PHYS-4602 Homework 7 Due 6 March 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/FFJiJMNt9Czgo72` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor*.

1. **Momentum Differentiates the Position Operator**

   (a) For any three operators A, B, C, show that

   $$[A, BC] = [A, B]C + B[A, C] \ . \tag{1}$$

   (b) Then prove by induction that

   $$[A, B^n] = n[A, B]B^{n-1} \ , \tag{2}$$

   if $[A, B]$ commutes with B (for $n > 0$).

   (c) Finally, show using (2) that $[p, f(x)] = -i\hbar df/dx$, where x and p are 1D position and momentum operators with $[p, x] = -i\hbar$. Assume $f(x)$ can be written as a Taylor series.

2. **Gaussian Wavepacket**

   Here we consider the Gaussian wavepacket in 1D at a single instant $t = 0$, ignoring its time evolution. The state is

   $$|\psi\rangle = \int_{-\infty}^{\infty} dx \ Ae^{-ax^2}|x\rangle \ . \tag{3}$$

   Some of these results may be useful on future assignments.

   (a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates. The textbook cover also has a formula for Gaussian integrals.

   (b) Since the wavefunction is even, $\langle x\rangle = 0$. Find $\langle x^2\rangle$. *Hint:* You can get a factor of x^2 next to the Gaussian by differentiating it with respect to the parameter a.

   (c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x + b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

   (d) Find $\langle p\rangle$ and $\langle p^2\rangle$ and show that this state saturates the Heisenberg uncertainty principle. You should not have to do any integrations.