PHYS-4303 Homework 3 Due 17 Oct 2023

This homework is due to `https://uwcloud.uwinnipeg.ca/s/dcYrc2Yys2jsSrz` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **Symmetrizing Wavefunctions**

   Consider states of three particles with mixed symmetry such that they are antisymmetric when two of the particles are exchanged but have no particular symmetry when exchanging the third particle with either of the specified two. That is, $|\psi_{12}(1,2,3)\rangle = -|\psi_{12}(2,1,3)\rangle$, $|\psi_{23}(1,2,3)\rangle = -|\psi_{23}(1,3,2)\rangle$, and $|\psi_{13}(1,2,3)\rangle = -|\psi_{13}(3,2,1)\rangle$. Assume that each of these states is normalized.

   (a) Suppose we are considering the spin states of three spin-1/2 particles. Then we can take

   $$\begin{aligned} |\psi_{12}\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right) \\ |\psi_{23}\rangle &= \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle\right) \ . \end{aligned}$$

   Show that $|\psi_{13}\rangle = |\psi_{12}\rangle + |\psi_{23}\rangle$ is antisymmetric under exchange of spins 1 and 3 and is normalized.

   (b) Using the spin states from the last part, including your answer, show that

   $$|\psi_{12}(1,3,2)\rangle = |\psi_{13}(1,2,3)\rangle \ , \ \ |\psi_{23}(2,1,3)\rangle = |\psi_{13}(1,2,3)\rangle \ , \ \ |\psi_{12}(3,2,1)\rangle = -|\psi_{23}(1,2,3)\rangle \ . \tag{1}$$

   (c) Suppose the three particles have both spin (ψ) and flavor (ϕ). Assume that (1) applies for both spin and flavor. Show that the state $|1,2,3\rangle = |\psi_{12}\rangle|\phi_{12}\rangle + |\psi_{23}\rangle|\phi_{23}\rangle + |\psi_{13}\rangle|\phi_{13}\rangle$ is symmetric under the exchange of any two of the particles. This gives us a spin/flavor state that we can use for baryon wavefunctions.

2. **Derivatives Have Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

   $$a'^{\mu} = \Lambda^{\mu}{}_{\nu} a^{\nu} \quad \text{and} \quad a_{\mu} = \Lambda^{\nu}{}_{\mu} a'_{\nu} \ , \tag{2}$$

   where $\Lambda^{\mu'}{}_{\nu}$ is the usual Lorentz transformation matrix from $S \to S'$.

   (a) Using the fact that the spacetime position x^{μ} is a 4-vector, find the partial derivatives $\partial x'^{\mu}/\partial x^{\nu}$ in terms of $\Lambda^{\mu}{}_{\nu}$. *Hint:* For two positions as measured in the same frame, $\partial x^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$ (think about why).

   (b) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the multivariable chain rule to show that

   $$\frac{\partial f}{\partial x^{\mu}} = \Lambda^{\nu}{}_{\mu} \frac{\partial f}{\partial x'^{\nu}} \ . \tag{3}$$

   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_{\mu} f \equiv \partial f/\partial x^{\mu}$.

3. **Some Short Practice Calculations**

Calculate the following quantities. You should get a number for each answer.

(a) $g_{\mu\nu}g^{\mu\nu}$

(b) $g^{\mu\nu}g^{\lambda\rho}\epsilon_{\mu\nu\lambda\rho}$

(c) $\epsilon_{\mu\nu\lambda\rho}\epsilon^{\mu\nu\lambda\rho}$

Based on a problem by Carroll In the next two calculations, define the tensor $X^{\mu\nu}$ and vector V^μ by

$$\left[\quad X^{\mu\nu} \quad \right] = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix}, \quad V^\mu = (-1, 2, 0, -2) \tag{4}$$

in some inertial frame S. Then calculate the following:

(d) $X^\mu{}_\mu$ (This is called the *trace* of X.)

(e) $X^{\mu\nu}V_\mu V_\nu$