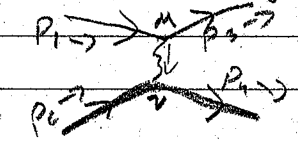
⑥ Applications + Calculations

- First examples
  - Electron - Muon Scattering
    - + Amplitude given by 1 diagram

$\mathcal{M} =$ 

There is no "crossed" diagram b/c the e^- and μ^- are different particles

+ To get amplitude, trace backward on each fermion line and connect w/ propagator

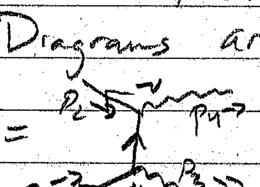
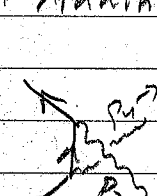
$$\mathcal{M} = \bar{u}(3)(ie\gamma^\mu)u(1)\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)\bar{u}(4)(ie\gamma^\nu)u(2)$$

+ If we know incoming spins and want to ask about probability of specific outgoing spins, we look up solutions of Dirac eqn for each $u_s(\vec{p})$ + plug in
+ We generally want to <u>average</u> over incoming spins (assume equal likelihood of each spin incoming) and <u>sum</u> over outgoing (want total probability of all options). Therefore, we will calculate

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\sum_{s_1,s_2}\sum_{s_3,s_4}|\mathcal{M}|^2$$

Need to see how to do this

- Electron - Positron Annihilation
  - + Diagrams are

$\mathcal{M} =$  $+$ 

$\mathcal{M}_1$ $\quad$ $\mathcal{M}_2$

We can switch photons, Diagrams add b/c <u>not</u> switching fermions

+ Feynman rules give

$$\mathcal{M} \equiv \mathcal{M}_1 + \mathcal{M}_2 = \bar{v}(2)(ie\gamma^\nu)\frac{i(\not{p}_1-\not{p}_3+m)}{(p_1-p_3)^2-m^2}(ie\gamma^\mu)u(1)\,\varepsilon^*_\mu(3)\varepsilon^*_\nu(4)$$

$$+\ \bar{v}(2)(ie\gamma^\mu)\frac{i(\not{p}_1-\not{p}_4+m)}{(p_1-p_4)^2-m^2}(ie\gamma^\nu)u(1)\,\varepsilon^*_\mu(3)\varepsilon^*_\nu(4)$$

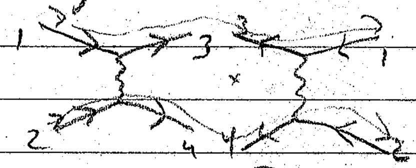
+ Note that the square is $|M|^2 = |M_1|^2 + |M_2|^2 \pm M_1^* M_2 \pm M_2^* M_1$
We also want $\langle |M|^2 \rangle$, averaged over $e^\pm$ spins and summed over photon polarizations. We will have factors like
$$\sum_{pol} \left(\varepsilon_\mu^*(3)\, \varepsilon_\nu(3)\right)\left(\varepsilon_\nu^*(4)\, \varepsilon_\rho(4)\right) \to g_{\mu\lambda} g_{\nu\rho}$$
in each term $\to g_{\mu\lambda} g_{\nu\rho}$

- Fermion Simplification

  - Conjugation
    + In diagrams, we get factors like $\bar{u}(3)\Gamma u(1)$ where Γ = product of γ matrices. So in $|M|^2$ we also have $(\bar{u}(3)\Gamma u(1))^*$
    + First, notice that the product is a 1x1 matrix, so transposing it is the same thing
    $$(\bar{u}(3)\Gamma u(1))^* = (\bar{u}(3)\Gamma u(1))^\dagger = u(1)^\dagger \Gamma^\dagger (\gamma^0)^\dagger u(3)$$
    + Define $\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$ and notice $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^0)^2 = 1$ from earlier. Then we have
    $$= u^\dagger(1)\gamma^0 \bar{\Gamma} u(3) = \bar{u}(1)\bar{\Gamma} u(3)$$
    So the complex conjugate is the Dirac conjugate
    + Can show $\bar{\gamma}^\mu = \gamma^\mu$ so $\overline{\gamma^\mu \gamma^\nu \ldots \gamma^\lambda} = \gamma^\lambda \ldots \gamma^\nu \gamma^\mu$
    (see HW)

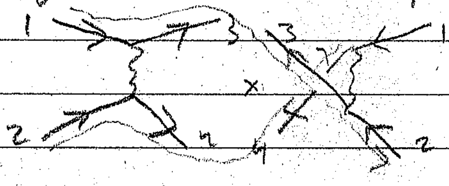
  - Assemble into traces
    + Including the conjugates, we have factors like
    $$\bar{u}(3)\Gamma u(1)\, \bar{u}(1)\bar{\Gamma} u(3) = \text{number} = 1\times 1 \text{ matrix}$$
    + This is also a trace b/c 1x1 matrix = its trace
    + Traces obey the cyclic property
    $$\mathrm{Tr}(AB) = \sum_{i,j} A_{ij} B_{ji} = \sum_{i,j} B_{ji} A_{ij} = \mathrm{Tr}(BA)$$
    + Therefore
    $$\bar{u}(3)\Gamma u(1)\, \bar{u}(1)\bar{\Gamma} u(3) = \mathrm{Tr}\left[\bar{u}(3)\Gamma u(1)\bar{u}(1)\bar{\Gamma} u(3)\right]$$
    $$= \mathrm{Tr}\left[(u(3)\bar{u}(3))\Gamma (u(1)\bar{u}(1))\bar{\Gamma}\right]$$

+ Can think about this graphically as "tracing around" the diagram × its reverse, always matching same fermions



gives 2 traces

+ If there are 2 diagrams, the trace of cross terms may be more complicated



gives 1 long trace

- Sum Over Spins
  + We saw that we can take $\sum_{i=1} \varepsilon^{*}_{\mu}(p)\varepsilon_{\nu}(p) \Rightarrow -g_{\mu\nu}$ for the same photon when the diagram is squared.
  + For fermions, use completeness relations

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

  + In our example above

$$\sum \mathrm{Tr}\left[(u(3)\bar{u}(3)\,\Gamma\,(u(1)\bar{u}(1))\,\bar{\Gamma}\right] = \mathrm{Tr}\left[(\not{p}_3 + m)\,\Gamma\,(\not{p}_1 + m)\,\bar{\Gamma}\right]$$

where Γ and $\bar{\Gamma}$ may have γ^{μ} factors from vertices and other $(\not{p} + m)$ factors from propagators

- Simplify and Evaluate Traces
  + Trace properties: linear + cyclicly, simplify + evaluate
  + Metric sum $g_{\mu\nu}g^{\nu\lambda} = \delta_{\mu}^{\lambda}$, $g_{\mu\nu}g^{\mu\nu} = 4$
  + Anticommutator $\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$
  + You can use those to prove "contraction identities"

$$\gamma_{\mu}\gamma^{\mu} = 4 \qquad \gamma_{\mu}\gamma^{\nu}\gamma^{\mu} = -2\gamma^{\nu}$$

$$\gamma_{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\mu} = 4g^{\nu\lambda}, \quad \gamma_{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\rho}\gamma^{\mu} = -2\gamma^{\rho}\gamma^{\lambda}\gamma^{\nu}$$

and similar w/ vectors contracted to them

+ Since there are 4 spin indices (γ matrices are 4×4), $\mathrm{Tr}(1) = 4$. Can also prove that

$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$, $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda})$

Tr (any odd # of γ) $= 0$, $\mathrm{Tr}(\gamma^5) = 0$,

$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$, $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4i\,\epsilon^{\mu\nu\lambda\rho}$

— Examples, take 2

• Electron - Muon Scattering Again

+ ~~Amplitude was~~

$$\mathcal{M} = \frac{ie^2 g_{\mu\nu}}{(p_1-p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma^\nu u(2)\right]$$

$t \longrightarrow (p_1-p_3)^2$

+ When squared we get

$$|\mathcal{M}|^2 = \left(\frac{e^4}{t^2}\right) g_{\mu\nu}\, g_{\lambda\rho} \times \begin{array}{l}\left[\bar{u}(3)\gamma^\mu u(1)\bar{u}(1)\gamma^\lambda u(3)\right] \\ \times\left[\bar{u}(4)\gamma^\nu u(2)\bar{u}(2)\gamma^\rho u(4)\right]\end{array}$$

Note that we had to use different summation indices in $\mathcal{M}^*$ than $\mathcal{M}$. That's why we have $g_{\lambda\rho}$, etc

+ Now do spin sum + average $\langle|\mathcal{M}|^2\rangle$.

The traces we get are

$\mathrm{Tr}\left[\gamma^\mu(\not{p}_1+m)\gamma^\lambda(\not{p}_3+m)\right]$ — $m = e^-$ mass

and $\mathrm{Tr}\left[\gamma^\nu(\not{p}_2+M)\gamma^\rho(\not{p}_4+M)\right]$ — $M = \mu^-$ mass

+ Because odd γ traces $= 0$, each of these is the sum of a 4-γ trace and a 2-γ trace

$$\mathrm{Tr}\left[\gamma^\mu(\not{p}_1+m)\cdot\gamma^\lambda(\not{p}_3+m)\right] = 4m^2 g^{\mu\lambda} + 4\left(p_1^\mu p_3^\lambda - p_1\cdot p_3\, g^{\mu\lambda} + p_3^\mu p_1^\lambda\right)$$

etc

+ Put it together:

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{4e^4}{4t^2}\right)\Big[4m^2M^2 + 2M^2\, p_1\cdot p_3 - 2m^2\, p_2\cdot p_4$$

$$+ 2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) + 0\Big]$$

factor $\frac{1}{4}$ from incoming spin average

+ The differential cross section in CM frame is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}\langle|\mathcal{M}|^2\rangle \;;\quad |\vec{p}_f| = |\vec{p}_i| \text{ b/c } e^-\mu^- \to e^-\mu^-$$

Since $M \gg m$, CM frame $\approx$ μ rest frame if electron $E \ll M$ also.

Then $E_{cm} = M$, μ^- stays at rest $p_2^\mu = p_4^\mu = (M, \vec{0})$

Write $p_1^\mu = (E, 0, 0, p) \Rightarrow p_3^\mu = (E, 0, p\sin\theta, p\cos\theta)$

Then $p_1\cdot p_2 = p_2\cdot p_3 = p_1\cdot p_4 = p_3\cdot p_4 = ME$, $p_2\cdot p_4 = M^2$,

$p_1\cdot p_3 = E^2 - p^2\cos\theta$, $t = (p_1-p_3)^2 = 2m^2 - 2E^2 + 2p^2\cos\theta = -2p^2(1-\cos\theta)$

After simplification $\quad = -2p^2(1-\cos\theta) = -4p^2\sin^2(\theta/2)$

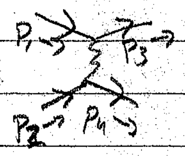
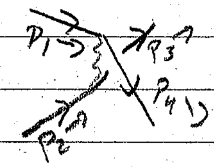
$$\langle|\mathcal{M}|^2\rangle = \frac{2e^4}{2p^4\sin^4\theta/2}\left[2m^2M^2 - 2M^2(E^2 - p^2\cos\theta) + 2M^2E^2\right]$$

so

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2 M}{p^2\sin^2\theta/2}\right)^2 \left(m^2 + p^2\cos^2\theta/2\right) \text{ after simplification}$$

This is <u>Mott scattering</u> (EM scattering of light off heavy particle)

- Electron - Electron Scattering
  + This has t-channel + u-channel diagrams

$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2 \equiv$  $-$  Subtract b/c of crossed e^- lines

+ $\mathcal{M}_1$ is the same as for $e\mu$ scattering with $M \to m$

$$\mathcal{M}_2 = \frac{ie^2 g_{\mu\nu}}{(p_1-p_4)^2}\left[\bar{u}(4)\gamma^\mu u(1)\right]\left[\bar{u}(3)\gamma^\nu u(2)\right]$$

+ Then we need $|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 - \mathcal{M}_1\mathcal{M}_2^* - \mathcal{M}_2\mathcal{M}_1^*$

$|\mathcal{M}_1|^2$ is as before; $|\mathcal{M}_2|^2$ is like $|\mathcal{M}_1|^2$ with $3 \leftrightarrow 4$.

The new one is

$$\mathcal{M}_1\mathcal{M}_2^* = \frac{e^4}{tu}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(1)\gamma^\nu u(4)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right] \times \left[\bar{u}(2)\gamma_\nu u(3)\right]$$

$$\langle\mathcal{M}_1\mathcal{M}_2\rangle^2 = \frac{e^4}{4tu}\,\mathrm{Tr}\left[(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_4 + m)\gamma_\mu(\not{p}_2 + m)\gamma_\nu\right]$$

+ The trace is a sum over all even γ terms from distributing

$$= \mathrm{Tr}\left(\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\not{p}_4\gamma_\mu\not{p}_2\gamma_\nu\right)$$

$$+\, m^2 \left(\mathrm{Tr}\left[\not{p}_3 \gamma^\mu \not{p}_1 \gamma^\nu \gamma_\mu \gamma_\nu\right] + \dots \text{ (6 terms)} \right)$$

$$+\, m^4\, \mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma_\mu \gamma_\nu\right]$$

+ We can evaluate these with contraction identities. Look at 1st one

$$\mathrm{Tr}\left[\not{p}_3 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_4 \gamma_\mu \not{p}_2 \gamma_\nu\right] = -2\,\mathrm{Tr}\left[\not{p}_3 \not{p}_4 \gamma^\nu \not{p}_1 \not{p}_2 \gamma_\nu\right]$$

$$= -8(p_1\cdot p_2)\,\mathrm{Tr}\left[\not{p}_3 \not{p}_4\right] = -32\,(p_1\cdot p_2)(p_3\cdot p_4)$$

- Electron-Positron Annihilation
  + We have

$$\mathcal{M}_1 = \left(-ie^2/t-m^2\right) \bar{v}(2)\gamma^\nu(\not{p}_1 - \not{p}_3 + m)\gamma^\mu u(1)\, \epsilon_\mu(3)^* \epsilon_\nu(4)^*$$

and

$$\mathcal{M}_2 = \left(-ie^2/u-m^2\right) \bar{v}(2)\gamma^\mu(\not{p}_1 - \not{p}_4 + m)\gamma^\nu u(1)\, \epsilon_\mu(3)^* \epsilon_\nu(4)^*$$

+ Then

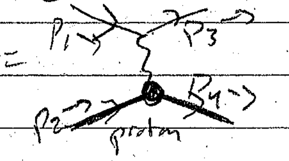
$$\langle |\mathcal{M}_1|^2 \rangle = \frac{1}{4}\left(\frac{e^4}{(t-m^2)^2}\right) \mathrm{Tr}\Big[(\not{p}_2 - m)\gamma^\nu(\not{p}_1 - \not{p}_3 + m)\gamma^\mu(\not{p}_1 + m) \times \gamma_\mu(\not{p}_1 - \not{p}_3 + m)\gamma_\nu\Big]$$

b/c. $\langle \epsilon_\mu(3)^* \epsilon_\lambda(3)\rangle \to -g_{\mu\lambda}$, etc.

There are 4 similar traces. In each, you have to distribute terms and take those with even # of γ matrices

+ You can start to see why computerizing these calculations is useful!

- Elastic Electron-Proton Scattering
  + Diagrammatically, this is like $e + \mu \to e + \mu$ scattering except we don't know the rule for the photon-proton vertex b/c the proton is a composite particle.

$\mathcal{M} =$ 

+ By comparison, we know

electron factor

$$\langle |\mathcal{M}|^2 \rangle = \frac{4e^4}{t^2}\left[p_1^\mu p_3^\nu - p_1\cdot p_3\, g^{\mu\nu} + p_3^\mu p_1^\nu + m^2 g^{\mu\nu}\right] K_{\mu\nu}$$

where $K_{\mu\nu}$ is a proton form factor that contains info about the structure of a proton.

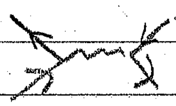
+ By general arguments, $K_{\mu\nu}$ can depend only on p_2^μ and p_4^μ. It is usually written in terms of $p_2^\mu = p^\mu$ and $q^\mu = p_4^\mu - p_2^\mu$ = photon momentum

$$K_{\mu\nu} \equiv -K_1 g_{\mu\nu} + K_2 p_\mu p_\nu / M^2 + K_4 q_\mu q_\nu / M^2 + K_5 (p_\mu q_\nu + p_\nu q_\mu)/M^2$$

where K_1, K_2, K_4, K_5 are functions of q^2

+ It's possible to see $q^\mu K_{\mu\nu} = 0 \Rightarrow K_4 = \frac{M^2}{q^2}K_1 + \frac{1}{4}K_2$, $K_5 = -\frac{1}{2}K_2$
You can now write the cross section in terms of K_1 and K_2 and compare to experiment to see evidence that protons are not point particles!

- Electron-Positron Annihilation to Hadrons
  + The Feynman diagram for $e^+e^- \to \mu^+ + \mu^-$ is the same as for $e^+ + e^- \to q + \bar{q}$ (when considering QED only)



except quarks have fractional charge q and 3 colors each (to sum over)

  + At high energies compared to initial + final masses,

$$\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{m_q < E} 3q^2$$

  + If we slowly increase collision energy, this is

$$3\left((2/3)^2 + (1/3)^2 + (1/3)^2\right) = 2 \quad \text{for } E < m_c \ (u, d, s \text{ quarks})$$
$$+\,3(2/3)^2 = 10/3 \quad \text{for } m_c < E < m_b \ (u, d, s, c)$$
$$+\,3(1/3)^2 = 11/3 \quad \text{for } m_b < E \quad \text{etc.}$$

  + Matches experimental data + demonstrates existence of 3 colors

  + Modified slightly by QCD (weak) effects.