+1)} \qquad n=2,3,---$

or

$$\boxed{a_{n+2} = \frac{(n-\ell)(n+\ell+1)}{(n+2)(n+1)}a_n}$$

Now $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x_5^5 + \cdots$

Let us obtain a number of these coefficients such that we see the pattern.

$$a_{n+2} = \frac{(n-\ell)(n+\ell+1)a_n}{(n+2)(n+1)}$$

$n=0$

$$a_2 = \frac{-\ell(\ell+1)a_0}{2\cdot 1}$$

$n=1$ $\quad a_3 = \frac{(1-\ell)(\ell+2)a_1}{3\cdot 2} = \frac{-(\ell-1)(\ell+2)a_1}{3\cdot 2}$

$n=2$

$$a_4 = \frac{(2-\ell)(\ell+3)a_2}{4\cdot 3} = \frac{(2-\ell)(\ell+3)}{4\cdot 3}\left[\frac{-\ell(\ell+1)}{2\cdot 1}\right]a_0$$

$$= \frac{(\ell-2)\ell(\ell+1)(\ell+3)a_0}{4\cdot 3\cdot 2\cdot 1}$$

$n=3$

$$a_5 = \frac{(3-\ell)(\ell+4)a_3}{5\cdot 4} = \frac{(3-\ell)(\ell+4)}{5\cdot 4}\left[-\frac{(\ell-1)(\ell+2)a_1}{3\cdot 2}\right]$$

$$= \frac{(\ell-3)(\ell-1)(\ell+2)(\ell+4)a_1}{5\cdot 4\cdot 3\cdot 2}$$

Now collect up terms involving a_0 [even powers] & a_1 [odd powers]

& write

$$y(x) = a_0\left[1 - \frac{\ell(\ell+1)}{2!}x^2 + \frac{(\ell-2)\ell(\ell+1)(\ell+3)}{4!}x^4 - \cdots\right]$$

$$+ a_1\left[x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \frac{(\ell-3)(\ell-1)(\ell+2)(\ell+4)}{5!}x^5 - \cdots\right]$$

We can show that for $|x|<1$, this series converges. however at $|x|=1$, the series diverges. We need a soln at $\pm 1 = x$ because $x = \cos\theta$ & $\theta = 0, \pi$ are perfectly good points in our spherical coordinate system.

Once again,

$$y(x) = a_0 y_1 + a_1 y_2$$

for $|x|<1$ represents the complete soln to the Legender differential equation.

for a valid soln(s) at $|x|=1$

if we choose l to be an integer, odd or even, the series representing y_1 & y_2 will terminate and we will obtain polynomials:

hence if we look at the recursion relationship

$$a_{n+2} = \frac{(n-l)(n+l+1)a_n}{(n+2)(n+1)}$$

look at $y_1 = 1 - \frac{l(l+1)}{2!}x^2 + \frac{(l-2)l(l+1)(l+3)x^4}{4!} - \ldots$

if l is an even integer y_1 will terminate there where $n = l$. & we will obtain an even polynomial.

Similarly,

$$y_2 = x - \frac{(l-1)(l+2)x^3}{3!} + \frac{(l-3)(l-1)(l+2)(l+4)x^5}{5!} - \ldots$$

here if l is an odd integer, y_2 will terminate there where $n = l$ & we will obtain an odd polynomial.

N.B. you can only have either odd/even polynomials for the soln, but not both.

these polynomials are called Legendre Polynomials and we write them as $P_l(x)$.

FIGURE 8.6
A plot of Legendre polynomials P_1, P_2, P_3, and P_4. (See Table 8.1.)

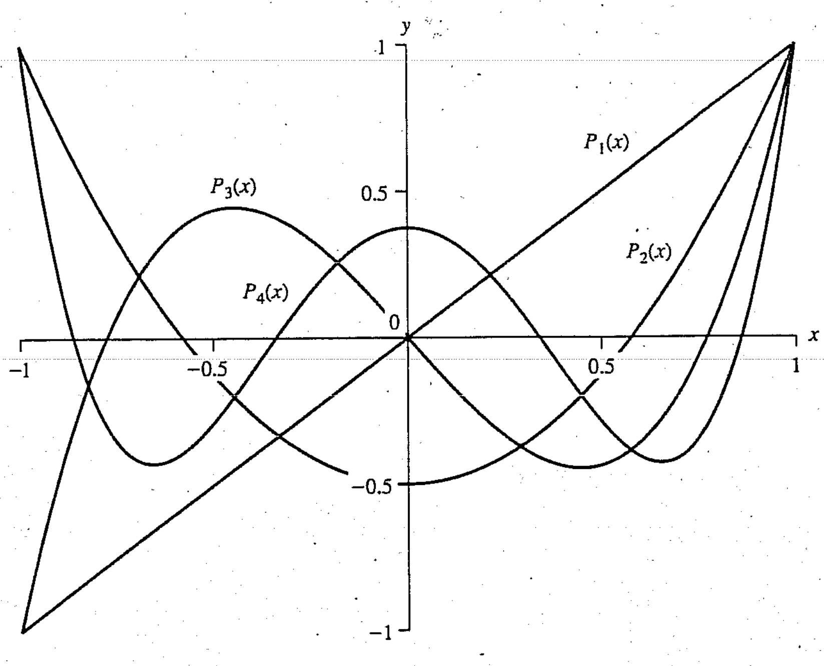


TABLE 8.1
Legendre polynomials.

l	$P_l(x)$
0	1
1	x
2	$(3x^2 - 1)/2$
3	$(5x^3 - 3x)/2$
4	$(35x^4 - 30x^2 + 3)/8$

Mathematical Physics

Lecture, January 2002

The Principle of Superposition
Harmonic oscillator with arbitrary applied force

An important property of the harmonic oscillator is that its motion $x(t)$, when subject to an applied force $F(t)$ which can be regarded as the sum of two or more other forces $F_1(t), F_2(t)$---- is the sum of the motions $x_1(t), x_2(t),$---- which it would have if each of the forces $F_n(t)$ were acting separately.

This is the principle of superposition $\Longleftrightarrow$ linearity

This principle applies to small (linearized) mechanical vibrations, electrical vibrations, sound waves, electromagnetic waves and all physical phenomena governed by linear differential equations

Theorem

Let the (finite or infinite)* set of fn's $x_n(t)$, $n = 1,2,3$

be solutions of the equations

$$m\ddot{x}_n + b\dot{x}_n + kx_n = F_n(t)$$

and let

$$F(t) = \sum_n F_n(t).$$

Then the fn

$$x(t) = \sum_n x_n(t). \qquad (1)$$

satisfies the equation

$$m\ddot{x} + b\dot{x} + kx = F(t). \qquad (2)$$

Proof: substitute (1) into (2)

$$m\ddot{x} + b\dot{x} + kx = m\sum_n \ddot{x}_n + b\sum_n \dot{x}_n + k\sum_n x_n$$

$$= \sum(m\ddot{x}_n + b\dot{x}_n + kx_n)$$

$$= \sum_n F_n(t)$$

$$= F(t) \quad \text{as required}$$

This theorem enables us to find a soln

of 2 whenever the force $F(t)$

can be expressed as a sum of forces $F_n(t)$ for which

the soln's of $m\ddot{x}_n + b\dot{x}_n + kx_n = F_n(t)$ can be found.

In particular, whenever $F(t)$ can be written

as a sum of sinusoidally oscillating terms

$$F(t) = \sum_n C_n \cos(\omega_n t + \phi_n)$$

C_n replaces F_0
ϕ_n replaces θ_0

$$x(t) = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}} \cos(\omega t + \theta_0 - \theta)$$

in our case now, we write

$$x(t) = \sum_n \frac{C_n}{m} \frac{\cos(\omega_n t + \phi_n - \theta_n)}{\left[(\omega_n^2-\omega_0^2)^2+4\gamma^2\omega_n^2\right]^{1/2}}$$

where $\theta_n = \tan^{-1} \dfrac{2\gamma\omega_n}{\omega_0^2-\omega_n^2}$

& for an "underdamped" oscillator

$$x(t) = x_h(t) + x_{inh}(t) = Be^{-\gamma t}\cos(\omega_1 t + \delta) + \sum_n \frac{C_n}{m} \frac{\cos(\omega_n t + \phi_n - \theta_n)}{\left[(\omega_n^2-\omega_0^2)^2+4\gamma^2\omega_n^2\right]^{1/2}}$$

Now look at.

$$F(t) = \sum_n C_n \cos(\omega_n t + \phi_n)$$

$$= \sum_n C_n \left\{ \cos\omega_n t \cos\phi_n + \sin\omega_n t \sin\phi_n \right\}$$

set $A_n = C_n \cos\phi_n$

$B_n = -C_n \sin\phi_n$.

$$\boxed{F(t) = \sum_n A_n \cos\omega_n t + B_n \sin\omega_n t}$$

E/ Now look at

$$x(t) = \sum_n \frac{C_n}{m} \frac{\cos(\omega_n t + \phi_n - \theta_n)}{\left[(\omega_n^2 - \omega_0^2)^2 + 4\gamma^2\omega_n^2\right]^{1/2}}$$

$$\cos([\omega_n t - \theta_n] + \phi_n) = \cos(\omega_n t - \theta_n)\cos\phi_n - \sin(\omega t - \theta_n)\sin\phi_n$$

& hence $C_n \cos(\omega_n t + \phi_n - \theta_n) = A_n \cos(\omega_n t - \theta_n) + B_n \sin(\omega_n t - \theta_n)$

or

$$x(t) = \sum_n \frac{A_n \cos(\omega_n t - \theta_n) + B_n \sin(\omega_n t - \theta_n)}{m\left[(\omega_n^2 - \omega_0^2)^2 + 4\gamma^2\omega_n^2\right]^{1/2}}$$

Now look at

$$C_n \cos([\omega_n t + \phi_n] - \theta_n) = C_n[\cos(\omega_n t + \phi_n)\cos\theta_n + \sin(\omega_n t + \phi_n)\sin\theta_n]$$

$$\text{set } A_n = C_n \cos\theta_n \quad B_n = C_n \sin\theta_n.$$

$$F(t) = A_n$$

$$F(t) = F_0 \left| \sin \omega_0 t \right| \qquad (-L, L)$$

$$F(t) = \cancel{A_0} \frac{A_0}{2} + \sum_{n=1} A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L}.$$

$$A_0 = \frac{1}{2L} \int_{-L}^{L} F_0 \sin \omega_0 t \, dt \qquad \frac{2\pi}{\omega} = L$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx.$$

$$A_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx.$$

$$= \frac{2}{L} \int_{0}^{\frac{\cancel{2}\pi}{\omega}} F_0 \sin \omega_0 t \, dt.$$

$$= \left[\frac{2}{\frac{\pi}{\omega_0}} \cdot F_0 \right] \left(- \frac{\cos \omega_0 t}{\omega_0} \right) \Bigg|_{0}^{\frac{\pi}{\omega_0}}.$$

$$+2$$

$$= \frac{2\cancel{\omega_0}}{\pi} F_0 \frac{2}{\cancel{\omega_0}}$$

$$= \frac{4 F_0}{\pi}$$