Symmetries In Particle Physics

• Symmetries, Conservation Laws, + Group Theory

- A symmetry is a transformation that leaves the laws of physics (Hamiltonian, etc) the same.
  - Symmetry transformations change the quantum numbers of the state (or other feature of wavefunction)
    - + Rotation around x-axis changes ang. momentum m quantum number
    - + Electromagnetism has a symmetry multiplying the wavefunction by a complex phase (charged particles)
  - There are spacetime symmetries and internal symmetries
    - + Spactime symmetries act on coordinates. Include reflections, translations of the origin, rotations, etc
    - + Internal symmetries change other properties like wavefunction phase or particle type
- Noether's Theorem: A symmetry implies the existence of a conservation law (no proof in this class)
  - In QM, the conserved quantity (aka conserved charge) is a Hermitian operator
    - + The expectation value of this operator is constant
    - + If the initial state is an eigenstate of the operator, the state remains an eigenstate with the same eigenvalue.

- Examples:
  + Translations of time origin ⇒ conserved energy
  + Translations of space origin ⇒ conserved momentum
  + Rotations ⇒ angular momentum
  + Electromagnetic phase rotation ⇒ electric charge

  + We will see other spacetime + internal symmetries and use their conservation laws.

— Symmetries are groups
- A symmetry transformation acts on physical quantities
  + Ex A rotation acts on position vectors $\vec{x} \to R\cdot\vec{x}$ for rotation matrix R
  + Repeated transformations defines a "multiplication rule"
    ex $R_2\cdot(R_1\cdot\vec{x}) = (R_2\cdot R_1)\cdot\vec{x}$ (by composition)

- The set of symmetry transformations obey the definition of an abstract mathematical group

  + A group G w/elements g_i obeys the following rules:
    1) Closure: $g_1 g_2 = g_3 \in G$, ie, a product of 2 group elements is a group element
    2) Identity: There is an element 1 st $1\cdot g = g\cdot 1 = g \;\forall g\in G$
    3) Inverse: $\forall g\in G$, there is $g^{-1}\in G$ s.t. $g^{-1}\cdot g = g\cdot g^{-1} = 1$
    4) Associativity: $g_1\cdot(g_2\cdot g_3) = (g_1\cdot g_2)\cdot g_3$

  + Group multiplication generally does not commute. Groups that commute are called Abelian

  + Groups may be finite (ex $\mathbb{Z}_2 \equiv \{1,-1\}$ with multiplication) or infinite (the positive rational numbers)

  + Groups may be discrete (like the above, or see math class) or continuous (Lie groups)

+ Different physical symmetries can be described by the same mathematical group

- Many Lie groups are defined as sets of matrices (w/ matrix multiplication)

  + $U(n)$ = $n \times n$ complex unitary matrices $U^\dagger = U^{-1}$ $(U^\dagger = (U^*)^T)$
  + $SU(n)$ = "special" $U(n)$ matrices meaning $\det U = 1$

  + $O(n)$ = $n \times n$ real orthogonal matrices $O^T = O^{-1}$
  + $SO(n)$ = special $O(n)$ matrices

  + 3D rotations are $SO(3)$, which is almost $= SU(2)$
  + We will see $SU(2)$ and $SU(3)$ as internal symmetries

- A <u>representation</u> of a group G is a map $g \to M(g)$ of group elements to matrices that have the same multiplication rule as the group: $M(g) \cdot M(h) = M(g \cdot h)$

  + For any G, the <u>trivial rep.</u> is $g \to I$ (all $\to$ identity matrix)

  + For a group of matrices, representing the matrix g by itself is the <u>fundamental rep.</u>

  + But there are other reps!
    <u>Ex</u> There are spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ rotation matrices that give reps. of every matrix dimension for $SO(3)$ (really $SU(2)$)

  + If a physical quantity transforms by matrix multiplication w/ matrices of a particular rep, we say it transforms "in" that rep.
    <u>For ex</u>, $\vec{x} \to R \cdot \vec{x}$ for positions when rotated, so $\vec{x}$ is "in the fundamental" of $SO(3)$

$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$ for matrices A

- Suppose we have a representation of unitary matrices for a Lie group, meaning $M(g) = U$ is unitary

  + We can write each $U(g) \equiv \exp\left(i \sum_a \theta^a T^a\right)$ where θ^a are a set of "rotation angles" that identify the group element g and the T^a are Hermitian matrices called <u>generators</u> for that rep.

  + The generators obey commutation relations
$$\left[T^a, T^b\right] = 2i \sum_c f^{abc} T^c$$
Lie algebra

    where f^{abc} are <u>structure constants</u> that depend only on the group and <u>not</u> the representation. f^{abc} is totally antisymmetric and controls group multiplication.

- <u>Example</u> $SO(3)$ + $SU(2)$, which have the same multiplication rule

  + These have a dimension $(2s+1)$ rep. for each value $s = 0, \frac{1}{2}, 1, \ldots$ for angular momentum (spin). (But these symmetries can represent more than just spin.)
  + The $s=0$ rep is trivial. All generators $= 0$. The state of a particle is given by a 1D vector.

  + The $s = \frac{1}{2}$ rep acts on 2D spinor variables. The generators are $\frac{1}{2}\vec{\sigma}$ where σ^i are <u>Pauli matrices</u>
$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
    which are familiar as the spin-$\frac{1}{2}$ angular momentum operators.

  + You probably worked out the $s=1$ spin matrices in quantum mechanics. Remember commutation rule $[S^i, S^j] = 2i\,\epsilon^{ijk} S^k \Rightarrow$ structure constants $= \epsilon^{ijk}$
  + The allowed values of angular momentum are really determined by group theory of $SU(2)$ and $SO(3)$!