PHYS-4602 Homework 5 Due 14 Feb 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/FFJiJMNt9Czgo72` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **2-Qubit Gates**

   Consider a 2 qubit system. Choose a basis for the 2 qubit Hilbert space and use it for all parts of this problem.

   (a) Write the CNOT gate operator as a matrix in that basis and show that it is unitary.

   (b) Consider the 1 qubit gate NOT acting only on the first qubit of our two. Write this gate (call it NOT_1) as a matrix in your 2-qubit basis.

2. **Cloning Means FTL Communication** *based on a problem by Wilde*

   Suppose that Alice and Bob are at two ends of an EPR/Bell experiment. In other words, they are at rest with respect to each other and separated by 5 lightyears, and each receives one of a pair of entangled electrons with total spin state $s = 0$ simultaneously (in their common rest frame). By prior agreement, Alice measures either the S_z or S_x spin of her electron as soon as she receives it, but Bob does not know which spin she measures.

   After Alice's measurement (in their rest frame time), Bob's electron is in some state $|\psi\rangle_B$. Suppose, in contradiction to the no-cloning theorem, Bob can clone his electron's state onto a large number N of other electrons. (For example, Bob can do some quantum operation that takes his $N+1$ electrons from state $|\psi\rangle_B|\uparrow\rangle_1\cdots|\uparrow\rangle_N$ to state $|\psi\rangle_B|\psi\rangle_1\cdots|\psi\rangle_N$.) What measurement(s) can Bob do on his extra N electrons that will tell him with great certainty whether Alice measured the S_z or S_x spin of her electron? Explain your answer. (Note that Bob can accomplish his measurement before Alice can tell him her measurement choice, so they can establish faster-than-light communication in this way. This is a good reason for the no-cloning theorem!)

3. **Qutrit Teleportation** *from Blumel 7.6.2*

   Suppose we wish to teleport the state $|\psi\rangle = a|1\rangle + b|0\rangle + c|-1\rangle$ of a spin $s = 1$ particle (where the states are labeled by quantum number m). Label this as spin number 1, and prepare spins 2 and 3 in the spin singlet entangled state

   $$|\alpha\rangle_{23} = \frac{1}{\sqrt{3}}\left(|1\rangle_2|-1\rangle_3 - |0\rangle_2|0\rangle_3 + |-1\rangle_2|1\rangle_3\right) \ . \tag{1}$$

   Each of these particles is a "qutrit," a state in a 3D Hilbert space.

   (a) Consider the state

   $$|\phi\rangle_{12} = \frac{1}{\sqrt{3}}\left(|1\rangle_1|-1\rangle_2 + |0\rangle_1|0\rangle_2 + |-1\rangle_1|1\rangle_2\right) \tag{2}$$

   for spins 1 and 2. Find ${}_{12}\langle\phi|(|\psi\rangle_1|\alpha\rangle_{23})$ and normalize it. This is the state of spin 3 if we measure spins 1 and 2 to be in state $|\phi\rangle$. *Hint:* Normalization of $|\psi\rangle$ means that $|a|^2 + |b|^2 + |c|^2 = 1$.

(b) Given the measurement in the previous part, spin 3 is now in the state from your answer. Give a unitary operator that turns the state of spin 3 into $|\psi\rangle_3$ (and check that the operator is unitary). You may write your answer in matrix form if you specify a basis.

(c) Consider the entangled state $|\alpha\rangle_{23}$. What is the von Neumann entropy of the reduced density matrix of spin 3?