Quantum Electrodynamics (QED)

This is the real-world theory describing the interaction of photons w/ charged particles. We have to learn Feynman rules for fermions + photons. Part of SM

Feynman Rules + ~~Calculation Methods~~

- Fermion lines (spin $\frac{1}{2}$)
  - ~~Properties of fermions w~~
    + None of the fermions we know are their own antiparticle, so fermion lines don't end (charge conservation)
    + Fermions have spin 1/2, so they must be represented by spinors. What spinors do we use?
  - ~~Notation~~: We will use Feynman slash notation. ~~to~~ If a^μ is a 4-vector, $\not{a} \equiv a_\mu \gamma^\mu$ for Dirac γ matrices
  - Dirac Equation
    + We know real particles satisfy $p^2 - m^2 = 0$, $\Rightarrow$ Klein-Gordon eqn $-(\partial_\mu \partial^\mu + m^2)\phi = 0$
    + For a fermion, solution is $\psi = (\text{spinor}) \times e^{ip\cdot x}$. KG eqn does not determine the spinor part
    + Dirac noticed we can write
      $$p^2 - m^2 = (\not{p} - m)(\not{p} + m) \text{ acting on spinors}$$
      a fermionic particle should satisfy $\not{p} = \pm m$
    + The Dirac equation is
      $$(i\gamma^\mu (\partial/\partial x^\mu) - m)\psi = (i\not{\partial} - m)\psi = 0$$
    + ~~The momentum space~~
  - External lines + solutions to Dirac eqn
    + The Dirac eqn is an eigenvalue problem
      $$\not{p} u = m u \quad \text{and} \quad \not{p} v = -m v$$
      Each has 2 solutions, one for spin up, one for spin down
    + The u solutions go with particle lines

      incoming fermion $\rightarrow\!\bigcirc \; = u_s(p)$ s = spin up or down

      outgoing fermion $\bigcirc\!\rightarrow \; = \bar{u}_s(p)$

+ The v solutions go with antiparticles

incoming antifermion $\longleftarrow\!\!\circ = \bar{v}_s(p)$

outgoing $\circ\!\!\longleftarrow = v_s(p)$

+ Useful properties

normalization $\bar{u}_s u_{s'} = 2m\,\delta_{ss'}$, $\bar{v}_s v_{s'} = -2m\,\delta_{ss'}$, $\bar{u}v = \bar{v}u = 0$

Completeness $\sum_s u_s(p)\bar{u}_s(p) = \not{p}+m$, $\sum_s v_s(p)\bar{v}_s(p) = \not{p}-m$

∘ Internal line / propagator

+ The internal line relates a final spinor ϕ to initial ψ

$$\psi \circ\!\longrightarrow\!\circ\, \phi$$

+ Follows from a "forced Dirac oscillator"

$$(i\not{\partial} - m)\phi = \psi \Longrightarrow (\not{p}-m)\phi = \psi, \quad \phi = (\not{p}-m)^{-1}\psi$$

+ Feynman rule is

$$\underset{p\rightarrow}{\circ\!\longrightarrow\!\circ} = \frac{i}{\not{p}-m} = \frac{i(\not{p}+m)}{p^2-m^2}$$

+ The states of identical fermions are antisymmetric. So two diagrams related by swapping identical incoming (outgoing) fermion/antifermion lines or an incoming fermion/antifermion with an outgoing antifermion/fermion have a relative minus sign. (This is "Pauli exclusion principle")

— Photons

• Relativistic E + M

+ The $\vec{E}$ and $\vec{B}$ fields fit into relativistic antisymmetric tensor $F^{\mu\nu}$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

+ Maxwell eqns become

$$\vec{\nabla}\cdot\vec{B} = 0 \text{ and } \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \epsilon_{\mu\nu\lambda\rho}\partial^\nu F^{\lambda\rho} = 0$$

$$\vec{\nabla}\cdot\vec{E} = \rho \text{ and } \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J} \Rightarrow \partial_\mu F^{\mu\nu} = J^\nu$$

for $J^\nu = (\rho, \vec{J})$

+ Suppose we define a vector potential A^μ s.t.

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

Then

$$\epsilon_{\mu\nu\lambda\rho}\partial^\nu F^{\lambda\rho} = 2\epsilon_{\mu\nu\lambda\rho}\partial^\nu\partial^\lambda A^\rho = 0 \quad \text{automatically}$$

b/c partial derivatives commute (ν,λ are symmetric above)

+ There is a gauge invariance $A^\mu \to A^\mu + \partial^\mu\lambda$ for function λ leaves $F^{\mu\nu}$ unchanged (check!)

use $\partial_\mu\partial^\mu\lambda = -\partial_\mu A^\mu$ $\to$ We can choose Lorentz gauge to make $\partial_\mu A^\mu = 0$

+ In Lorentz gauge, remaining Maxwell eqn is

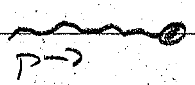
$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu\partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \partial_\mu\partial^\mu A^\nu = J^\nu$$

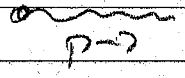
• External lines are photons in vacuum

For $J^\nu = 0$ + We can make an additional gauge shift by any λ with $\partial_\mu\partial^\mu\lambda = 0$ to set $A^0 = 0 \Rightarrow \vec\nabla\cdot\vec A = 0$. This is Coulomb gauge.

+ Therefore, free photons (ie, $J^\nu = 0$) satisfy KG eqn $\partial_\mu\partial^\mu A^\nu = 0 \Rightarrow A^\mu = \epsilon^\mu(p)\, e^{-ip\cdot x}$ for polarization vector $\epsilon^\mu(p)$. In Coulomb gauge, $\epsilon^0 = 0$, $\vec p\cdot\vec\epsilon = 0$. So for $\vec p$ along z axis, the 2 choices are $\epsilon_x^\mu = (0,1,0,0)$, $\epsilon_y^\mu = (0,0,1,0)$

+ As usual, the external lines should go with polarization


= incoming photon = $\epsilon_s^\mu(p)$

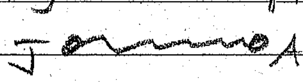

= outgoing photon = $\epsilon_s^\mu(p)^*$

The complex conjugate allows us to use circular polarization, $\epsilon_+^\mu = (0,1,i,0)/\sqrt2$, $\epsilon_-^\mu = (0,1,-i,0)/\sqrt2$ etc

+ In $|M|^2$ calculations, can replace $\sum_s \epsilon_s^\mu(p)\epsilon_s^\nu(p)^* \to -g^{\mu\nu}$ (but it's not $=$)

• Internal photon lines / Photon propagators

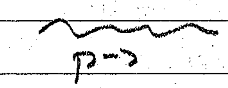
+ This is telling us find A^μ from a current J^ν



+ The Maxwell eqn is the KG eqn with a vector index

$$g^{\mu\nu}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A_\nu = J^\mu$$

As usual, solving for A_ν gives the propagator

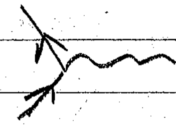


$$\frac{-i g_{\mu\nu}}{p^2} \quad \text{b/c mass} = 0$$

+ We've actually swept a lot under the rug about gauge invariance

— The Vertex

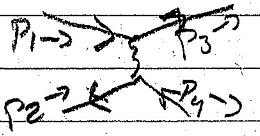
• We need a factor that can connect spinors to vectors



$$= iqe\,\gamma^\mu$$

+ Each vertex inserts a Dirac γ matrix
+ e = (absolute value of) electron charge
+ q = # of charge units

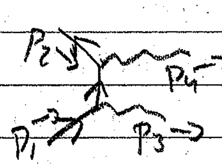
• In the end each diagram gives a scalar
+ Track backwards along each fermion line to soak up all spinor (matrix) indices
+ Each photon vector index is contracted w/attached vertex
+ Example 1 for e^+e^- scattering, first diagram



$$= \bar{u}_s(3)(ie\gamma^\mu)u_{s'}(1)\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)\bar{v}_{s''}(2)(ie\gamma^\nu)v_{s'''}(4)$$

numbers indicate momentum + spin/polarization

+ Example 2 1 diagram for e^+e^- annihilation



$$= \bar{v}_s(2)(ie\gamma^\mu)\left(\frac{i(\not{p}_1-\not{p}_3+m)}{(p_1-p_3)^2-m^2}\right)(ie\gamma^\nu)u_{s'}(1)$$

$$\times\, \epsilon^\nu_{s''}(p_3)^*\, \epsilon^\mu_{s'}(4)^* \quad \times \epsilon_\nu$$