PHYS-3202 Homework 4 Due 18 Oct 2023

This homework is due to `https://uwcloud.uwinnipeg.ca/s/H4t44ogzdTkskyD` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **Average Energy**

   Consider a harmonic oscillator with mass m, spring constant k, and frequency $\omega_0 = \sqrt{k/m}$ (and no damping). The motion of this system is described by $x(t) = A\cos(\omega_0 t - \phi)$, where A and ϕ are constants.

   (a) Show that the period of oscillation is $T = 2\pi/\omega_0$.

   (b) Calculate the average kinetic energy over one period. Note that the time average of any quantity X over a time T is
   $$\langle X\rangle = \frac{1}{T}\int_0^T dt\, X(t)\ . \tag{1}$$
   Hint: first use the double angle formula to show that $\langle\cos^2(\theta)\rangle = \langle\sin^2(\theta)\rangle = 1/2$.

   (c) Calculate the average potential energy over one period. How does it compare to the average kinetic energy?

2. **Hanging Spring**

   Consider a mass m on a spring with potential energy $kx^2/2$, where $x = 0$ is the equilibrium extension. Suppose the spring is hung from the ceiling (with x increasing downwards).

   (a) Write the potential energy as a function of x with the inclusion of gravity and find the new equilibrium point x_0.

   (b) Rewrite the potential energy in terms of $y = x - x_0$. From the form of the potential energy only, argue that the motion of the hanging spring is harmonic oscillation around $y = 0$ and find the frequency of oscillation. Do not solve any differential equations — just compare the potential energy you find to the potential energy of a harmonic oscillator.

3. **Critically Damped Oscillator**

   Consider a critically damped harmonic oscillator with mass m, natural frequency ω_0, and damping coefficient $\alpha = \omega_0$. The oscillator has initial conditions $x = L, \dot{x} = 0$ at $t = 0$.

   (a) Show that $x(t) = (A + Bt)\exp(-\alpha t)$ solves the equation of motion. Then find the solution $x(t)$ that also satisfies the given initial conditions.

   (b) Without using the solution for $x(t)$, find the work done on the oscillator by the damping force from $t = 0$ to $t = \infty$. You may use the fact that $x \to 0$ as $t \to \infty$.