Rotation, Part I

Rigid Objects

- Properties of rigid bodies

  - A rigid body is a collection of a number of particles held together w/positions $\vec{r}_i$
    + The distance between two particles in a rigid body is fix-ed: $|\vec{r}_i - \vec{r}_j| = \text{constant}$
    + Rigid bodies can not flex or bend. They can have an overall linear motion (translation), and they can rotate.

  - Center of mass position
    + The center of mass (CM) position $\vec{R}$ is the mass-weighted average position

$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i \quad \text{where } M = \sum_i m_i = \text{total mass}$$

    + Total momentum of the object is

$$\vec{P} = \sum_i p_i = \sum_i m_i \dot{\vec{r}}_i = \frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right) = M\dot{\vec{R}}$$

    + This is the same as a point object of mass M at the CM position
    + For a continuous object, replace sums by integrals over density

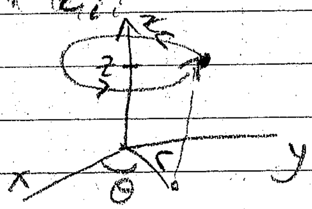
$$M = \int d^3\vec{r}\, \rho(\vec{r}) \equiv \int dm, \quad \vec{R} = \frac{1}{M}\int d^3\vec{r}\, \rho(\vec{r})\vec{r} \equiv \frac{1}{M}\int dm\, \vec{r}$$

  - Rotating around a fixed axis

    + Say the object rotates around the z-axis. Each component particle has $z_i = \text{const}$ and circular motion in x_i, y_i

+ In cylind

That $\vec{r}_i$'s $(\vec{r}_i = (r_i\cos\theta_i)\hat{i} + (r_i\sin\theta_i)\hat{j} + z_i\hat{k})$
and $\dot{z}_i$



+ For the object to be rigid, $\dot{r}_i = 0$ for each particle and $\dot{\theta}_i = \omega$ is the same for all the parts of the object (it is rigid)

+ As usual, we define the angular velocity vector pointing along the axis of rotation.
In our case, $\vec{\omega} = \omega\hat{k}$. We can check
$$\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i \text{ for every particle}$$

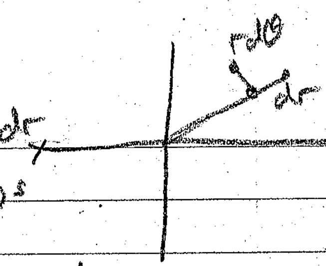
+ Check that distances between particles are constant:
$$\frac{1}{2}\frac{d}{dt}\left(|\vec{r}_i - \vec{r}_j|^2\right) = (\dot{\vec{r}}_i - \dot{\vec{r}}_j)\cdot(\vec{r}_i - \vec{r}_j) = \left[\vec{\omega}\times(\vec{r}_i - \vec{r}_j)\right]\cdot(\vec{r}_i - \vec{r}_j)$$
$$= 0$$

- Moment of Inertia
  + Using cylindrical coordinates, each particle has velocity $\vec{v}_i = r_i\omega\,\hat{\theta}$, so the z-component of angular momentum is $L_{z,i} = m_i r_i^2 \omega$
  + The total $L_z = \sum_i m_i r_i^2 \omega \rightarrow \left(\int dm\, r^2\right)\omega$
  + The quantity in parentheses is moment of inertia around the axis
$$I \equiv \int dm\, r^2$$
where r = distance from the axis

– Integrating in curvilinear coordinates
Can use Jacobian determinant but let's take a more intuitive approach

- Plane Polar Coordinates
  + How do we measure distance?

Infinitesimal displacement along $\hat{r}$ is dr, along $\hat{\theta}$ is $r\,d\theta$. Pythagoras says



$$ds^2 = dr^2 + r^2 d\theta^2$$

+ This matches what we have for speed

$$v^2 = (ds/dt)^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

+ But these distances are the sides of a rectangular cell. So

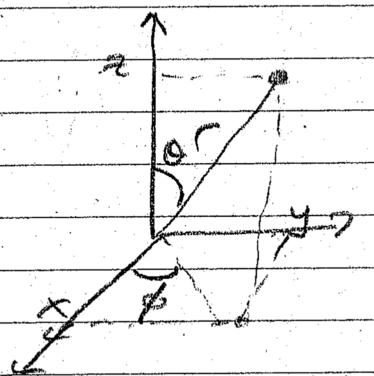
$$\int d^2\vec{r}\, f(\vec{r}) = \int dr \int d\theta \;\, r f(r,\theta)$$

+ Cylindrical coordinates in 3D are an easy generalization

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$\int d^3\vec{r}\, f(\vec{r}) = \int dz \int dr \int d\theta \;\, r f(r,\theta,z)$$

- Spherical Polar Coordinates



+ These are coordinates s.t.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

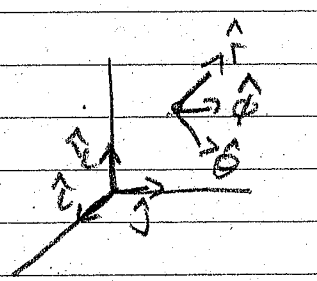
+ That means

r = distance from origin

θ = angle between $\vec{r}$ and z axis ($\hat{k}$)

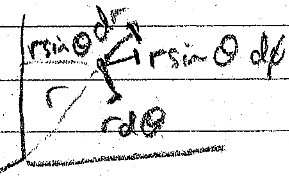
ϕ = angle from x axis ($\hat{i}$) to projection of $\vec{r}$ in the (xy)-plane

+ There are unit vectors pointing in the directions of increasing coordinates. Note that $\hat{\phi}$ is $\perp \hat{k}$



+ The infinitesimal distances for changes dr, $d\theta$, $d\phi$ are dr, $r\,d\theta$, $r\sin\theta\, d\phi$, so

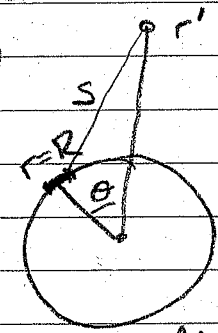
$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$



+ These distances mean 3D volume integration is

$$\int d^3\vec{r}\, f(\vec{r}) = \int dr \int d\theta \int d\phi\; r^2 \sin\theta\, f(r,\theta,\phi)$$

- Example of spherical coord integration: gravitational potential energy near a spherical shell



+ The shell has mass M spread uniformly over a sphere of radius $r=R$.
+ We want the potential energy of a mass m at point r' along the z axis
+ The potential energy due to the little mass dM at position (θ,ϕ) on the shell is $dV = -G m\, dM/s$
  where $s^2 = R^2 + (r')^2 - 2r'R\cos\theta$ (law of cosines)
+ The total potential energy is the integral

$$V = -Gm\int \frac{dM}{s} = -Gm \int \frac{(MR^2\, d\theta\, d\phi\, \sin\theta/4\pi R^2)}{s}$$

$$= \frac{-GMm}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \frac{\sin\theta}{\left[R^2+(r')^2-2r'R\cos\theta\right]^{1/2}}$$

$u = \cos\theta$

$$= -(GMm/2)\int_{-1}^{1} \frac{du}{\left[R^2+(r')^2-2r'Ru\right]^{1/2}}$$

$$= -\frac{GMm}{2}\left(\frac{-1}{r'R}\right)\left[R^2+(r')^2-2r'Ru\right]^{1/2}\Bigg|_{u=-1}^{u=+1}$$

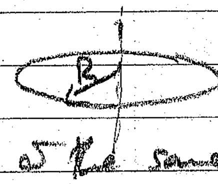
$$= \frac{-GMm}{2r'R}\left(|R+r'| - |R-r'|\right)$$

$$= -GMm \begin{cases} 1/r' & \text{for } r'>R \\ 1/R & \text{for } r'<R \end{cases}$$

+ This shows that the potential energy due to a spherical distribution of mass is the same as from a point mass (if you are outside)

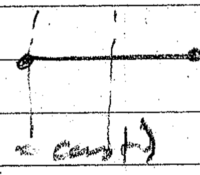
- Example Moment of Inertia Calculations

- Circular hoop of mass M and radius R with axis $\perp$ hoop through the center.



  + Each mass element of the hoop is at the same distance from the axis $\Rightarrow$ integral becomes multiplication. Note $z=0$ everywhere, $r=R=$ const.
  + Result is $I = MR^2$

- Thin Uniform Rod of mass M, length L
  + Axis at end point $\perp$ rod gives

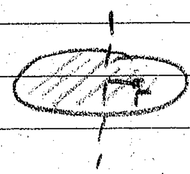


$$I_{end} = \int_0^L dr \left(\frac{M}{L}\right) r^2 = \frac{1}{3} ML^2 \quad (z, \theta = \text{const})$$

  + Axis through center of rod $\perp$ rod

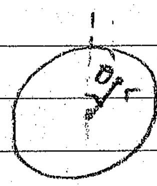
$$I_{cm} = \int_{-L/2}^{L/2} dx \left(M/L\right) x^2 = \frac{1}{12} ML^2$$

- Thin uniform disk of mass M, radius R
  + Let's take the axis through the center $\perp$ the disk. Using cylindrical coords w/ $z=0$,



$$I = \int_0^R dr \int_0^{2\pi} d\theta \, r \left(\frac{M}{\pi R^2}\right) r^2 = \frac{1}{2} MR^2$$

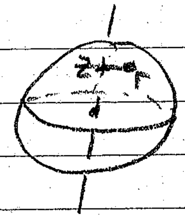
  + Axis through center but in disk: Use plane polar coords. in disk, so distance from axis $= r\sin\theta$.



Then
$$I = \int_0^R dr \int_0^{2\pi} d\theta \, r \left(\frac{M}{\pi R^2}\right) (r\sin\theta)^2 = \frac{1}{4} MR^2$$

b/c we can use
$$\int_0^{2\pi} d\theta \sin^2\theta = \frac{1}{2} \int_0^{2\pi} d\theta \left(1 - \cos(2\theta)\right) = \pi$$

- Uniform density sphere, mass M, radius R with axis through the center.



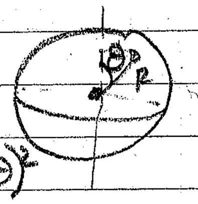
  + In cylindrical coordinates,

$$I = \left(\frac{M}{4\pi R^3/3}\right)\int_{-R}^{R} dz \int_0^{\sqrt{R^2-z^2}} dr \int_0^{2\pi} d\theta \, r\,(r^2)$$

$$= \frac{3M}{8R^3}\int_{-R}^{R} dz\,(R^2-z^2)^2 = \frac{3M}{8R^3}\left(2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5\right)$$

$$= 2MR^2/5$$

This is a little annoying



  + In spherical coordinates,

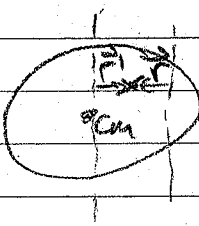
$$I = \left(\frac{M}{4\pi R^3/3}\right)\int_0^R dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, r^2 \sin\theta \,(r\sin\theta)^2$$

$$= \frac{3M}{2R^3}\left(\frac{1}{5}R^5\right)\int_0^{\pi} d\theta \sin^3\theta \qquad \text{Use } u=\cos\theta$$

so $$\int_0^{\pi} d\theta \sin^3\theta = \int_{-1}^{1} du\,(1-u^2) = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\Rightarrow \quad I = 2MR^2/5$$

- Parallel Axis Theorem
  + Suppose you have an axis through the CM of an object and another a distance L away that is parallel



  + In cylindrical coords, look at the vector to a point in the (x,y) plane from each axis. If these are $\vec{r}$ and $\vec{r}'$ (with latter from CM axis) with $\vec{L}$ the vector from the CM axis to the other, we get

$$\vec{r} = \vec{r}' - \vec{L}$$

+ Therefore, the moment of inertia for the other axis is

$$I = \int dm\, \vec{r}^{\,2} = \int dm\, (\vec{r}^{\,\prime 2} - 2\vec{L}\cdot\vec{r}^{\,\prime} + \vec{L}^2)$$

+ However $\vec{L}$ = constant and $\int dm\, \vec{r}^{\,\prime} = 0$ b/c CM position for CM origin.

$$\Rightarrow \quad I = \int dm (\vec{r}^{\,\prime})^2 + \int dm\, L^2 = I_{cm} + ML^2$$

+ This is the parallel axis theorem.
+ Check it works for the 2 thin rod calculations

- Motion Around a Single Axis

— Dynamics of Rotation

• Change of Angular momentum

+ The total angular momentum is the sum over all component particles

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$

+ So then

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i \quad \text{where } \vec{F}_i = \text{force on particle } i$$

But the force is

$$\vec{F}_i = \vec{F}_{i,ext} + \sum_j \vec{F}_{i,j}$$

= external force plus forces from the other particles in the object

+ By Newton's 3rd law $\vec{F}_{i,j} = -\vec{F}_{j,i}$ so.

$$\sum_{i,j} \vec{r}_i \times \vec{F}_{i,j} = \frac{1}{2} \sum_{i,j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{i,j}$$

+ If $\vec{F}_{ij}$ is central, it is $\propto (\vec{r}_i - \vec{r}_j)$, so the cross product $=0$. Then

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_{i,ext} = \vec{\tau}_{ext} = \text{external torque}$$

+ Change in angular momentum is due to external torque, ie, ang. mom. is conserved if $\vec{\tau}_{ext} = 0$.
No spontaneous rotation!

- Uniform Gravity acts on the CM
  + For $\vec{F}_G = m\vec{g}$, gravity on a rigid object is

$$\vec{F}_G = \sum_i m_i \vec{g} = M\vec{g}$$

  + The torque due to gravity is

$$\vec{\tau}_{ext} = \left(\sum_i m_i \vec{r}_i\right) \times \vec{g} = M\vec{R} \times \vec{g}$$

  + Force and torque act at the CM position on a mass M
  + Motion is CM in gravity + torque-free rotation around CM.

- Kinetic Energy
  + Suppose the object rotates around an axis through origin with no other motion. Then (my cylindrical coords)

$$T = \frac{1}{2}\sum_i m_i \vec{v}_i^{\,2} = \frac{1}{2}\sum_i m_i (r_i \omega \hat{\theta})^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$
$$= \frac{1}{2} I \omega^2$$

  + What if the object is translating + rotating around a particular axis through the CM?

    Let $\vec{r}_i = \vec{R} + \vec{r}_i{}'$ so $\vec{r}_i{}'$ = position relative to CM.

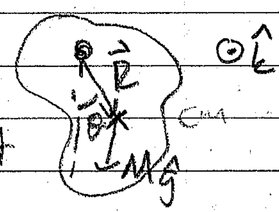
    Then

$$T = \frac{1}{2}\sum_i m_i \left(\dot{\vec{R}} + \dot{\vec{r}}_i{}'\right)^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \dot{\vec{R}}\cdot\left(\sum_i m_i \dot{\vec{r}}_i{}'\right)$$
$$+ \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i{}'^{\,2} = \frac{1}{2}M\dot{\vec{R}}^2 + 0 + \frac{1}{2} I_{cm}\omega^2$$

    ← 2nd term vanishes b/c avg. position relative to CM = 0 and 3rd term is same as above.

— Physical Pendulum

- A physical pendulum is an object hung from a pivot that can rotate around an axis through the pivot



  + If $\vec{R}$ = CM position, torque is

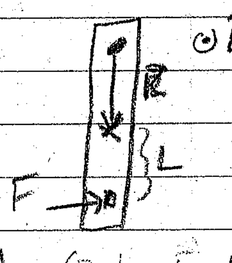
$$\vec{\tau} = M\vec{R} \times \vec{g} = -MRg\sin\theta\,\hat{k}$$

  + Angular momentum around axis is $\vec{L} = I\dot{\theta}\hat{k}$

    so

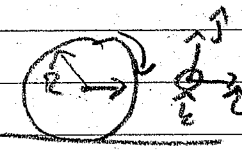
$$\dot{\vec{L}} = \vec{\tau}_{ext} \Rightarrow I\ddot{\theta} = -MRg\sin\theta \approx -MRg\theta.$$

+ This is harmonic motion like a regular pendulum, but replace the frequency $\sqrt{g/l} \rightarrow \sqrt{MRg/I}$

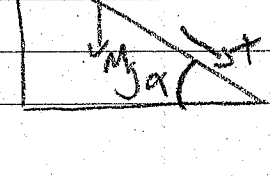


• "Baseball Bat Theorem"
  + An object pivots around the origin w/ CM a distance R away.
    A force $\vec{F}$ strikes the object at a distance L farther from the CM. (note $\vec{F} \perp$ ...)
  + Assuming straight line + right angles, $\vec{P} = \vec{F} + \vec{F}_{pivot}$
    Meanwhile, $\vec{\tau} = F(R+L)\hat{k} = I\dot{\omega}\hat{k}$ where I is around pivot.
  + Total force can include a force from the pivot point
  + w/ purely rotational motion $MR\dot{\omega}$ (CM motion),
    $$\vec{F} + \vec{F}_{pivot} = MR\dot{\omega}\hat{\theta}$$
  + When is $F_{pivot} = 0$? Divide eqns w/ $F_{piv} = 0$.
    $$R + L = I/MR.$$
    Using parallel axis theorem, $I = I_{cm} + MR^2$
    $$\Rightarrow \quad I_{cm} = MRL$$
  + For fixed pivot position, L gives location of "sweet spot"

— Rolling



• Rolling without slipping is when a circular object (hoop, ball, disk, cylinder) moves forward and rotates so that the contact point is still
  + If CM velocity is $V\hat{i}$ and angular velocity $-\omega\hat{k}$, the contact point velocity is $(V - \omega R)\hat{i} \Rightarrow V = \omega R$ for no-slip
  + The velocity of the top point is $2V\hat{i}$, etc. Generally, velocity is not tangent to circle
  + Any friction is static + does no work b/c there is no motion at contact point.
  + You can consider this as rotation around contact point

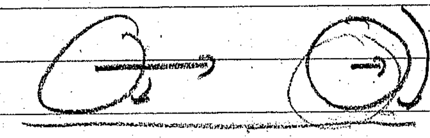


• Example: Rolling on incline
  + Linear motion has
    $$M\ddot{x} = Mg\sin\alpha - F_s$$ ← note static friction!

+ From CM point of view, friction also leads to torque
$$I_{cm}\ddot{\theta} = F_s R$$
and with $R\dot{\theta} = \dot{x}$, we get $(M + I_{cm}/R^2)\ddot{x} = Mg\sin\alpha$.

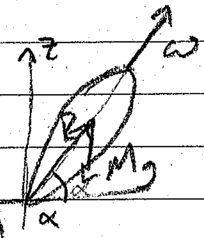
+ Acceleration is slower due to rolling. Why?
Change in potential energy is the same, but KE is
$$T = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_{cm}\omega^2 = \tfrac{1}{2}(M + I_{cm}/R^2)v^2$$
which is larger at same speed

• Rolling with slipping



+ Either the translation is too fast for the rotation or rotation is too fast for the translation
+ There is kinetic friction at the contact.
+ Friction speeds up or slows down rolling until slipping stops

– Precession: when the axis can move a little



• We have an object – often called a top – rotating quickly around an axis that can move slowly around a point
+ We have $\vec{L} = I\vec{\omega} = I\omega(\cos\alpha\,\hat{r} + \sin\alpha\,\hat{z})$
so $\dot{\vec{L}} = I\omega\cos(\alpha)\dot{\theta}\,\hat{\theta}$ assuming $\omega \approx$ const.
+ The torque due to gravity is
$$\vec{\tau} = \vec{R}\times M\vec{g} = MgR\cos\alpha\,\hat{\theta} = \dot{\vec{L}}$$
so $\vec{L}$ rotates slowly w/ angular velocity
$$\dot{\theta} = MgR/I\omega \leftarrow \text{precession}$$
+ This analysis really only works when $\dot{\theta} \ll \omega$ since it is an additional rotation (ie, $\vec{\omega}$ is not quite what we said here).

• The torque from gravity causes the top to precess, not fall over
+ This is why bicycles don't fall over while you ride.
+ A small torque from sun + moon cause earth's axis to precess once every ~26,000 yr.
"precession of equinoxes"