① Continuous Symmetries

- U(1)/Phase Symmetries
  - These multiply a particle state by a complex phase depending on the conserved charge.
    + Ex a particle w/ charge q ~~gets~~ factor $e^{iq\phi}$
      for EM symmetry $|p\rangle \to \exp(ie\phi)|p\rangle$, $|e\rangle \to e^{-ie\phi}|e\rangle$
    + The transformation is $e^{iQ\phi}$ where Q = charge operator is the generator
  - Charges add
    + The transformation acts on each particle
      $$e^{iQ\phi}(|a\rangle|b\rangle) = (e^{iQ\phi}|a\rangle\, e^{iQ\phi}|b\rangle)$$
    + In the exp, we can replace Q by its e'value
      Then
      $$e^{iQ\phi}(|a\rangle|b\rangle) \to \exp(i(q_a+q_b)\phi)|a\rangle|b\rangle$$
      if $|a\rangle$ + $|b\rangle$ are e'states
  - Examples in the SM
    + Electric Charge Q, measured in proton units
      Associated with a force
    + Baryon Number B: quarks have $+1/3$, antiquarks $-1/3$
      (anti)baryons have ± 1, mesons have 0
      Means protons are stable in SM, but must not be completely conserved ✱
    + Lepton number L: leptons +1, antileptons −1, others 0 ✱
    ✱ Actually, while SM Feynman diagrams conserve B+L separately, there are other processes that turn 3 (anti)baryons ⟷ 3 antileptons ⇝ conserve B−L.
      In fact in SM, only B−L (or μ−τ) numbers can be good quantum numbers.
    ★ We need B+L to be broken to have more matter than antimatter!

- SU(2) (or SO(3)): key example is rotation w/ angular momentum conserved

- Conservation Law
  + Generators J_1, J_2, J_3 (could be $\vec{L}$ = orbital ang. mom, $\vec{S}$ = spin, or something else) don't all commute

  + But $\vec{J}^2$ commutes with all. A state is labeled by $\vec{J}^2$ e'value $j(j+1)$ and J_3 e'value m (remember I am using $\hbar = 1$)

  + Conservation means j and m stay the same

- Lightning "spin-½" ie $j = ½$ review. Called <u>doublet</u>
  + As we said before, generators are $\vec{J} = \vec{\sigma}/2$ for Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  + States in the rep are 2 component "spinors"

$$|j = \tfrac{1}{2}, m = +\tfrac{1}{2}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |j = \tfrac{1}{2}, m = -\tfrac{1}{2}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  + Eigenstates of J_1, J_2 are superpositions of those

  + A rotation of angle θ around axis $\hat{n}$ acts as $\exp(\theta \hat{n} \cdot \vec{\sigma}/2)$ on the "spinor"

- Addition of "Angular Momentum"
  + Here we will state results; derivation is in QM class PHYS-3301

  + If you have 2 systems with reps ("spins") j_1 and j_2, what rep describes the total system?

+ The j quantum number runs between $|j_1-j_2|$ to (j_1+j_2) in steps of 1. ie $|j_1-j_2|, |j_1-j_2|+1, \ldots (j_1+j_2)-1, (j_1+j_2)$

+ For a given state with m_1+m_2, $M = m_1+m_2$ since the J_z operators add

+ Given a state $|j_1 m_1\rangle|j_2 m_2\rangle$, what is it in the $|j,m\rangle$ basis? Ex

$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle = |1,+1\rangle$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = (|1,0\rangle + |0,0\rangle)/\sqrt{2}$$
$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$$

You get the change of basis by reading across a row of the appropriate table of Clebsch-Gordan coefficients. Remember to put a square root on the entries.

+ Reading down the Clebsch-Gordan table gives $|j,m\rangle$ in terms of $|j_1 m_1\rangle|j_2 m_2\rangle$. Ex

$$\left.\begin{array}{l} |1,1\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle \\ |1,0\rangle = (|\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle)/\sqrt{2} \\ |1,-1\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{array}\right\} \text{triplet}$$
$$|0,0\rangle = (|\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle)/\sqrt{2} \quad \text{singlet}$$

+ To add 3 or more, add a pair, then add that to the next one.
+ See examples in the text!
+ In group theory, this is asking how the "product" of reps. breaks into a sum of "irreducible" reps.

∘ Isospin: an internal symmetry almost like "weak charge" that the strong force respects.

+ Both u + d quarks are very light compared to any hadrons + contribute about the same way to hadron masses

+ So say that (u, d) makes an isospin $I = \frac{1}{2}$ doublet

$$u = |I = \tfrac{1}{2}, I_3 = +\tfrac{1}{2}\rangle, \quad d = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

and $(\bar{u}, \bar{d})$ also make a doublet

$$\bar{d} = -|\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \quad \bar{u} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

+ Particles made of $u, d, \bar{u}, \bar{d}$ fit into isospin multiplets given by "adding" isospin

Ex Mesons are combination of $2 \times I = \frac{1}{2}$, we get

$$|I = 1, I_3 = +1\rangle = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = -u\bar{d} = \pi^+$$

$$|I = 1, I_3 = 0\rangle = \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle\right)/\sqrt{2}$$

$$= (u\bar{u} - d\bar{d})/\sqrt{2} = \pi^0$$

$$|1, -\tfrac{1}{2}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \pi^-$$

$\Rightarrow$ the pions form an $I = 1$ triplet

+ the same is true of ρ vector mesons

But adding $I = \frac{1}{2}$ to $I = \frac{1}{2}$ also gives a singlet

$$|0, 0\rangle = \left(|\tfrac{1}{2}, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, +\tfrac{1}{2}\rangle\right)/\sqrt{2}$$

$$= (u\bar{u} + d\bar{d})/\sqrt{2} \simeq \eta \text{ or } \eta'$$

This is not quite true b/c of strange quark

$$\eta = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}, \text{ etc}$$

Ex Baryons are 3 quarks, or $3 \times I = \frac{1}{2}$ if made out of $u + d$. These give $I = 3/2$ or $I = 1/2$

The $I = 3/2$ particles are Δ baryons

$\Delta^{++} = uuu$, $\Delta^+ = uud + \ldots$, $\Delta^0 = udd + \ldots$, $\Delta^- = ddd$

while $I = 1/2$ are nucleons $p^+ = uud + \ldots$; $n^0 = udd + \ldots$

+ Strange quarks have $I = 0$, so "addition" of I is easier. Ex Kaons have $I = \frac{1}{2}$

$$\left.\begin{aligned} K^+ &= |\tfrac{1}{2}, \tfrac{1}{2}\rangle = u\bar{s}, \quad K^0 = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = d\bar{s} \\ \bar{K}^0 &= |\tfrac{1}{2}, +\tfrac{1}{2}\rangle = -s\bar{d}, \quad K^- = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = s\bar{u} \end{aligned}\right\} \text{2 doublets}$$

etc

+ For scattering or decays, isospin symmetry means that (for strong force processes)

a) Isospin I and I_3 can't change.
b) There is 1 probability amplitude for each value of I (indep. of I_3). This applies to many processes.

Ex: $\pi^+ + p = |1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{3}{2},+\tfrac{3}{2}\rangle$; $\pi^- + n = |\tfrac{3}{2},-\tfrac{3}{2}\rangle$
so the amplitudes
$$\mathcal{M}(\pi^+ + p \to \pi^+ + p) = \mathcal{M}(\pi^- + n \to \pi^- + n) \equiv \mathcal{M}_3$$
Meanwhile
$$\pi^- + p = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle,\quad \pi^0 + n = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$
So
$$\mathcal{M}(\pi^- + p) = \tfrac{1}{3}\mathcal{M}_3 + \tfrac{2}{3}\mathcal{M}_1 \text{ vs } \mathcal{M}(\pi^0 + n) = \tfrac{2}{3}\mathcal{M}_3 + \tfrac{1}{3}\mathcal{M}_1$$
$$\text{vs } \mathcal{M}(\pi^- + p \to \pi^0 + n) = \tfrac{\sqrt{2}}{3}(\mathcal{M}_3 - \mathcal{M}_1)$$
Also helps with nuclear scattering.

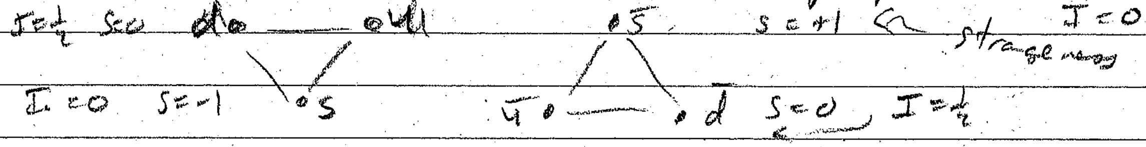
– SU(3) symmetries

- Color Symmetry: the symmetry of the strong force
  - The 3 colors form the fundamental rep of SU(3)
    $$|R\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix},\ |G\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix},\ |B\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}\ ;\ |\text{color}\rangle \to U|\text{color}\rangle$$
  - The 3 anticolors of antiquarks make an antifundamental a $\bar{3}$ rep
  - Quarks always must combine to form a singlet (trivial) rep.
    quark+antiquark $\to (|R\bar{R}\rangle + |G\bar{G}\rangle + |B\bar{B}\rangle)/\sqrt{3}$
    quark+quark+quark $\to (|RGB\rangle - |RBG\rangle + |BRG\rangle - |BGR\rangle + |GBR\rangle - |GRB\rangle)/\sqrt{6}$

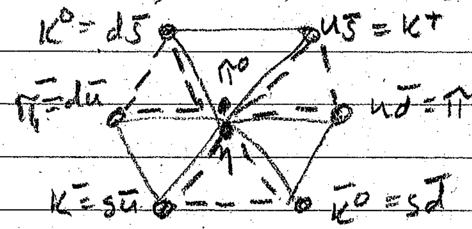
- Flavor SU(3):
  - Consider adding the strange quark to the mix, so (u,d,s) form SU(3) fundamental
  - Antiquarks form antifundamental
  - Mesons can form any rep $3\times\bar{3} = 8+1$
    Baryons can form any rep $3\times3\times3 = 10+8+8+1$
  - Not a great symmetry b/c s quarks are much heavier!

- Application: Hadron Wavefunctions + Eightfold Way
  - States are composed of various factors
    - + Space = wavefunction in $\vec{r}$, orbital ang mom.
    - + Spin + Color + Flavor ($SU(3)$)

- Mesons: Quark + Antiquark
  - + We'll consider spatial ground states, but excited states exist.
  - + Total spin can be $S=0$ (pseudoscalar mesons) or $S=1$ (vector mesons), all else similar
  - + Color is always the $3\times\bar{3}$ singlet
  - + To understand flavor, take graphical version



Then the (scalar) mesons are



$I=\frac{1}{2}$, $S=+1$
$I=1$, $S=0$
$I=0$
$I=\frac{1}{2}$, $S=-1$

These are the octet states. There is a 3rd particle η' at the center, which is the singlet

- + You can add c, b, and t quarks to get 3D (or higher) plots, but the symmetry is bad due to heavy quark masses. Notable mesons are
  D and D_s ($c+\bar{u},\bar{d},\bar{s}$); B, B_s, B_c (b and other)
  Charmonium ($c\bar{c}$ inc. η_c, J/ψ), Bottomonium

- Baryons: 3 indistinguishable quarks.
  - + By Pauli exclusion principle, total state must be antisymmetric when exchanging any pair
  - + Spatial ground state should be symmetric (no orbital ang. mom)

+ Color must be the totally antisymmetric 3-quark singlet
$\Rightarrow$ Spin $\times$ flavor state must be symmetric

+ Possible spins are $s = \frac{3}{2}$ (totally symmetric) and 2 combinations of $s = \frac{1}{2}$ w/ mixed symmetry

+ For $s = \frac{3}{2}$, flavor must be totally symmetric. This is the SU(3) <u>decuplet</u> rep

Δ^- Δ^0 Δ^+ Δ^{++} $S=0,\ I=3/2$
Σ^* $S=-1,\ I=1$
Ξ^* $S=-2,\ I=1/2$
Ω^- $S=-3,\ I=0$

+ For $s = \frac{1}{2}$, the flavor state is a mixed symmetry <u>octet</u> of SU(3).

n p $I=\frac{1}{2},\ S=0$
Σ^- Σ^0 Λ Σ^+ $\left.\begin{matrix} I=1 \\ I=0 \end{matrix}\right\} S=-1$
Ξ^- Ξ^0 $I=\frac{1}{2},\ S=-2$

These particles have states that are a linear superposition of the corresponding mixed symmetry spin & flavor states. See textbook

+ There are known baryons w/ c + b quarks. Again, symmetry is not good, but there are diagrams

- These patterns are called the <u>Eightfold Way</u> (after the Eightfold Path of Buddhism).
  + Murray Gell-Mann proposed strangeness as a quantum number.
  + He + Yuval Ne'emann organized the isospin multiplets into these SU(3) multiplets
  + This was a way to organize many seemingly random hadrons. Led to idea of quarks + color!
  + Charm, Bottom, Top discovered from 1970s to 1995
  + More recently, discovery of tetraquarks + pentaquarks