Special Relativity

4-Vectors, Dot Product, and Tensors

- Lorentz Transformations
  - Boost in x-direction
    - Two reference frames S + S' w/ parallel axes
    - Origins coincide when $t = t' = 0$
    - S' moves at speed v along x relative to S
    - Coordinate transformation is

$$t' = \gamma(t - vx), \quad x' = \gamma(x - vt), \quad \gamma = 1/\sqrt{1-v^2}$$
$$y' = y, \quad z' = z \qquad (\text{recall } c = 1)$$

(inverse by reversing sign of v)

  - Boosts in other directions are related by rotations

  - Consequences
    - Loss of simultaneity for different frames
    - Lorentz contraction
    - Time dilation: If 2 events occur at x a time t apart in S, they happen $t' = \gamma t$ apart in S'.
      $\Rightarrow$ proper time (time in frame where events are at same place) is the smallest.
    - Velocity addition formula

  - Can define rapidity ϕ by $\tanh\phi \equiv v$
    $\Rightarrow \cosh\phi = \gamma, \quad \sinh\phi = \gamma v$

  - Lorentz transformations are linear + can be written with matrix multiplication

- 4-vectors
  - Consider a spacetime position $(t, \vec{x})$
    - We can define a 4-vector position x^μ with $x^0 = t$, $x^i = x^i$ for $i = 1, 2, 3$

+ The Lorentz boost transformation is

$$x'^{\mu} = \sum_{\nu} \Lambda^{\mu}{}_{\nu} x^{\nu} \quad \text{where} \quad \Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for boost along x direction

+ From now on, use Einstein summation notation: that a repeated raised-lowered index pair is summed

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

+ Also works for infinitesimal displacements $dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$

• Proper time

+ The proper time between 2 events separated by dx^{μ} is

$$d\tau^2 = (dx^0)^2 - d\vec{x}^2 = dt^2/\gamma^2$$

+ Proper time is the time measured on a single clock that is at the time + location of both events

+ $d\tau^2 = 0$ for events separated along a light ray
and $d\tau^2 < 0$ for events farther apart than light can travel in the time between them

+ $d\tau^2$ is Lorentz invariant - same as measured in any inertial frame

• General 4-vectors are 4-component objects a^{μ}, $\mu = 0,1,2,3$

+ The Lorentz boost transformation is $a'^{\mu} = \Lambda^{\mu}{}_{\nu} a^{\nu}$

+ There is a relativistic dot product

careful of sign!

$$a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

This is a Lorentz invariant quantity

+ $d\tau^2$ is the square of dx^{μ} for this dot product.
For any a^{μ}, we use the following nomenclature
$a^2 > 0$ timelike, $a^2 = 0$ lightlike, $a^2 < 0$ spacelike

- The Metric
  - We can write the dot product as a double sum
    + $a \cdot b \equiv g_{\mu\nu} a^\mu b^\nu$ where $[g_{\mu\nu}] = \begin{array}{cc} & \nu \rightarrow \\ \mu \downarrow & \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} \end{array}$
    + $g_{\mu\nu}$ is known as the <u>metric</u>
    + The <u>inverse metric</u> $g^{\mu\nu}$ is the matrix inverse of $[g_{\mu\nu}]$ and has the same components
  - Raising + Lowering Indices
    + Define a lowered index version of a 4-vector
      $$a_\mu \equiv g_{\mu\nu} a^\nu$$
      (Book calls these <u>covariant</u> 4-vectors, a bit archaic)
    + You can raise the index with inverse metric
      $$a^\mu = g^{\mu\nu} a_\nu \quad (a^\mu \text{ called } \underline{\text{contravariant}})$$
    + Can write dot product as
      $$a \cdot b = g_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu = g^{\mu\nu} a_\mu b_\nu$$
  - What does invariance of the dot product mean?
    + Look at dot product in S' frame

$$a' \cdot b' = g_{\mu\nu} a'^\mu b'^\nu = g_{\mu\nu} (\Lambda^\mu{}_\lambda a^\lambda)(\Lambda^\nu{}_\rho b^\rho)$$

$$= (g_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho) a^\lambda b^\rho$$

But this must $= g_{\lambda\rho} a^\lambda b^\rho$ for all a^λ, b^ρ

$$\Longrightarrow \quad g_{\lambda\rho} = g_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho \qquad (\star)$$

+ Equation ($\star$) defines Lorentz transformations. These include all boosts + rotations (put the rotation matrix inside Λ). Can also include parity + time reflections, though normally we don't consider those.

+ We can also find the transformation of lowered indices

$$a' \cdot b' = a'_\mu b'^\mu = a'_\mu \Lambda^\mu{}_\nu b^\nu \Rightarrow a_\nu = a'_\mu \Lambda^\mu{}_\nu$$
$$\Rightarrow a'_\mu = a_\nu (\Lambda^{-1})^\nu{}_\mu$$

lowered indices transform with the inverse Lorentz transformation

- Tensors
  - The metric has 2 lower indices
    + What if we imagine it has different values in different reference frames? Then (*) is

$$g_{\lambda\rho} = g'_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho$$

    + This looks like 2 copies of the lower-index 4-vector transformation
    + Of course $g_{\mu\nu} = g'_{\mu\nu} \Rightarrow$ the metric is invariant

  - We define a tensor as an object with multiple indices, each of which transforms like a vector index

    + ~~Ex~~ $T_{\mu\nu}{}^\lambda \rightarrow T'_{\mu\nu}{}^\lambda = T_{\alpha\beta}{}^\rho (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu \Lambda^\lambda{}_\rho$
      etc
    + Contracted (summed) pairs of lowered/raised indices are invariant like the dot product. Only un-contracted indices transform

$$T'_{\mu\nu}{}^\lambda a'^\mu b'^\nu = T_{\alpha\beta}{}^\rho a^\alpha b^\beta \Lambda^\lambda{}_\rho$$

    + Physical equations must be covariant: all un-contracted indices must be the same in all terms (Give correct + incorrect examples)

  - Example tensors:

invariant (same in all frames)
    + The metric $g_{\mu\nu}$
    + Levi-Civita tensor $\epsilon_{\mu\nu\lambda\rho} = \pm 1$ if μ,ν,λ,ρ all different, $= 0$ otherwise, totally antisymmetric
    + EM field strength $F^{0i} = -E_i$, $F^{ij} = -\epsilon^{ijk} B_k$