PHYS-3203 Homework 1 Due 25 Jan 2024

This homework is due to `https://uwcloud.uwinnipeg.ca/s/Re9qoZBqcD8F5oe` by 10:59PM on the due date. Your file(s) must be in PDF format; they may be black-and-white scans or photographs of hardcopies (all converted to PDF), PDF prepared by LaTeX, or PDF prepared with a word processor *using an equation editor.*

1. **More on the Brachistochrone**

   Consider a particle moving along a brachistochrone path, written parametrically as

   $$x = a\,(\theta - \sin\theta)\ ,\quad y = a\,(1 - \cos\theta)\ , \tag{1}$$

   under the influence of gravity (without friction). Note that y increases downward.

   (a) Show that the time for the particle to move from $x = y = 0$ at $\theta = 0$ to a point parameterized by $\theta = \theta_0$ is

   $$T = \frac{1}{\sqrt{2g}} \int_0^{\theta_0} d\theta \sqrt{\frac{(dx/d\theta)^2 + (dy/d\theta)^2}{y}}\ . \tag{2}$$

   (b) The minimum of the curve at $x = \pi a, y = 2a$ is at $\theta = \pi$. Show that the time for the object to reach the bottom is $T = \pi\sqrt{a/g}$.

   (c) Write the Euler-Lagrange equations for the functional (2) for both functions $x(\theta)$ and $y(\theta)$. You may use a prime to indicate the derivative with respect to θ. *Note:* do not plug in the solution (1).

2. **Maximizing Area**

   Consider a closed curve enclosing the origin in two dimensions described by the function $r(\phi)$ in plane polar coordinates. *Hint:* you may find examples 5.8 and 5.9 in Cline to be relevant.

   (a) Show that this curve encloses an area

   $$A = \frac{1}{2} \int_0^{2\pi} d\phi\, r(\phi)^2 \tag{3}$$

   and has circumference

   $$L = \int_0^{2\pi} d\phi \sqrt{r^2 + (r')^2}\ , \tag{4}$$

   where the prime indicates a derivative with respect to ϕ. *Hint:* Use the infinitesimal area element and Pythagorean length in polar coordinates.

   (b) Write down a functional that you can optimize to find the curve $r(\phi)$ that maximizes the enclosed area A for a fixed circumference L.

   (c) Show that $r(\phi)$ constant (that is, $r' = 0$) is a solution of the Euler-Lagrange equation, which proves that a circle is the shape of maximum area for a given circumference. *Hint:* find the allowed value of r if $r' = 0$.

3. **Harmonic Oscillator**

   Consider a simple harmonic oscillator moving in one dimension with restoring force $F = -kx$. Write the Lagrangian for the harmonic oscillator and show that the Euler-Lagrange equation is Newton's second law $m\ddot{x} = -kx$.