PHYS-4602 Homework 4 Due 7 Feb 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/FFJiJMNt9Czgo72` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor*.

1. **Thermal State**

   The density matrix for a quantum system in thermal equilibrium at temperature T is

   $$\rho = e^{-H/k_BT}/Z \text{ where } Z \equiv \mathrm{Tr}\left(e^{-H/k_BT}\right) \tag{1}$$

   is the *partition function*, H is the Hamiltonian, and k_B is the Boltzmann constant.

   (a) First, show that the partition function is

   $$Z = \sum_n e^{-E_n/k_BT} \ , \tag{2}$$

   where the E_n are the energy eigenvalues.

   (b) Show that another expression for the density operator is

   $$\rho = \frac{1}{Z}\sum_n e^{-E_n/k_BT}|E_n\rangle\langle E_n| \ , \tag{3}$$

   where $|E_n\rangle$ are the energy eigenvectors.

   (c) Show that the so-called *thermofield double state*

   $$|\psi\rangle = \frac{1}{\sqrt{Z}}\sum_n e^{-E_n/2k_BT}|E_n\rangle_1|E_n\rangle_2 \ , \tag{4}$$

   where $1, 2$ indicate two copies of the quantum system, is a purification of the thermal state by showing that the partial trace of $|\psi\rangle\langle\psi|$ over system 2 is ρ.

2. **Partial Traces and Entropy**

   Consider a system of two qubits with density operator

   $$\rho = \frac{1}{2}(|0\rangle_1|0\rangle_2)({}_1\langle 0|{}_2\langle 0|) + \frac{1}{2}(|+\rangle_1|1\rangle_2)({}_1\langle +|{}_2\langle 1|) \ , \tag{5}$$

   where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

   (a) Find the reduced density operator and von Neumann entropy of qubit #2 by taking the partial trace over qubit #1.

   (b) Find the reduced density operator and von Neumann entropy of qubit #1. Is the entropy the same as in part (a)? Why or why not?

   (c) Suppose the system actually has a third qubit such that the state of the 3-qubit system is a pure state (that is, the 3-qubit system is a purification of ρ). Write some example of such a 3-qubit pure state. What is the entropy of the third qubit considered on its own?