Mathematical Physics

Lecture, Monday, January 14, 2002

The driven harmonic Oscillator

(1) $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$ inhomogeneous diff equation

Theorem:

If $x_{inh}(t)$ is a soln of an inhomogeneous linear equation (1) above, and $x_h(t)$ is a soln of the corresponding homogeneous eq [.], then

ie $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$ homogeneous

x_h solves

Then $\boxed{x(t) = x_h(t) + x_{inh}(t)}$ is also a soln of the inhomogeneous equation

N.B. this theorem applies whether the coefficients in the equation are constants or fn' of t. Just substitute directly.

we already know the soln's to the homogeneous equation — Now let us get a soln for

$$\boxed{F(t) = F_0\cos(\omega t + \theta_0)}$$

or $$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t + \theta_0)} \quad [I] \quad \left[\begin{matrix} F_0, \omega, \theta_0 \\ \text{are known} \end{matrix}\right]$$

θ_0 is constant specifying the phase of the applied force.

i.e. F_0, ω, θ_0 are <u>known</u>, ω is called the driver frequency

From physical considerations, we expect that one solution will be a steady state oscillation of the coordinate x at the <u>same</u> <u>frequency</u> ω as the applied force

(a) $$\boxed{x(t) = A_s\cos(\omega t + \theta_s)}$$ assumed form for soln —

$$\boxed{\text{Our task is to obtain } A_s, \theta_s}$$

The amplitude A_s an θ_s of the oscillations in x will have to be determined by substituting (a) into [I]

— Procedure straight forward leads to correct answer

The algebra is simpler if we write the force as the real part of a complex fn

$$F(t) = \mathrm{Re}\{\bar{F}_0 e^{i\omega t}\}$$

$$\bar{F}_0 = F_0 e^{i\theta_0}$$

Thus if we can find a sol'n of $\bar{x}(t)$ of

$$\boxed{m\frac{d^2\bar{x}}{dt^2} + b\frac{d\bar{x}}{dt} + k\bar{x} = \bar{F}_0 e^{i\omega t}} \quad (\mathrm{II})$$

then by splitting the equation into real and imaginary parts, we can show that the real part of $\bar{x}(t)$ will satisfy [I]; we assume a soln of the form

$$\boxed{\bar{x}(t) = \bar{x}_0 e^{i\omega t}}$$

here we go! $\dot{\bar{x}} = \bar{x}_0 i\omega e^{i\omega t}$

$$\ddot{\bar{x}} = -\omega^2 \bar{x}_0 e^{i\omega t}$$

Substitute into [II] → over

before doing so, write:

$$\frac{d^2\bar{x}}{dt^2} + 2\gamma\frac{d\bar{x}}{dt} + \omega_0^2\bar{x} = \overline{F_0}/m\, e^{i\omega t}$$

$$\ddot{\bar{x}} + 2\gamma\dot{\bar{x}} + \omega_0^2\bar{x} = \frac{\overline{F_0}}{m}e^{i\omega t} \qquad (II)$$

Substitute into above (II)

$$-\omega^2\bar{x}_0 e^{i\omega t} + 2\gamma i\omega\bar{x}_0 e^{i\omega t} + \omega_0^2\bar{x}_0 e^{i\omega t} = \frac{\overline{F_0}}{m}e^{i\omega t}$$

or $\bar{x}_0\left[\omega_0^2 - \omega^2 + i2\gamma\omega\right] = \overline{F_0}/m$ — we have cancelled the common factor $e^{i\omega t}$.

response fn

Solve: $$\boxed{\bar{x}_0 = \frac{\overline{F_0}/m}{(\omega_0^2-\omega^2)+i2\gamma\omega}}$$ & $$\boxed{\bar{x}(t) = \bar{x}_0 e^{i\omega t}}$$

Let us work on $\bar{x}_0$: write denominator in polar form

$$\boxed{[\omega_0^2-\omega^2] + i2\gamma\omega = \left[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2} e^{i\theta} \quad \text{where } \tan\theta = \frac{2\gamma\omega}{\omega_0^2-\omega^2}}$$

→ over

or $\bar{x}_0 = \dfrac{\bar{F}_0/m}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2} e^{i\theta}}$

$$= \frac{F_0/m\, e^{i\theta_0} e^{-i\theta}}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}$$

& $\bar{x}(t) = \bar{x}_0 e^{i\omega t}$

$$\bar{x}(t) = \frac{F_0/m\, e^{i[\omega t+\theta_0-\theta]}}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}$$

this is the soln' to the complex equation (II).

Now $x(t) = \mathrm{Re}\left\{\bar{x}(t)\right\} = \dfrac{F_0/m \cos(\omega t+\theta_0-\theta)}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}$.

real soln

therefor comparing This result with

$$x_{inh}(t) = A_s \cos(\omega t+\theta_s)$$

we must conclude $A_s = \dfrac{F_0/m}{\left[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2\right]^{1/2}}$

& $\boxed{\theta_s = \theta_0-\theta}$

Now for the case of an "underdamped oscillator"

the homogeneous soln has the form

$$x_h(t) = B e^{-\gamma t} \cos(\omega_1 t + \delta)$$

B, δ used to avoid confusion

where B, δ are constants of integration {to be determined from initial conditions}

$\therefore$ the complete soln'

$$x(t) = x_h(t) + x_{inh}(t)$$

$$\boxed{x(t) = B e^{-\gamma t} \cos(\omega_1 t + \delta) + A_s \cos(\omega t + \theta_s)}$$

$$\boxed{v(t) = -\gamma B e^{-\gamma t} \cos(\omega_1 t + \delta) - \omega_1 B e^{-\gamma t} \sin(\omega_1 t + \delta) - \omega A_s \sin(\omega t + \theta_s)}$$

Now put in the initial conditions

at $t = 0$ $\quad x(0) = x_0$

$v(0) = v_0$

we have

$$\boxed{\begin{aligned} x_0 &= B\cos\delta + A_s \cos\theta_s \\ v_0 &= -\gamma B \cos\delta - \omega_1 B \sin\delta - \omega A_s \sin\theta_s \end{aligned}}$$

we now solve for B, δ $\longrightarrow$ over

$$B\cos\delta + 0B\sin\delta = (x_o - A_s\cos\theta_s)$$

$$+\gamma B\cos\delta + \omega_1 B\sin\delta = -(v_o + \omega A_s\sin\theta_s)$$

solve for $B\cos\delta$, $B\sin\delta$

$$B\cos\delta = \frac{\begin{vmatrix} (x_o - A_s\cos\theta_s) & 0 \\ -(v_o + \omega A_s\sin\theta_s) & \omega_1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ \gamma & \omega_1 \end{vmatrix}} = \frac{\omega_1(x_o - A_s\cos\theta_s)}{\omega_1}$$

or

$$\boxed{B\cos\delta = (x_o - A_s\cos\theta_s)}$$

$$B\sin\delta = \frac{\begin{vmatrix} 1 & (x_o - A_s\cos\theta_s) \\ \gamma & -(v_o + \omega A_s\sin\theta_s) \end{vmatrix}}{\omega_1} = \frac{-(v_o + \omega A_s\sin\theta_s) - \gamma(x_o - A_s\cos\theta_s)}{\omega_1}$$

Now write $\dfrac{B\sin\delta}{B\cos\delta} = -\left\{\dfrac{(v_o + \omega A_s\sin\theta_s) + \gamma(x_o - A_s\cos\theta_s)}{\omega_1(x_o - A_s\cos\theta_s)}\right\}$

$$\tan\delta = -\left\{\frac{(v_o + \omega A_s\sin\theta_s}{\omega_1(x_o - A_s\cos\theta_s)} + \frac{\gamma}{\omega_1}\right\}$$

before continuing "check units"

$$\gamma = \frac{b}{2m} \qquad F = bv \Rightarrow b = \frac{F}{v} = \frac{N \cdot s}{m}$$

$$\gamma = \frac{N \cdot s}{m \cdot kg}$$

$$\frac{\gamma}{\omega} = \frac{N \cdot s}{m \cdot kg \cdot s^{-1}} = \frac{N \cdot s^2}{m \cdot kg} = \frac{kg \cdot m \cdot s^2}{s^2 \cdot m \cdot kg} = 1$$

$$\frac{v_0 + \omega A_s \sin\theta_s}{\omega_1 (x_0 - A_s \cos\theta_s)} = \frac{\frac{m}{s} \cdot \frac{m}{s}(\,)}{\frac{m}{s} \cdot \frac{m}{s}} = 1$$ right on the button

$$\therefore \tan\delta = -\left\{ \frac{(v_0 + \omega A_s \sin\theta_s)}{\omega_1 (x_0 - A_s \cos\theta_s)} + \frac{\gamma}{\omega_1} \right\} \Rightarrow \delta \text{ is known.}$$

---

get B

but $B\cos\delta = (x_0 - A_s \cos\theta_s)$

$$B = \frac{(x_0 - A_s \cos\theta_s)}{\cos\delta}$$

but since $\tan\delta$ is known so is $\cos\delta$

the problem is completely solved —

Mathematical Physics.

Lecture, Wednesday, January 16, 2002 —

The particular (inhomogeneous) soln' — 2nd order linear diff eq.

$\boxed{\text{Method of Undetermined Coefficients}}$

See pg 147 → 149.

Please look at examples worked out —

Example # 24 pg 201 Fletcher

Find general soln $\frac{d^2y}{dx^2} - y = x^2$

Homogeneous: $\frac{d^2y}{dx^2} - y = 0$ try $y = e^{pt}$ $\frac{dy}{dx} = pe^{pt}$, $\frac{d^2y}{dx^2} = p^2e^{pt}$.

or $(p^2-1)e^{pt} = 0$ $p^2 - 1 = 0 \Rightarrow p^2 = 1$, $\Rightarrow p = \pm 1$

$$\therefore \boxed{y(x) = Ae^{x} + Be^{-x}}$$

Now for $y_p = y_{inh}$ → try $y_p = ax^2 + bx + c$

$\frac{dy_p}{dx} = 2ax + b$, $\frac{d^2y_p}{dx^2} = 2a$

Substitute

$$2a - [ax^2 + bx + c] = x^2$$

$$2a - ax^2 - bx - c = x^2$$

$bx = 0$

$\therefore -ax^2 = x^2 \Rightarrow a = -1$ & $2a - c = 0$

$\boxed{c = 2a = -2}$

Hence $\boxed{y_{total} = y_h + y_p = Ae^{x} + Be^{-x} - x^2 - 2.}$

<u>check:</u> $\frac{dy_t}{dx} = Ae^{x} - Be^{-x} - 2x$

$$\frac{d^2y_t}{dx^2} = Ae^{x} + Be^{-x} - 2$$

$\therefore \quad \frac{d^2y}{dx^2} - y = x^2$

$$\cancel{Ae^{x}} + \cancel{Be^{-x}} \cancel{-2} - \cancel{Ae^{x}} - \cancel{Be^{-x}} + x^2 + \cancel{2} = x^2 \quad \checkmark \quad \text{checks}$$

Example. #25 pg 201 Fletcher.

Find general soln to

$$\frac{d^2y}{dx^2} + y = 2\cos x$$

<u>homogeneous:</u>

$$\frac{d^2y}{dx^2} + y = 0 \quad SHO \Rightarrow y_h = A\cos x + B\sin x$$

Now if we chose $y_p = C\cos x$ — this solves the homogeneous equation. (see rules pg 148)

try $\boxed{y_p = Cx\cos x + Dx\sin x}$

$$\frac{dy_p}{dx} = C\cos x - Cx\sin x + D\sin x + Dx\cos x.$$

$$\frac{d^2y_p}{dx^2} = -C\sin x - \left[C\sin x + Cx\cos x\right] + D\cos x + \left[D\cos x - Dx\sin x\right]$$

$$= -C\sin x - C\sin x - Cx\cos x + D\cos x + D\cos x - Dx\sin x$$

$$\frac{d^2y_p}{dx^2} = -2C\sin x + 2D\cos x - \left[Cx\cos x + Dx\sin x\right]$$

$\longrightarrow$ over

Now substitute

$$-2C\sin x + 2D\cos x - \left[\cancel{Cx\cos x + Dx\sin x}\right] + \left[\cancel{Cx\cos x + Dx\sin x}\right] = 2\cos x.$$

$$2C\sin x = 0 \Rightarrow C = 0$$

$$2D\cos x = 2\cos x \Rightarrow D = 1$$

Hence $\boxed{y_t = y_h + y_p = A\cos x + B\sin x + x\sin x.}$ check this out !

Example # 26 pg 201 Fletcher.

Find general soln

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x$$

Homogeneous: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ $\quad p^2 - 2p + 1 = 0$

$$p = \frac{+2 \pm \sqrt{4-4}}{2}$$

$p = 1$ only one soln

hence $\boxed{y_h = (A + Bx)e^x}$

try

$$y_p = Cx + D$$

$$\frac{dy_p}{dx} = C \qquad \frac{d^2y_p}{dx^2} = 0$$

Substitute

$\therefore$ $0 - 2C + Cx + D = 2x \Rightarrow Cx = 2x \Rightarrow C = 2$

$$-2C + D = 0 \Rightarrow D = 2C = 4.$$

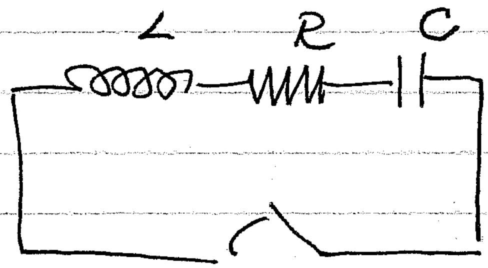
$$\boxed{\therefore y_t = y_h + y_p = (A + Bx)e^x + 2x + 4}$$

check it out

Mathematical Physics.

Lecture January 2002

The Electrical Analog of the Damped Harmonic Oscillator



homogeneous
at $t=0$

$$\begin{cases} q(0) = Q_0 \\ i(0) = 0 \end{cases}$$

$$-L\frac{di}{dt} - Ri - \frac{q}{C} = 0$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

$$2\gamma = \frac{R}{L}$$

$$\gamma = \frac{R}{2L}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

form is the same

Mechanics	Electrical Circuit
x	
v	
m	
b	
k	
$\omega_0^2 = \frac{k}{m}$	
$\gamma = \frac{b}{2m}$	
$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$	

hence all soln's will be the same.

Case I $\quad \gamma^2 > \omega_0^2$

$$q(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$$

$\omega_0^2 > \gamma^2$

Case II $\quad q(t) = A e^{-\gamma t} \cos(\omega_1 t + \theta)$

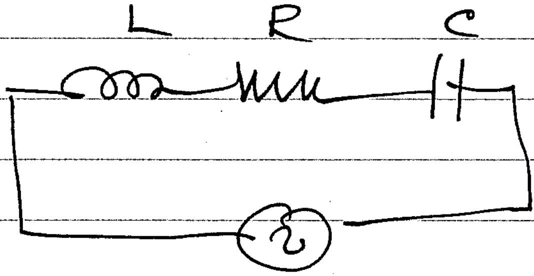
$\omega_0^2 = \gamma^2$

Case III $\quad q(t) = (A + Bt) e^{-\gamma t}$

Initial Conditions

$$\boxed{\begin{aligned} &\text{at } t=0 \quad q(0) = Q_0 \\ &\left.\frac{dq}{dt}\right|_{t=0} = i(0) = I_0 \end{aligned}}$$

Driven LRC Circuit



$V = V_0 \cos(\omega t + \theta_0)$

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \cos(\omega t + \theta_0)}$$

Mechanical circuit	Electrical Circuit.
x	q
v	i
m	L
b	R
k	$\frac{1}{C}$
$\omega_0^2 = k/m$	$\omega_0^2 = \frac{1}{LC}$
$\gamma = \frac{b}{2m}$	$\gamma = \frac{R}{2L}$.
$\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$	$\omega_1 = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$
$= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$.	
driver: F_0	V_0
ω = driver freq	ω

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + y = 0 \qquad x = 0 \qquad p(x) = \frac{1}{2x}.$$

$$\lim (x-0)\frac{1}{2x} = 1/2.$$

$$y(x) = x^r \sum_{n=0} a_n x^n \qquad \text{regular point}$$

$$y(x) = \sum_{n=0} a_n x^{n+r}$$

$$\frac{dy}{dx} = \sum_{n=0} a_n (n+r) x^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2} + \sum_{n=0} \frac{a_n}{2}(n+r) x^{n+r-2} + \sum_{n=0} a_n x^{n+r} = 0$$

shift the summation in 3rd term

$$\sum_{n-2=0} a_{n-2} x^{n+r-2}$$

Now write out $n=0$, $n=1$ terms explicitly

$$a_0 (r)(r-1) x^{r-2} + a_1 (r+1)(r) x^{r-1} + \frac{a_0}{2} r x^{r-2}$$

$$+ \frac{a_1}{2}(r+1) x^{r-1} + \cancel{\sum_{n=2}}$$

$$+ \sum_{n=2} \left[a_n (n+r)(n+r-1) + \frac{a_n (n+r)}{2} + a_{n-2} \right] x^{n+r-2} = 0$$

Equate coeff of powers of x to zero.

$$\left[a_0 (r)(r-1) + \frac{a_0 r}{2}\right] x^{r-2} + \left[a_1 (r+1)(r) + \frac{a_1}{2}(r+1)\right] x^{r-1}$$

$$a_0\left[r(r-1) + \frac{r}{2}\right] = 0$$

$$a_0\left[r(r-1/2)\right] = 0$$ $\boxed{\text{indicial equation}}$

$$a_1\left[r^2 + r + \frac{r+1}{2}\right] = 0$$

$$a_1\left[r^2 + 3/2\, r + 1/2\right] = 0$$

$$a_n\left\{[n+r][n+r-1] + \frac{[n+r]}{2}\right\} = -a_{n-2}$$

$$a_n\left[n+r\{n+r-1+1/2\}\right] = -a_{n-2}$$

$$a_n\left[(n+r)^2 - n - r + 1/2\, n + 1/2\, r\right] = -a_{n-2}$$

$$a_n\left[(n+r)^2 - n/2 - r/2\right] = -a_{n-2}$$

or $\boxed{a_n = \dfrac{-a_{n-2}}{[n+r]^2 - n/2 - r/2}}$ recursion indicial equation

$a_0 \neq 0$ by assumption

then $r=0$, $r=1/2$

look at.

$$a_1\left[r^2+\frac{3r}{2}+1/2\right]=0$$

if $r=0 \Rightarrow a_1=0$

$r=1/2$

$$a_1\left[\left(1/2\right)^2+3/4+1/2\right]=a_1=0$$

$\therefore$ $r=0$ look at recursion

$$a_n=\frac{-a_{n-2}}{n^2-n/2}=\frac{-a_{n-2}}{n(n-1/2)}$$

and ~~$y_1(x)=a_0$~~

$n=2$ $\qquad a_2=\frac{-a_0}{2(2-1/2)}=\frac{-a_0}{3}$

$n=3$ $\qquad a_3=\frac{-a_1}{n(n-1/2)}=0 \qquad a_1=0$

$n=4$ $\qquad a_4=\frac{-a_2}{4(4-1/2)}=\frac{+a_0}{3(16-2)}=\frac{a_0}{3(14)}=\frac{a_0}{42}$

or $\boxed{y_1(x)=a_0\left(1-\frac{x^2}{3}+\frac{x^4}{42}-\cdots\right)}$

for $r = 1/2$

$$b_n = \frac{-b_{n-2}}{(n+r)^2 - n/2 - r/2}$$

$$b_n = \frac{-b_{n-2}}{(n+1/2)^2 - n/2 - 1/4}$$

$$= \frac{-b_{n-2}}{n^2 + n + \cancel{1/4} - n/2 - \cancel{1/4}}$$

$n=2$ $$\boxed{b_n = \frac{-b_{n-2}}{n(n+1/2)}}$$

a_0 $$b_2 = \frac{-b_0}{2(3/2)} = \frac{-b_0}{3} = -a_0$$

$$y = x^{1/2} \sum_{n=0} b_n x^n$$

$$= x^{1/2}\left[b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots\right]$$

$b_1 = 0$

$n=2$ $$b_2 = \frac{-b_0}{2(5/2)} = -\frac{b_0}{5}$$

$n=3$ $$b_3 = 0$$

$n=4$ $$b_4 = \frac{-b_2}{4(4+1/2)} = \frac{+b_0}{(5)(4)(4+1/2)} = \frac{b_0}{(20)(\frac{9}{2})} = \frac{b_0}{90}$$

$\therefore$ $$\boxed{y_2 = a_0 x^{1/2}\left[1 - \frac{x^2}{5} + \frac{x^4}{90} - \cdots\right]}$$

y_1, y_2 two independent soln to diff equation

Example #36 Fletcher pg 202

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

$$\frac{d^2y}{dx^2} - \left(\frac{x}{x-1}\right)\frac{dy}{dx} + \frac{y}{x-1} = 0 \qquad \lim_{x\to 1}(x-1)\frac{x}{(x-1)} = 1 \text{ finite.}$$

$$\lim_{x\to 1}(x-1)^2\frac{1}{x-1} = \text{finite} = 0.$$

$\therefore$

$$y(x) = x^r \sum_{n=0} a_n x^n$$

$$= \sum_{n=0} a_n x^{n+r}$$

$$\frac{dy}{dx} = \sum_{n=0} a_n (n+r) x^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$x\frac{d^2y}{dx^2} - \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

Substitute

$$x\sum_{n=0} a_n (n+r)(n+r-1)x^{n+r-2} - \sum_{n=0} a_n (n+r)(n+r-1)x^{n+r-2}$$

$$+ x\sum_{n=0} a_n (n+r) x^{n+r-1} + \sum_{n=0} a_n x^{n+r} = 0$$

or

$$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$+ \sum_{n=0} a_n (n+r) x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0.$$

$\boxed{\text{Shift summation index}}$

$$\sum_{\substack{\cancel{n+1=0} \\ n-1=0 \\ n=1}} a_{n-1} (n-1+r)(n+r-2) x^{n+r-2} - \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$+ \sum_{\substack{n-2=0 \\ n=2}} a_{n-2} (n-2+r) x^{n+r-2} + \cancel{\sum_{n-2=0} a_{n-2} (n+r-2)(n+r)}$$

$$+ \sum_{\substack{n-2=0 \\ n=2}} a_{n-2} x^{n+r-2} = 0.$$

write out $n=0,1$ explicitly

$$-a_0 (r)(r-1) x^{r-2} - a_1 (r+1)(r) x^{r-1}$$

$$+ a_0 (r)(r-1) x^{r-1} + \sum_{n=2} (\quad)$$

$$\boxed{\begin{aligned} &-a_0(r[r-1]) = 0 \\ &-a_1 (r+1)(r) + a_0 (r)(r-1) = 0 \end{aligned}}$$

Mathematical Physics.

Lecture, Friday / Mon January 18/21/'02

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

Series Soln – 2nd order diff. equations.

$p(x)$, $q(x)$ have no singularities

Example: pg 201 Fletcher #31.

$$y(x) = \sum a_n x^n$$

Find the first four nonzero terms of the series soln

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + (x-4)y = 0$$

done in class Friday Jan. 18,'02

Try $y = \sum_{n=0} a_n x^n$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n n(n-1) x^{n-2}$$

Substitute:

$$\sum_{n=2} a_n n(n-1) x^{n-2} + \sum_{n=1} a_n n x^{n-1} + x\sum_{n=0} a_n x^n - 4\sum_{n=0} a_n x^n = 0$$

$$\| \quad \sum_{n=0} a_n x^{n+1}$$

Shift summation indices. $n+2=2$ $\quad n+1=1$ $\quad n-1=0$

$$\sum_{n=0} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0} a_{n+1}(n+1)x^n + \sum_{n=1} a_{n-1} x^n - 4\sum_{n=0} a_n x^n = 0$$

Now write out $n=0$ term explicitly

$$a_2(2)(1) + a_1(1) - 4a_0 + \sum_{n=1}$$

$$2a_2 + a_1 - 4a_0 + \sum_{n=1}\left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_{n-1} - 4a_n\right]x^n = 0$$

Then a_0, a_1 as constants of integration.

$$2a_2 + a_1 - 4a_0 = 0$$

$$2a_2 = 4a_0 - a_1$$

$$\boxed{a_2 = 2a_0 - a_1/2.}$$

&

$n=1$ $$a_{n+2}(n+2)(n+1) + a_{n+1}(n+1) + a_{n-1} - 4a_n = 0$$

$$a_{n+2} = \frac{\left[4a_n - a_{n-1} - (n+1)a_{n+1}\right]}{(n+1)(n+2)}$$

$$\therefore \quad a_3 = \frac{\left[4a_1 - a_0 - 2a_2\right]}{(2)(3)}$$

but $a_2 = 2a_0 - a_1/2$

$$\therefore a_3 = \frac{\left[4a_1 - a_0 - 2\left(2a_0 - \frac{a_1}{2}\right)\right]}{(2)(3)}$$

$$\boxed{a_3 = \frac{4a_1 - a_0 - 4a_0 + a_1}{6} = \frac{5a_1}{6} - \frac{5a_0}{6}}$$

$n=2 \Rightarrow$ $$a_4 = \frac{\left[4a_2 - a_1 - (3)a_3\right]}{(3)(4)}$$

$$a_4 = \frac{4\left(2a_0 - a_1/2\right) - a_1 - 3\left[\frac{5a_1}{6} - \frac{5a_0}{6}\right]}{12}$$

$$a_4 = \frac{8a_0 - 2a_1 - a_1 - \frac{5a_1}{2} + \frac{5a_0}{2}}{12}$$

or

$$a_4 = \frac{16a_0 - 4a_1 - 2a_1 - 5a_1 + 5a_0}{24}$$

$$= \frac{21a_0}{24} - \frac{11a_1}{24}$$

$$a_4 = \frac{7a_0}{8} - \frac{11a_1}{24}$$

etc.

$$y(x) = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ---$$

$$= a_0 + a_1 x + \left(2a_0 - \frac{a_1}{2}\right)x^2 + \left(\frac{5a_1}{6} - \frac{5a_0}{6}\right)x^3 + \left(\frac{7a_0}{8} - \frac{11a_1}{24}\right)x^4 + ---$$

$$y(x) = a_0\left[1 + 2x^2 - \frac{5x^3}{6} + \frac{7x^4}{8} + ---\right] + a_1\left[x - \frac{x^2}{2} + \frac{5x^3}{6} - \frac{11x^4}{24} + ---\right]$$

as required

#32 Fletcher pg 201 do this on Monday Jan 21, 02 and move on to Boundary Value problems —

Find the first six non-zero terms of series soln

$$\frac{d^2y}{dx^2} - 3x^3\frac{dy}{dx} + 4xy = 0$$

$$\boxed{\frac{d^2y}{dx^2} - 3x^3\frac{dy}{dx} + 4xy = 0}$$

try $y(x) = \sum_{n=0} a_n x^n$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n n(n-1) x^{n-2}$$

substitute:

$$\sum_{n=2} a_n n(n-1) x^{n-2} - 3x^3 \sum_{n=1} a_n n x^{n-1} + 4x \sum_{n=0} a_n x^n = 0$$

$$\sum_{n=2} a_n n(n-1) x^{n-2} - 3 \sum_{n=1} a_n n x^{n+2} + 4 \sum_{n=0} a_n x^{n+1} = 0$$

Now get everything to power x^n — shift summation indices

$$\sum_{\substack{n+2=2 \\ n=0}} a_{n+2}(n+2)(n+1)x^n - 3\sum_{n-2=1} a_{n-2}(n-2)x^n + 4\sum_{n-1=0} a_{n-1}x^n = 0$$

or $$\boxed{\sum_{n=0} a_{n+2}(n+2)(n+1)x^n - 3\sum_{n=3} a_{n-2}(n-2)x^n + 4\sum_{n=1} a_{n-1}x^n = 0}$$

Now write out $n=0,1,2$ terms explicitly

$$a_2 2 + a_3(3)(2)x + a_4(4)(3)x^2 + 4a_0 x + 4a_1 x^2$$

$$+ \sum_{n=3}\left[a_{n+2}(n+2)(n+1) - 3a_{n-2}(n-2) + 4a_{n-1}\right]x^n = 0$$

Now clean this up —

$$2a_2 + 6a_3x + 4a_0x + 12a_4x^2 + 4a_1x^2 + \sum_{n=3}\left[a_{n+2}(n+2)(n+1) - 3a_{n-2}(n-2) + 4a_{n-1}\right]x^n = 0$$

Coefficients of all powers of x^n must equal zero separately

$$2a_2 = 0, \quad x(6a_3 + 4a_0) = 0, \quad (12a_4 + 4a_1)x^2 = 0$$

$$a_0, a_1 \quad a_2 = 0 \quad a_3 = -\tfrac{2}{3}a_0, \quad a_4 = -\frac{a_1}{3}$$

and for $n=3 \rightarrow$ up

$$a_{n+2} = \frac{\left[3(n-2)a_{n-2} - 4a_{n-1}\right]}{(n+1)(n+2)}$$

$n=3$

$$a_5 = \frac{\left[3(1)a_1 - 4a_2\right]}{(4)(5)} = \frac{3}{20}a_1 \quad \text{since } a_2 = 0$$

$n=4$

$$a_6 = \frac{\left[3(2)a_2 - 4a_3\right]}{(5)(6)} = \frac{-4a_3}{30} = \frac{-4\left(-\tfrac{2}{3}a_0\right)}{30}$$

or

$$a_6 = \frac{8a_0}{90} = \frac{4a_0}{45} \qquad \text{etc.}$$

$$\therefore y(x) = \sum_{n=0} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$= a_0 + a_1x + 0 - \tfrac{2}{3}a_0x^3 - \frac{a_1}{3}x^4 + \frac{3}{20}a_1x^5 + \frac{4a_0}{45}x^6 + \cdots$$

$$y(x) = a_0\left[1 - \tfrac{2}{3}x^3 + \frac{4}{45}x^6 + \cdots\right] + a_1\left[x - \frac{x^4}{3} + \frac{3}{20}x^5 + \cdots\right]$$

as required

#33 Fletcher pg201

Find the first six nonzero terms of the series soln

$$\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} - 3y = 0 \qquad \text{no singularities}$$

try $y(x) = \sum_{n=0} a_n x^n$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n n(n-1) x^{n-2}$$

substitute:

$$\sum_{n=2} a_n (n)(n-1) x^{n-2} + x^3 \sum_{n=1} a_n n x^{n-1} - 3\sum_{n=0} a_n x^n$$

$$\sum_{n=2} a_n n(n-1) x^{n-2} + \sum_{n=1} a_n n x^{n+2} - 3\sum_{n=0} a_n x^n = 0$$

now shift summation indices to get all terms in x^n

$$\sum_{\substack{n+2=2\\n=0}} a_{n+2}(n+2)(n+1)x^n + \sum_{n-2=1} a_{n-2}(n-2)x^n - 3\sum_{n=0} a_n x^n = 0$$

or

$$\sum_{n=0} a_{n+2}(n+2)(n+1)x^n + \sum_{n=3} a_{n-2}(n-2)x^n - 3\sum_{n=0} a_n x^n = 0$$

Now write out $n=0,1,2$ terms explicitly

$$\cancel{a_2 2 + a_3 6 + a_4 8}$$

$$a_2 2 + a_3 6x + a_4 8x^2 - 3a_0 - 3a_1 x - 3a_2 x^2$$

$$+ \sum_{n=3} \left[a_{n+2}(n+2)(n+1) + a_{n-2}(n-2) - 3a_n\right]x^n = 0$$

each coeff of powers of x must equal zero separately:

$2a_2 - 3a_0 = 0$; $(6a_3 - 3a_1)x = 0$; $(8a_4 - 3a_2)x^2 = 0$

& $\left[a_{n+2}(n+2)(n+1) + a_{n-2}(n-2) - 3a_n\right] = 0$

for $n=3 \longrightarrow$

hence $2a_2 = 3a_0$ $\quad 6a_3 = 3a_1$ $\quad 8a_4 = 3a_2. = 3\left(\frac{3}{2}\right)a_0$

a_0, a_1 $\quad \boxed{a_2 = 3/2\, a_0}$ $\quad \boxed{a_3 = a_1/2}$ $\quad \cancel{a_4 = 3/8\, a_2}$ $\Rightarrow 8a_4 = \frac{9}{2}a_0$

$a_4 = \frac{9}{16}a_0$

$\boxed{a_4 = \frac{3}{8}a_0}$

&

$$a_{n+2} = \frac{-(n-2)a_{n-2} + 3a_n}{(n+1)(n+2)}.$$

for $n=3$ & up

$$\boxed{a_{n+2} = \frac{3a_n - (n-2)a_{n-2}}{(n+1)(n+2)}}$$

$n=3$ $\quad a_5 = \frac{3a_3 - (1)a_1}{(4)(5)} = \frac{3\left(\frac{a_1}{2}\right) - a_1}{20} = \frac{a_1}{40}$

$n=4$ $\quad a_6 = \frac{3a_4 - (2)(a_2)}{(5)(6)} = \frac{3\left(\frac{3}{8}a_0\right) - 2\left(\frac{3}{2}\right)a_0}{30}$

$$= \frac{\frac{9a_0}{8} - 3a_0}{30} = \frac{\frac{(9-24)a_0}{8}}{30} = \frac{-15a_0}{(8)(30)}$$

$$= -\frac{a_0}{16}$$

etc $\longrightarrow$ over

hence

$$y(x) = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= a_0 + a_1 x + \tfrac{3}{2} a_0 x^2 + \tfrac{1}{2} a_1 x^3 + \tfrac{3}{8} a_0 x^4 + \frac{a_1}{40} x^5 - \frac{a_0}{16} x^6 + \cdots$$

$$= a_0\left[1 + \tfrac{3}{2}x^2 + \frac{3}{8}x^4 - \frac{1}{16}x^6 + \cdots\right] + a_1\left[x + \frac{x^3}{2} + \frac{x^5}{40} + \cdots\right]$$

as required.

#34 Fletcher pg 202

Find the first four nonzero terms of the series solns for.

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$ no singular points.

try

$$y(x) = \sum_{n=0} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n n(n-1) x^{n-2}$$

Substitute.

$$\sum_{n=2} a_n n(n-1) x^{n-2} + x\sum_{n=1} a_n n x^{n-1} - \sum_{n=0} a_n x^n = 0.$$

shift ~~the~~ summation indices

$$\sum_{\substack{n+2=2 \\ n=0}} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1} a_n n x^n - \sum_{n=0} a_n x^n$$

Write out $n=0$ term explicitly.

$$a_2 2 - a_0 + \sum_{n=1}\left[a_{n+2}(n+2)(n+1) + a_n n - a_n\right]x^n = 0$$

$$\Rightarrow a_2 2 - a_0 = 0 \quad \&$$

$$a_{n+2}(n+2)(n+1) + (n-1)a_n = 0$$

$n=1$

$$\text{or} \quad \boxed{a_{n+2} = +\frac{(1-n)a_n}{(n+1)(n+2)}}$$

So a_0, a_1, $a_2 = a_0/2$

$n=1$

$$a_3 = 0$$

$n=2$ $\quad a_4 = \dfrac{a_2}{3\cdot 4} = \dfrac{a_0}{2\cdot 3\cdot 4} = \dfrac{a_0}{24}$

$n=3$ $\quad a_5 = \dfrac{(-2)a_3}{(4)(5} = 0$ since $a_3 = 0$

$n=4$ $\quad a_6 = \dfrac{-3a_4}{(5)(6)} = \dfrac{-3a_0}{(30)(24)} = \dfrac{-a_0}{240}$ etc

$$y(x) = \sum_{n=0} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$= a_0 + a_1 x + \frac{a_0}{2}x^2 + 0 + \frac{a_0}{24}x^4 + 0 - \frac{a_0}{240}x^6 + \cdots$$

$$= a_0\left[1 + \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{240} + \cdots\right] + a_1 x$$

as required

35 Fletcher pg 202

Find the first four nonzero terms of the series soln's for

$$\frac{d^2y}{dx^2} - (x-1)y = 0 \qquad \text{no singular points.}$$

try $y(x) = \sum_{n=0} a_n x^n$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n n(n-1) x^{n-2}$$

substitute:

$$\sum_{n=2} a_n n(n-1)x^{n-2} - x\sum_{n=0} a_n x^n + \sum_{n=0} a_n x^n = 0.$$

$$\sum_{n=2} a_n n(n-1)x^{n-2} - \sum_{n=0} a_n x^{n+1} + \sum_{n=0} a_n x^n = 0.$$

shift summation index

$$\sum_{\substack{n+2=2\\n=0}} a_{n+2}(n+2)(n+1)x^n - \sum_{\substack{n-1=0\\n=1}} a_{n-1}x^n + \sum_{n=0} a_n x^n = 0.$$

Write out the $n=0$ term explicitly

$$a_2 2 + a_0 + \sum_{n=1}\left[a_{n+2}(n+2)(n+1) - a_{n-1} + a_n\right]x^n = 0$$

or $2a_2 + a_0 = 0$

& for $n=1 \longrightarrow$ $$\boxed{a_{n+2} = \frac{a_{n-1} - a_n}{(n+1)(n+2)}}$$

$a_0,\ a_1 \qquad 2a_2 = -a_0 \Rightarrow a_2 = -\frac{a_0}{2}$

$n=1$

$$a_3 = \frac{a_0 - a_1}{(2)(3)} = \frac{a_0}{6} - \frac{a_1}{6}$$

$n=2 \qquad a_4 = \frac{a_1 - a_2}{(3)(4)} = \frac{a_1 + \frac{a_0}{2}}{12} = \frac{a_1}{12} + \frac{a_0}{24}$

etc —

$$y(x) = \sum_{n=0} a_n x^n = a_0 + a_1 x - \frac{a_0 x^2}{2} + \left(\frac{a_0}{6} - \frac{a_1}{6}\right)x^3 + \left(\frac{a_0}{24} + \frac{a_1}{12}\right)x^4 + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right]$$

as required.

Method of ~~Fot~~ Frobenius

$$y(x) = x^r \sum_{n=0} a_n x^n = \sum_{n=0} a_n x^{n+r}.$$

$a_0 \neq 0$

try this on

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$

try $y(x) = \sum_{n=0} a_n x^{n+r}.$

$$\frac{dy}{dx} = \sum_{n=0} a_n (n+r) x^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$x\frac{d^2y}{dx^2} - \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

$$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$- \sum_{n=0} a_n (n+r) x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0$$

$$y(x,t) = \sum_{n=1} \left(A_n' \cos \frac{n\pi v t}{L} + B_n' \sin \frac{n\pi v t}{L} \right) \sin \frac{n\pi x}{L}$$

Show above is of the form $f(x \pm vt)$

recall $\sin\theta \cos\phi = \frac{1}{2}\left[\sin(\theta+\phi) + \sin(\theta-\phi) \right]$

Look at $\cos\frac{n\pi v t}{L} \sin\frac{n\pi x}{L} = \frac{1}{2} \sin\left\{ \frac{n\pi}{L}(x+vt) \right\} + \frac{1}{2} \sin\left\{ \frac{n\pi}{L}(x-vt) \right\}$

what about $\sin\frac{n\pi v t}{L} \sin\frac{n\pi x}{L}$.

$$\sin\theta \sin\phi = \frac{1}{2}\left\{ \cos(\theta-\phi) - \cos(\theta+\phi) \right\}$$

$$\sin\frac{n\pi v t}{L} \sin\frac{n\pi x}{L} = \frac{1}{2}\left\{ \cos\left[\frac{n\pi}{L}(x-vt)\right) - \frac{1}{2} \cos\left(\frac{n\pi}{L}(x+vt)\right) \right\}$$

Mathematical Physics.

Lecture, Friday January 18, 2002

Series Soln:

Assume soln of the form of a power series.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

e.g. S.H.O in x

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n(n)(n-1) x^{n-2}$$

∴ Substituting

$$\sum_{n=2} a_n n(n-1) x^{n-2} + k^2 \sum_{n=0} a_n x^n = 0$$

shift summation index in 1st term from $n \to n+2$

$$\sum_{n+2=2} a_{n+2} (n+2)(n+1) x^n + k^2 \sum_{n=0} a_n x^n = 0$$

or $$\sum_{n=0} \left[a_{n+2}(n+2)(n+1) + k^2 a_n\right] x^n = 0$$

or

$$a_{n+2} = \frac{-k^2 a_n}{(n+2)(n+1)}$$

This is a 2nd order differential equation — we need two constants say a_0, a_1

a_0, a_1 to be constants of integration

$\therefore$ Get soln' in terms of a_0, a_1

$n=0$

$$\boxed{a_2 = -\frac{k^2 a_0}{2\cdot 1}}$$

$n=1$

$$\boxed{a_3 = \frac{-k^2 a_1}{3\cdot 2}}$$

$n=2$

$$\boxed{a_4 = -\frac{k^2 a_2}{4\cdot 3} = -\frac{k^2(-k^2 a_0)}{4\cdot 3\cdot 2\cdot 1} = \frac{k^4 a_0}{4\cdot 3\cdot 2\cdot 1}}$$

$n=3$

$$a_5 = -\frac{k^2 a_3}{5\cdot 4} = \frac{-k^2}{5\cdot 4}\left(\frac{-k^2 a_1}{3\cdot 2}\right).$$

$$\boxed{a_5 = \frac{k^4 a_1}{5\cdot 4\cdot 3\cdot 2}}$$

@ $n=4$

$$\boxed{a_6 = \frac{-k^2 a_4}{6\cdot 5} = -\frac{k^2(k^4 a_0)}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = -\frac{k^6 a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}$$

$n=5$

$$\boxed{a_7 = \frac{-k^2 a_5}{7\cdot 6} = \frac{-k^2 k^4 a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}}$$

etc

$$\therefore \quad y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$$

$$= a_0 + a_1 x - \frac{k^2 a_0 x^2}{2\cdot 1} - \frac{k^2 a_1 x^3}{3\cdot 2} + \frac{k^4 a_0 x^4}{4\cdot 3\cdot 2\cdot 1} + \frac{k^4 a_1 x^5}{5\cdot 4\cdot 3\cdot 2}$$

$$-\frac{k^6 a_0 x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} - \frac{k^6 a_1 x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \cdots$$

Hence we conclude as follows:

$$y = \sum_{n=0} a_n x^n$$

$$= a_0\left[1 - \frac{k^2x^2}{2!} + \frac{k^4x^4}{4\cdot3\cdot2\cdot1} - \frac{k^6x^6}{6\cdot5\cdot4\cdot3\cdot2\cdot1} + \cdots\right]$$

$$+ a_1\left[x - \frac{k^2x^3}{3\cdot2} + \frac{k^4x^5}{5\cdot4\cdot3\cdot2} - \frac{k^6x^7}{7\cdot6\cdot5\cdot4\cdot3\cdot2} + \cdots\right].$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0\left[1 - \frac{k^2x^2}{2!} + \frac{k^4x^4}{4!} - \frac{k^6x^6}{6!} + \cdots\right]$$

$$+ \frac{a_1}{k}\left[kx - \frac{k^3x^3}{3!} + \frac{k^5x^5}{5!} - \frac{k^7x^7}{7!} + \cdots\right]$$

$$\boxed{y = a_0 \cos kx + \frac{a_1}{k} \sin kx.}$$ as required

Series soln:

$$\frac{d^2y}{dx^2} - k^2 y = 0$$

$$y = \sum_{n=0} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2} a_n n(n-1) x^{n-2}$$

Substitute

$$\sum_{n=2} a_n n(n-1) x^{n-2} - k^2 \sum_{n=0} a_n x^n = 0$$

shift index in 1st term from $n \longrightarrow n+2$

$$\therefore \sum_{n+2=2} a_{n+2}(n+2)(n+1) x^n - k^2 \sum a_n x^n = 0$$

$$\text{or} \quad \sum_{\substack{n+2=2 \\ n=0}} \left[a_{n+2}(n+2)(n+1) - k^2 a_n \right] x^n = 0$$

$$\Rightarrow a_{n+2} = \frac{k^2 a_n}{(n+2)(n+1)}$$

2nd order diff eq — need two const of integration

Say a_0, a_1

$\longrightarrow$ over

$n=0$

$$a_2 = \frac{k^2 a_0}{2\cdot 1}$$

$n=1$

$$a_3 = \frac{k^2 a_1}{3\cdot 2}$$

$n=2$

$$a_4 = \frac{k^2 a_2}{\cancel{3\cdot 2}\,4\cdot 3} = \frac{k^2}{\cancel{4}\cdot\cancel{3}}\,\frac{k^2 a_0}{2\cdot 1}$$

$$a_4 = \frac{k^4 a_0}{4\cdot 3\cdot 2\cdot 1}$$

$n=3$

$$a_5 = \frac{k^2 a_3}{5\cdot 4} = \frac{k^2\, k^2 a_1}{5\cdot 4\cdot 3\cdot 2} = \frac{k^4 a_1}{5\cdot 4\cdot 3\cdot 2}$$

$n=4$

$$a_6 = \frac{k^2 a_4}{6\cdot 5} = \frac{k^2\, k^4 a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{k^6 a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$n=5$

$$a_7 = \frac{k^2 a_5}{7\cdot 6} = \frac{k^2\, k^4 a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} = \frac{k^6 a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

etc!! ~~$n=6$~~

etc

$\therefore$

$$y = \sum_{n=0} a_n x^n = a_0 + a_1 x + \frac{k^2 x^2 a_0}{2\cdot 1} + \frac{k^2 a_1 x^3}{3\cdot 2} + \frac{k^4 x^4 a_0}{4\cdot 3\cdot 2\cdot 1}$$

$$+ \frac{k^4 x^5 a_1}{5\cdot 4\cdot 3\cdot 2} + \frac{k^6 x^6 a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{k^6 x^7 a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \cdots \text{ etc}$$

$$y = \sum_{n=0} a_n x^n = a_0\left[1 + \frac{k^2 x^2}{2!} + \frac{k^4 x^4}{4!} + \frac{k^6 x^6}{6!} + \cdots\right]$$

$$+ \frac{a_1}{k}\left[kx + \frac{k^3 x^3}{3!} + \frac{k^5 x^5}{5!} + \frac{k^7 x^7}{7!} + \cdots\right]$$

Now have a look at.

$$e^{kx} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} + \frac{k^5x^5}{5!} + \frac{k^6x^6}{6!} + \frac{k^7x^7}{7!} + \cdots$$

$$e^{-kx} = 1 - kx + \frac{k^2x^2}{2!} - \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} - \frac{k^5x^5}{5!} + \frac{k^6x^6}{6!} - \frac{k^7x^7}{7!}$$

$$e^{kx} + e^{-kx} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} + \cdots$$

$$+ 1 - kx + \frac{k^2x^2}{2!} - \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} - \frac{k^5x^5}{5!} + \cdots$$

$$= 2 + 2\frac{k^2x^2}{2!} + 2\frac{k^4x^4}{4!} + 2\frac{k^6x^6}{6!} + \cdots$$

or

$$\frac{e^{kx} + e^{-kx}}{2} = \cosh(kx) = 1 + \frac{k^2x^2}{2!} + \frac{k^4x^4}{4!} + \frac{k^6x^6}{6!} + \cdots$$

Now look at.

$$\frac{e^{kx} - e^{-kx}}{2} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} + \frac{k^5x^5}{5!} + \frac{k^6x^6}{6!} + \frac{k^7x^7}{7!} + \cdots$$

$$- \left(1 - kx + \frac{k^2x^2}{2!} - \frac{k^3x^3}{3!} + \frac{k^4x^4}{4!} - \frac{k^5x^5}{5!} + \frac{k^6x^6}{6!} - \frac{k^7x^7}{7!} + \cdots\right)$$

$$= \frac{2kx + 2\frac{k^3x^3}{3!} + 2\frac{k^5x^5}{5!} + 2\frac{k^7x^7}{7!} + \cdots}{2}$$

$$\frac{e^{kx} - e^{-kx}}{2} = \sinh(kx) = kx + \frac{k^3x^3}{3!} + \frac{k^5x^5}{5!} + \frac{k^7x^7}{7!} + \cdots$$

Now put it all together

$$\boxed{y = \sum_{n=0} a_n x^n = a_0 \cosh kx + \frac{a_1}{k} \sinh kx.}$$ as required

Mathematical Physics

Lecture, January 2002.

Method of Undetermined Coefficients

— used to solve inhomogeneous equation
particular soln.

pg 147 – hand-out | No B.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2x^2$$

Assume soln $y_p(x) = Ax^2 + Bx + C$

$$\frac{dy_p}{dx} = 2Ax + B$$

$$\frac{d^2y_p}{dx^2} = 2A.$$

$$\therefore \quad 2A + (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2$$

$$(2A + B - 2C) + x\left[2A - 2B\right] + x^2\left[-2A\right] = 2x^2$$

Conclusion

$$-2A = 2 \Rightarrow A = -1$$

$$2A - 2B = 0 \Rightarrow -2 - 2B = 0$$
$$B = -1$$

$$2A + B - 2C = 0 \Rightarrow -2C = -\left[2A + B\right]$$

$$C = \frac{2A + B}{2} = \frac{-2(2) - 2}{2} = -3/2$$

$\therefore \boxed{y_p(x) = -x^2 - x - 3/2}$ Check it out.

Special Problem

Suppose inhomogeneous term is a soln of homogeneous.

e.g. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

e^x solves $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

$e^x + e^x - 2e^x = 0.$

~~y = e~~ ~~y = e^{px}~~

$y = e^{px}$

$\frac{dy}{dx} = pe^{px}$

$\frac{d^2y}{dx^2} = p^2e^{px}$

$\therefore (p^2 + p - 2)e^{px} = 0$

$p^2 + p - 2 = 0$

$p = \frac{-p \pm \sqrt{p^2 + 8}}{2}$

diff roots

Now what.

See pg 148 handout.

$y_p(x) = axe^x$

$\frac{dy_p x}{dx} = ae^x + axe^x$

$\frac{d^2y_p}{dx^2} = ae^x + ae^x + axe^x$

$= 2ae^x + axe^x$

$\therefore \quad 2ae^x + axe^x + ae^x + axe^x - 2axe^x = e^x$

$3ae^x = e^x$

$3a = 1 \Rightarrow a = 1/3$

$\therefore \boxed{y_p(x) = \frac{x}{3}e^x}$

double root

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

$$p^2 - 2p + 1 = 0$$

$$p = \frac{2 \pm \sqrt{4 - 4}}{2} = 1$$

$\therefore$ in inhomogeneous — we try $y_p(x) = ax^2e^x$.

for $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$$\frac{dy_p}{dx} = 2axe^x + ax^2e^x$$

$$\frac{d^2y_p}{dx^2} = 2ae^x + 2axe^x + 2axe^x + ax^2e^x$$
$$= 2ae^x + 4axe^x + ax^2e^x.$$

Substituting

$\therefore$

$$2ae^x + 4axe^x + ax^2e^x - 2\left(2axe^x + ax^2e^x\right) + ax^2e^x = e^x$$

$$2ae^x + \cancel{4axe^x} + \cancel{ax^2e^x} - \cancel{4axe^x} - \cancel{2ax^2e^x} + \cancel{ax^2e^x} = e^x$$

$$2ae^x = e^x$$

$$2a = 1 \Rightarrow a = 1/2$$

$$\therefore \boxed{y_p(x) = \frac{x^2}{2}e^x}$$