PHYS-4602 Homework 10 Due 27 March 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/FFJiJMNt9Czgo72` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **Fermi's Golden Rule**

   Consider a sinusoidal perturbation Hamiltonian $H_1 = Ve^{-i\omega t} + V^\dagger e^{+i\omega t}$. In the class notes, we found the probability for a transition from state $|1\rangle$ to $|2\rangle$ as a function of time and frequency ω. In the following, define $\hbar\omega_0 = E_2 - E_1$, the difference of the energy eigenvalues of the unperturbed Hamiltonian H_0. We will investigate the transition probability near $\omega = \omega_0$ at large t (at least as long as the probability stays small).

   (a) At a fixed (and large) time, the probability is

   $$P = \frac{4|V_{21}|^2}{\hbar^2}\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}\ . \tag{1}$$

   Using L'Hospital's rule or just a power series expansion, find the peak transition probability as a function of time.

   (b) Find the values of ω where the probability first vanishes on either side of $\omega = \omega_0$. The difference in these two values tells us the width of the peak.

   (c) For large enough times, approximate the transition probability as a rectangle with the peak value from part (a) and width given by half the difference in part (b). Integrate this approximate probability function and argue that

   $$P \to \frac{2\pi|V_{21}|^2}{\hbar^2} t\delta(\omega_0-\omega) \tag{2}$$

   as $t \to \infty$.

   This problem shows two things: first, transitions occur only to states at energies related by the perturbation frequency and, second, that there is a constant transition rate (probability per unit time) to the appropriate states. The relationship (2) is known as *Fermi's Golden Rule.* (There is of course a more rigorous derivation possible.)

2. **Magnetic Resonance Spin Flips**

   Consider a spin-1/2 particle (for example, a proton) with gyromagnetic ratio γ in the presence of a magnetic field

   $$\vec{B} = B_0\hat{z} + B_1\cos(\omega t)\hat{x} - B_1\sin(\omega t)\hat{y} \tag{3}$$

   at its fixed position. This is a magnetic field with a fixed z component and another component rotating in the x, y plane.

   (a) Write the Hamiltonian either as a matrix or in terms of spin operators and show that it takes the form $H = H_0 + Ve^{-i\omega t} + V^\dagger e^{i\omega t}$. *Hint:* Recall that the Hamiltonian for a charged spin in a magnetic field is $-\gamma\vec{B}\cdot\vec{S}$, and write S_x, S_y in terms of spin raising and lowering operators $S_\pm = S_x \pm iS_y$ as defined in GS chapter 4.

(b) Assume that the rotating field B_1 is much smaller than B_0. If the spin is initially spin up at $t = 0$, find the transition probability to spin down at a later time t using perturbation theory. *Hint:* Consider the states in the Hamiltonian H_0 and their energy differences first.

(c) It is also possible to find this transition probability exactly. With the initial conditions given in part (b), the solution of the time-dependent Schrödinger equation is

$$\begin{aligned} \langle\uparrow|\Psi(t)\rangle &= e^{i\omega t/2}\left[\cos(\alpha t/2) - i\frac{(\omega - \gamma B_0)}{\alpha}\sin(\alpha t/2)\right] \\ \langle\downarrow|\Psi(t)\rangle &= ie^{-i\omega t/2}\frac{\gamma B_1}{\alpha}\sin(\alpha t/2) \end{aligned} \tag{4}$$

with $\alpha = \sqrt{\gamma^2 B_1^2 + (\omega - \gamma B_0)^2}$. Use (4) to find the transition probability from spin up ($|\uparrow\rangle$) to spin down ($|\downarrow\rangle$). Find the conditions that this probability is one. Finally, show that it reduces to the perturbation theory result when $\gamma B_1 \ll \omega - \gamma B_0$.