PHYS-4602 Homework 2 Due 24 Jan 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/FFJiJMNt9Czgo72` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor*.

1. **Measurement vs Time Evolution** *a considerable revision of Griffiths 3.33*

   Suppose a system has observable A with eigenstates $|a_1\rangle, |a_2\rangle$ of eigenvalues a_1, a_2 respectively and Hamiltonian H with eigenstates $|E_1\rangle, |E_2\rangle$ of energies E_1, E_2 respectively. The eigenstates are related by

   $$|a_1\rangle = \frac{1}{5}\left(3|E_1\rangle + 4|E_2\rangle\right)\ ,\quad |a_2\rangle = \frac{1}{5}\left(4|E_1\rangle - 3|E_2\rangle\right)\ . \tag{1}$$

   Suppose the system is measured to have value a_1 for A initially. Each of the following parts asks about a different possible set of subsequent measurements.

   (a) What is the probability of measuring energy E_1 immediately after the first measurement? Assuming we do get E_1, what is the probability of measuring a_1 again if we measure A again immediately after the measurement of energy?

   (b) Instead, consider immediately measuring A again after the first measurement. What are the probabilities for observing a_1 and a_2?

   (c) Finally, consider making the first measurement and then allowing the system to evolve for time t. If we then measure energy, what is the probability of finding energy E_1? If we instead measured A again, what is the probability we find a_1 again?

2. **Oscillating Spin**

   Consider a spin-1/2 particle like an electron. In terms of the S_z eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$, the eigenstates of the S_y operator are

   $$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)\ ,\quad |\leftarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - i|\downarrow\rangle\right)\ , \tag{2}$$

   where $|\rightarrow\rangle$ has eigenvalue $\hbar/2$ and $|\leftarrow\rangle$ has eigenvalue $-\hbar/2$. The electron is placed in a magnetic field along y, so the Hamiltonian is $H = -\gamma B S_y$. The electron has initial state $|\uparrow\rangle$.

   (a) What is the probability of finding $-\hbar/2$ as the result of a measurement of S_z at time t?

   (b) Find the expectation value $\langle S_y\rangle$ as a function of time.

   (c) As a matrix written with respect to the S_z eigenbasis $\{|\uparrow\rangle, |\downarrow\rangle\}$,

   $$S_y \simeq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\ . \tag{3}$$

   Show that the time evolution operator for this Hamiltonian can be written

   $$U(t) \simeq \cos\left(\frac{\gamma B t}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\sin\left(\frac{\gamma B t}{2}\right)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{4}$$

   in the S_z eigenbasis.

3. **MSW Effect for Neutrino Oscillation**

   Consider a simplified model of neutrino oscillation for two neutrino flavors. In vacuum, the Hamiltonian can be written in the mass eigenbasis $\{|\nu_1\rangle, |\nu_2\rangle\}$ as $H_{vac} \simeq \lambda \mathbb{I} + \Delta \sigma_z$ where λ, Δ are constants, $\mathbb{I}$ is the identity matrix, and σ_i are the Pauli matrices for $i = x, y, z$. The flavor states are

   $$|\nu_e\rangle = \cos\theta |\nu_1\rangle + \sin\theta |\nu_2\rangle \ , \quad |\nu_\mu\rangle = \sin\theta |\nu_1\rangle - \cos\theta |\nu_2\rangle \ . \tag{5}$$

   (a) According to the *MSW effect*, the neutrino Hamiltonian in dense matter (such as the interior of the sun) changes to a new Hamiltonian H_{MSW}. If the eigenstates $|\nu_A\rangle, |\nu_B\rangle$ of H_{MSW} are

   $$|\nu_A\rangle = \cos\alpha |\nu_1\rangle + \sin\alpha e^{i\phi} |\nu_2\rangle \ , \quad |\nu_B\rangle = \sin\alpha |\nu_1\rangle - \cos\alpha e^{i\phi} |\nu_2\rangle \ , \tag{6}$$

   write the flavor states $|\nu_e\rangle, |\nu_\mu\rangle$ as superpositions of $|\nu_A\rangle, |\nu_B\rangle$.

   (b) Suppose the new Hamiltonian is $H_{MSW} \simeq H_{vac} + \gamma \sigma_x$ (still written in the original basis), where γ is another constant. If we require $\tan\alpha > 0$, show that $\phi = \pi$ and $\tan\alpha = (\Delta/\gamma + \sqrt{1 + \Delta^2/\gamma^2})$. Assume that $|\nu_A\rangle$ has the smaller energy eigenvalue and that λ, Δ, γ are all positive.