Scattering + Feynman Rules

Decay Rates + Cross Sections

- Decays

- Decays are random
  + The chance of a decay of particle A to B, C, D, ... in time Δt does not depend on how long A has existed (all A particles are identical).
  + We can measure that probability by looking at a large number of particles:
    (prob / time) = (# decays/time) / (number of particles)
  + Mathematically, this is
    $$\Gamma = -(dN/dt)/N \quad \text{a} \quad N = N(0)\, e^{-\Gamma t}$$
    since $\Delta N = -(\#\text{ decays})$.

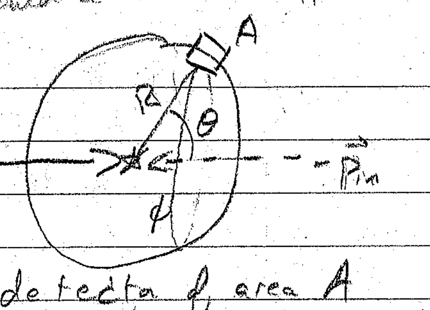
- Multiple Decay modes:
  + What if a particle can decay different ways like $A \to B+C$ or $A \to D+E+F$? Most particles have many decays listed in RPP.
  + Each decay process has separate rate Γ_i. The total decay rate is $\Gamma = \Sigma \Gamma_i$
  + The <u>branching ratio</u> for a mode is Γ_i/Γ. This is the fraction of particles that decay that way.
  + The lifetime is $\tau = 1/\Gamma$. We measure in the initial particle's rest frame; it time dilates in other frames.

- Cross Sections

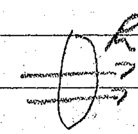
- This is when 2 particles enter the region + interact
  + They could just scatter / change directions $A+B \to A+B$ or create new/different particles $A+B \to C+D+\ldots$
  + The cross-section is the effective size (area) of B as a target for A (or vice versa). Depends on A, B, and the products
  + We'll use CM frame values. It can also depend on the frame

could be $C = A$ or differ

- Differential Cross Section $A+B \to C + \ldots$

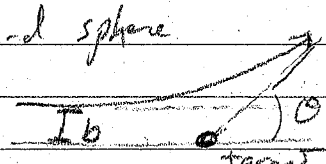


  + (In CM frame) set up detectors in spherical arrangement around interaction point.
  How many C particles enter a detector of area A located at angles θ, ϕ?
  + θ = <u>scattering angle</u>, the angle of $\vec{p}_C$ compared to $\vec{p}_A = \vec{p}_{in}$ (how much direction changes). ϕ = azimuthal angle around $\vec{p}_{in}$
  + The number of C particles going into detector is $N \propto A/R^2$ due to particles spreading out.
  + Like a length l on a circle of radius R takes up angle $\theta = l/R$ in radians, an area A on a sphere takes up <u>solid angle</u> $\Omega = A/R^2$ in <u>steradians</u>. There are 4π total steradians, and an infinitesimal "rectangular" solid angle is $d\Omega = \sin\theta \, d\theta \, d\phi$
  + Suppose we have a beam of incoming particles with <u>luminosity</u> $\mathcal{L} = \#/\text{time}/\text{area}$ passed through. Then number into detector is

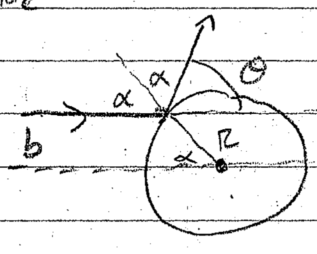


  $dN = \mathcal{L} \, d\sigma$, where $d\sigma$ = cross section area for scattering into detector
  + But, $dN \propto d\Omega$, so the physical quantity is the <u>differential cross section</u> $d\sigma/d\Omega \equiv \frac{1}{\mathcal{L}} dN/d\Omega$
  The <u>total cross section</u> is $\sigma = \int d\Omega (d\sigma/d\Omega)$
  This is a probabilistic definition.

- Classical Example: Scattering from a hard sphere
  + Classically, scattering angle depends on <u>impact parameter</u> b, the distance incident particle misses the scattering center

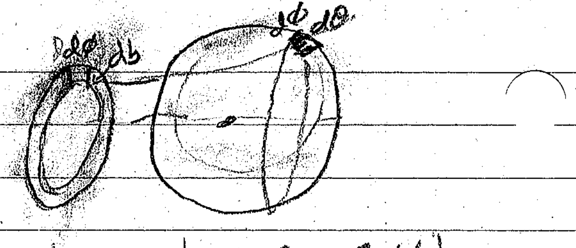


  + Consider elastic scattering from infinitely hard sphere of radius R



  + b and θ depend on angle α of reflection
  $b = R \sin\alpha, \quad \alpha = (\pi - \theta)/2 \implies b = R\cos(\theta/2)$

+ Generally, the incoming target area for scattering angle θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$ is



$$d\sigma = |b\, db\, d\phi| \quad \text{vs} \quad d\Omega = |\sin\theta\, d\theta\, d\phi|$$

Therefore

$$d\sigma/d\Omega = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \leftarrow \text{general}$$

+ For the hard sphere, $db/d\theta = -(R/2)\sin(\theta/2)$

So $\frac{d\sigma}{d\Omega} = R^2 \frac{\sin(\theta/2)\cos(\theta/2)}{2\sin\theta} = R^2/4$; $\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \pi R^2$

which is indeed the profile (cross section) area of sphere

— Fermi's Golden Rule: how to calculate Γ + σ

- Fermi's Golden Rule states that the probability rate for a transition from one quantum state is given by 2 ingredients:
  + squared probability amplitude for transition from given initial state to each possible final state: dynamics

depends on Hamiltonian →

  + integrated over all possible final states (phase space) which is universal kinematics
  + We'll try to justify but not rigorously derive

- A useful fact: $\delta(f(z)) = \sum_i \delta(z - z_i) / |f'(z_i)|$ where z_i are all roots of $f(z) = 0$.

- The phase space of a single particle counts momentum states
  + Relativistically, we want to integrate over 4-momenta
  + But energy is determined by $p^2 = m^2$ and $p^0 \geq 0$ (Particles are "on the mass shell" or "on-shell")
  + The phase space is therefore

2π factors from FT →

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\Theta(p^0) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p} \quad \text{where } E_p = \sqrt{\vec{p}^2 + m^2}$$

  + The factor of $1/2E$ is due to normalization. Think about densities. With a boost, volume is length contracted, $V \to V/\gamma$, so $n \to \gamma n$. Therefore the probability density of a particle $\propto E$, its energy

• Two Particle Decays $A \to B + C$.

+ Number of decays per time dN_A/dt given by Golden rule. This is ΓN_A with $N_A \propto 2E_A$ by normalization.

+ If we assume A at rest, Golden Rule gives

$$\Gamma = \left(\frac{1}{2m_A}\right) S \int \frac{d^3\vec{p}_B}{(2\pi)^3} \int \frac{d^3\vec{p}_C}{(2\pi)^3} \frac{1}{4E_B E_C} (2\pi)^4 \delta^4(p_A - p_B - p_C) |\mathcal{M}|^2$$

which is

a) Normalization factor

b) S = symmetry factor = 1 for $B \neq C$, $\frac{1}{2}$ for $B = C$ (identical particles)

c) phase space for each of $B + C$

d) restriction to part of phase space that conserves 4-momentum

e) squared probability amplitude ("matrix element")

+ For A at rest,

$$\delta^4(p_A - p_B - p_C) = \delta(m_A - E_B - E_C)\,\delta^3(\vec{p}_B + \vec{p}_C)$$

Can do $\vec{p}_C$ integral easily ($\vec{p}_C = -\vec{p}_B$)

$$\Gamma = \frac{S}{2m_A} \int \frac{d^3\vec{p}_B}{(2\pi)^3} \frac{1}{4E_B E_C} (2\pi)\,\delta(m_A - E_B - E_C) |\mathcal{M}|^2$$

+ In general $\mathcal{M}(\vec{p}_B)$. However, if we average over the possible spins of particle A, it must be spherically symmetric. Then write

$$d^3\vec{p}_B |\mathcal{M}|^2 = p^2\,dp\,d\Omega\,|\mathcal{M}(p)|^2$$

In polar coords. Integral over $d\Omega$ is easy $= 4\pi$. (You can keep spin of A as "polarized rate", but it's harder, so we'll always average over spin).

+ Now

$$\Gamma = \frac{S}{4m_A} \int \frac{dp}{2\pi} \frac{p^2}{E_B E_C} |\mathcal{M}|^2 \delta(m_A - E_B - E_C)$$

+ Use $E_B = \sqrt{p^2 + m_B^2}$, $E_C = \sqrt{p^2 + m_C^2}$ to get $\frac{d}{dp}(E_B + E_C) = \frac{p}{E_B} + \frac{p}{E_C}$

so

only $\neq 0$ for $m_A \geq m_B + m_C$ $\longrightarrow$ $$\delta(m_A - E_B - E_C) = \delta(p - p_f) \frac{E_B E_C}{p_f (E_B + E_C)}$$

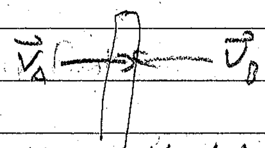
where $E_B + E_C = m_A$ by energy conservation with $p_f^2 = E_B^2 - m_B^2$

+ Then

$$\Gamma = \frac{S\,p_f}{8\pi m_A^2} |\mathcal{M}|^2 \qquad (*)$$

• Two-to-two Scattering Cross Sections $A+B\to C+D$

+ The scatterings per unit time per target A particle
$= (1/N_A)\, dN_{scatt}/dt = N_B\, |\vec{v}_A - \vec{v}_B|\, \sigma$
with $N_A \propto 2E_A$, $N_B \propto 2E_B$.



$|\vec{v}_A - \vec{v}_B|$ = relative velocity in lab frame
= how fast the particles approach each other in the lab

+ We have a factor
$$E_A E_B\, |\vec{v}_A - \vec{v}_B| = E_A E_B\, |\vec{p}_A/E_A - \vec{p}_B/E_B|$$
In CM frame, $\vec{p}_B = -\vec{p}_A$, so this is $= |\vec{p}_A|\,(E_A+E_B) = |\vec{p}_i|\, E_{CM}$
where E_{CM} = total CM frame energy, and $\vec{p}_i = \vec{p}_A$ is the momentum of 1 incoming particle in CM frame.

+ The Golden Rule looks similar to the lifetime
$$\sigma = \frac{S}{4E_{CM}|\vec{p}_i|}\int\frac{d^3\vec{p}_C}{(2\pi)^3}\int\frac{d^3\vec{p}_D}{(2\pi)^3}\frac{1}{4E_C E_D}(2\pi)^4\delta^4(p_A+p_B-p_C-p_D)\,|\mathcal{M}|^2$$
with the same type of factors.

+ The calculation is the same, except now $\mathcal{M}$ can depend on $\vec{p}_i$ and $\vec{p}_C$. If we take $\vec{p}_i$ along z, that means it depends on $p = |\vec{p}_C|$ and θ, the polar angle of $\vec{p}_C$.
So we end up with
$$\sigma = \frac{S}{64\pi^2}\,\frac{|\vec{p}_f|}{E_{CM}^2|\vec{p}_i|}\int d\Omega\, |\mathcal{M}|^2$$
b/c we can't do angular integrals yet.
Here
$|\vec{p}_f|$ solves energy conservation for $E_C + E_D = E_{CM}$.
(This is the same calculation as before.

+ We are left with differential cross section
$$d\sigma/d\Omega = \frac{S}{64\pi^2 E_{cm}^2}\,\frac{|\vec{p}_f|}{|\vec{p}_i|}\,|\mathcal{M}|^2 \qquad (*)$$

+ This is true for CM frame. But it is also invariant for boosts along the direction of $\vec{p}_i$ b/c transverse coordinates don't length contract.