PHYS-4602 Homework 1 Due 17 Jan 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/FFJiJMNt9Czgo72` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **The Last Eigenvector**

   A system with a three-dimensional Hilbert space has an operator represented by the matrix

   $$B \simeq \frac{b}{3}\begin{bmatrix} 1/2 & -1/2 & 2i \\ -1/2 & 1/2 & -2i \\ -2i & 2i & -1 \end{bmatrix} \tag{1}$$

   in some basis. Two of the eigenstates are represented by the column vectors

   $$|1\rangle \simeq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } |2\rangle \simeq \frac{1}{\sqrt{3}}\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix} . \tag{2}$$

   (a) Find the third eigenstate $|3\rangle$ as a column vector, including proper normalization. Use *only* relations between the eigenvectors, not the action of B on them. *Hint:* show that B is Hermitian first.

   (b) Find all the eigenvalues of B and write B as a matrix in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.

   (c) The operator $|3\rangle\langle 3|$ projects any vector $|\psi\rangle$ onto its component in the $|3\rangle$ direction. Write $|3\rangle\langle 3|$ as a matrix in the original basis.

2. **Diagonalization** *Based on Griffiths A.26*

   Consider a three-dimensional Hilbert space with orthonormal basis $|e_i\rangle$, $i = 1, 2, 3$. The operator A takes the matrix representation

   $$A = \sum_{i,j} |e_i\rangle\langle e_i|A|e_j\rangle\langle e_j| \simeq \begin{bmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{bmatrix} . \tag{3}$$

   You should be able to check yourself that A is Hermitian.

   (a) Find the eigenvalues a_i and corresponding eigenstates $|a_i\rangle$ ($A|a_i\rangle = a_i|a_i\rangle$) written in terms of their components $\langle e_j|a_i\rangle$. Choose the eigenstates to form an orthonormal eigenbasis; that is, choose any ambiguities such that $\langle a_i|a_j\rangle = \delta_{ij}$.

   (b) Write the state $|\psi\rangle = |e_1\rangle - i|e_3\rangle$ in the A eigenbasis (as a superposition of the $|a_i\rangle$).

3. **Commutators and Functions of Operators**

   (a) Suppose $|a\rangle$ is an eigenfunction of some operator A, $A|a\rangle = a|a\rangle$. Consider the inverse operator A^{-1} defined such that $AA^{-1} = A^{-1}A = 1$. Show that $|a\rangle$ is an eigenvector of A^{-1} with eigenvalue $1/a$ if $a \neq 0$ (if there is an eigenvalue $= 0$, A is not invertible).

(b) For any function $f(x)$ that can be written as a power series

$$f(x) = \sum_n f_n x^n \ , \tag{4}$$

we can define

$$f(A) = \sum_n f_n A^n \ , \tag{5}$$

where A^n denotes operating with A n times. Show that

$$f(A)|a\rangle = f(a)|a\rangle \ . \tag{6}$$

Does this result hold if the power series includes negative powers?

4. **Permutation Operator**

Consider an N-dimensional Hilbert space with orthonormal basis $\{|1\rangle, |2\rangle, \cdots |N\rangle\}$ and define the permutation operator S such that $S|n\rangle = |n+1\rangle$ for $1 \leq n < N$ and $S|N\rangle = |1\rangle$. (S is the same as translation by one site on a lattice of allowed positions.)

(a) Show that the state

$$|\lambda\rangle = \sum_{n=1}^{N} \lambda^{-n+1}|n\rangle \tag{7}$$

is an eigenstate of S with eigenvalue λ as long as λ takes one of N allowed values. Find those allowed values.

(b) Is S ever a Hermitian operator? If so, what are the values of N such that S is Hermitian?