Spinors

– Lorentz Transformations + Group Theory

- Lorentz transformations form a group
  + 4-vectors transform in the fundamental rep.
    $$a \to \Lambda a$$
  + Rotations are a subgroup (part of Lorentz group), each rep of Lorentz group is rep. of rotations
  + 4-vectors are fundamental (vector) rep of rotations. For other spins, we need other Lorentz reps

- Lorentz transformations + generators
  + For an object ψ in a Lorentz rep, transformation is $\psi \to M(\Lambda)\psi$, where $M(\Lambda)$ = matrix rep of Lorentz matrix Λ
  + We can write in terms of generators
    $$M(\Lambda) = \exp(i\vec{\theta}\cdot\vec{S}) \quad \text{for rotations}$$
    $$ = \exp(i\vec{\phi}\cdot\vec{B}) \quad \text{for boosts}$$
    where $\vec{\theta}$ tells axis + angle of rotation, $\vec{\phi}$ tells direction + rapidity of boost
  + For 4-vectors, generators along z direction are
    $$S^z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B^z = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$
    (Notice $\vec{B}$ is not Hermitian)
  + Generators have appropriate commutators, etc

– Spin-$\frac{1}{2}$

- For spin-$\frac{1}{2}$ states, we should have $\vec{S} \sim \vec{\sigma}/2$ (in terms of Pauli matrices) and $\vec{B}$ that satisfy the right commutators

- Dirac's Solution
  + Suppose we have a 4-vector of 4×4 matrices γ^μ (like $\vec{\sigma}$ is a 3D vector of 2×2 matrices) such that

$$(*) \qquad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\underbrace{1_{4\times 4}}_{\text{identity}}$$

+ A set of matrices that obey this anticommutation relation are called <u>Dirac Gamma matrices</u>

+ Then the commutator $J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ gives a rep of Lorentz generators with

$$B^i = \frac{i}{4}[\gamma^0, \gamma^i], \quad S^i = \frac{1}{2}\epsilon^{ijk}\left(\frac{i}{4}\right)[\gamma^j, \gamma^k] \qquad (\dagger)$$

+ The Griffiths book uses

$$\gamma^0 = \begin{bmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -I_{2\times2} \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0_{2\times2} & \sigma^i \\ -\sigma^i & 0_{2\times2} \end{bmatrix}$$

which gives

$$\vec{S} = \frac{1}{2}\begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}, \quad \vec{B} = \frac{i}{2}\begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \quad \text{by } (\dagger)$$

+ Other books use different conventions (choice of basis) What matters is (*). We'll try to keep to basis independent features

- A <u>spinor</u> ψ is a 4-component (mathematical) vector ← column that transforms as $\psi \to M_{\frac{1}{2}}(\Lambda)\psi$ where $M_{1/2}$ has generators given by γ matrices in $(\dagger)$
  + These are <u>not</u> 4-vectors!

  + The <u>Dirac conjugate</u> $\bar{\psi} \equiv \psi^\dagger\gamma^0$ ← row has inverse transformation $\bar{\psi} \to \bar{\psi}(M_{1/2})^{-1}$
  + Dirac matrices are Lorentz invariant (same always) when transforming 4-vector + spinor indices

- Combination of spinors
  + $\bar{\psi}\psi$ forms a Lorentz scalar (invariant)
  + $\bar{\psi}\gamma^\mu\psi$ makes a 4-vector
  + Suppose we define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0_{2\times2} & 1_{2\times2} \\ 1_{2\times2} & 0_{2\times2} \end{bmatrix}$
  + $\bar{\psi}\gamma^5\psi$ is pseudoscalar
  + $\bar{\psi}\gamma^\mu\gamma^5\psi$ is pseudovector
  + $\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$ = antisymmetric tensor (like $F^{\mu\nu}$)
  + These make 16 components