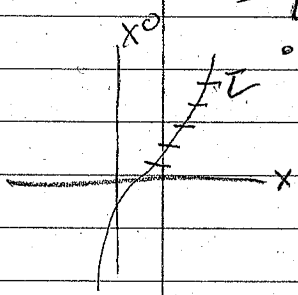
Momentum + Energy

- 4-velocity



- How do we make a 4-vector velocity?
  - Define $U^\mu = dx^\mu/d\tau$ as 4-velocity
  - This is how fast the spacetime position of the particle changes with respect to the particle's clock
  - U^μ is a 4-vector b/c dx^μ is a 4-vector and $d\tau$ is Lorentz invariant
- The components
  - We've seen $dt = \gamma d\tau$, so $U^0 = dt/d\tau = \gamma$ where γ is the γ factor of the particle's speed
  - Space components are
    $$U^i = (dt/d\tau)(dx^i/dt) = \gamma\, dx^i/dt$$
    $d\vec{x}/dt = \vec{v}$ is the "regular velocity" that gives the γ factor.

- Normalization $U^2 = (g_{\mu\nu}\, dx^\mu dx^\nu)/d\tau^2 = +1$
  This is always true, no matter the speed

- Momentum + Energy
  - Define momentum as 4-vector $p^\mu = mU^\mu$ where m = particle's rest mass
    - Components are $p^0 = \gamma m$, $p^i = \gamma m v^i$
    - The square is
      $$p^2 = m^2 U^2 = m^2 \quad (*)$$

  - Energy
    - At small velocity, $p^0 = \gamma m \cong m/\sqrt{1-v^2} \approx m + \frac{1}{2}mv^2 + \cdots$
    - This is rest mass energy plus kinetic energy (plus corrections)
    - We should define $p^0 = E$ = relativistic energy
    - Eqn $(*)$ becomes
      $$E^2 - \vec{p}^{\,2} = m^2 \text{ or } \boxed{E^2 = \vec{p}^{\,2} + m^2}$$

- Massless Particles
  + For a normal massive particle, $\vec{p}/p^0 = (\gamma m\vec{v})/(\gamma m) = \vec{v}$
  + But there is no problem taking the $m \to 0$ limit:
    $p^2 = m^2 \to 0$ just means p^μ is lightlike
  + This relationship becomes $E = |\vec{p}|$, so the particle's speed is $|\vec{v}| = |\vec{p}|/E = 1$ = speed of light
  + Massless particles travel at the speed of light always. They include photons, gluons, some hypothetical particles

recall $\hbar = 1$ →
- Quantum mechanics assigns $E = \omega$ for ω = angular freq. and $\vec{p} = \vec{k}$ for $\vec{k}$ = wavevector, $|\vec{k}| = 2\pi/\lambda$, λ = wavelength. Allows us to define 4-vector $k^\mu = (\omega, \vec{k})$

– Note on measurements + invariant quantities
- What is a particle's energy in my rest frame?
  + In my frame, my 4-velocity is $U^\mu = (1, 0, 0, 0)$ and the particle's momentum is $p^\mu = (E, \vec{p})$
  + The dot product is
    $$U \cdot p = E$$
  + But suppose we know the components of U^μ, p^μ in some other lab frame. Dot product is invariant, so $U \cdot p$ might "look different" but still tells me the particle's energy measured in <u>my frame</u>

- Doppler Effect
  + Similarly, if I observe a light ray / photon w/ wavevector k^μ, $U \cdot k = \omega_{obs}$, where U^μ = my (observer) 4-velocity (ω_{obs} = observed frequency)
  + But suppose I am moving in the emitter's rest frame
    $U^\mu = (\gamma, \gamma\vec{v})$, $k^\mu = (\omega_{em}, \vec{k})$
  + Then
    $$\omega_{obs} = U \cdot k = \gamma(\omega_{em} - \vec{v}\cdot\vec{k})$$

em $\vec{k}$ obs

If I am moving toward emitter, $\vec{v} \propto -\vec{k}$ and $|\vec{k}| = \omega_{em}$
so $\omega_{obs} = \gamma\omega_{em}(1+v) = \omega_{em}\sqrt{(1+v)/(1-v)}$
Similarly if I move away, $\vec{v} \propto \vec{k}$, so $\omega_{obs} = \omega_{em}\sqrt{(1-v)/(1+v)}$

- CM frame energy
  + Given a collection of particles, there is an inertial frame where total spatial momentum $= 0$
  + Nonrelativistically, this is the rest frame of center of mass, so we call it CM frame.
  + Consider total momentum 4-vector $\sum_i p_i^\mu$, i = which particle
    Space component totals to zero, so
    $$(\Sigma p)^2 = (\Sigma p^0)^2 = E_{tot,CM}^2$$
  + In other words, square of total 4-momentum (measured in any frame) = square of total CM frame energy

— Conservation of momentum + energy in particle collisions
- Using invariant quantities, like the dot product, simplifies most calculations
  + Note the difference between invariant (same in all frames) vs conserved (same before + after collision)
  + Use of invariants means we can analyze a problem in the easiest frame (often CM)

- <u>Example</u>: A pion at rest decays into a muon and antineutrino. What is the muon's energy?
  Assume antineutrino is massless
  + Call the 4-momenta p^μ for π, q^μ for μ^- and k^μ for $\bar{\nu}$
    Conservation is $p^\mu = q^\mu + k^\mu$
  + Subtract across, square, and remember $p^2 = m^2$ for a particle

$$k^2 = (p-q)^2 \Rightarrow 0 = m_\pi^2 + m_\mu^2 - 2p \cdot q$$

  + The pion is at rest, so $p^\mu = (m_\pi, 0)$ and $q^\mu = (E, \vec{q})$
    We find
    $$2m_\pi E = m_\pi^2 + m_\mu^2 \Rightarrow E = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

- <u>Example</u>: A particle of mass M decays into particles of masses $m_1, m_2, m_3, \ldots$ When is this possible?

+ In the initial particle rest frame, the minimum total final energy occurs if all final particles are at rest.
That's the sum of their masses. Initial energy is M, so
$$M \geq \sum_i m_i$$
+ What if the initial particle is moving? $p^\mu_{init} = \sum p_i^\mu$
and we know the square gives $M^2 = E^2_{tot,cm} \geq (\sum m_i)^2$
That works b/c of invariance!

- Example: Particles A and B collide. What is the minimum energy they need to create some specific set of other particles.

  + As above, we know that $E_{tot,cm} \geq \sum_i m_i$
  = total mass of final particles.

  + Suppose we set up A + B to collide in their CM frame.
  Then $E_A + E_B \geq \sum m_i$
  + What if B is at rest? Then $(p_A + p_B)^2 \geq (\sum_i m_i)^2$
  We can evaluate $p_A^\mu = (E_A, \vec{p}_A)$, $p_B^\mu = (m_B, \vec{0})$
  so
  $$(p_A + p_B)^2 = p_A^2 + p_B^2 + 2 p_A \cdot p_B = m_A^2 + m_B^2 + 2 m_B E_A$$
  Then
  $$E_A = \frac{1}{2m_B}\left((\sum m_i)^2 - m_A^2 - m_B^2 \right)$$
  + It's harder in terms of energy to create particles when you are aiming at a fixed target.

- The last two examples demonstrate the idea of thresholds ⇐ minimum energy requirement to do something